

Emergency Preparedness

Rare Events and the Persistence of Uncertainty*

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Abstract

Unexpected events can have lasting effects on uncertainty. This paper uses a model in which the realizations of ex-ante unlikely events endogenously result in lower levels of private information. Lower levels of information propagate within the model, as uncertainty makes it harder for agents to acquire information about future periods, resulting in uncertainty persistence. This model of uncertainty is applied to an economy with a financial market. Uncertainty reduces asset demand and expected wealth, while increasing dispersion of beliefs. Data on financial uncertainty, dispersion of beliefs, and risk appetite confirm the predictions of the model.

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1 Introduction

Unexpected events of economic relevance can cause persistent jumps in uncertainty. Uncertainty has financial implications: In financial markets, spikes in uncertainty are often accompanied by increased dispersion of beliefs, increased bid-ask spreads, increased volatility, and decreased leverage. By focusing on the aftermath of rare events, this paper presents and tests a model that explains patterns in uncertainty and the effects of persistent uncertainty on financial variables.

Consider the examples of events marked by green lines in Figure 1 panel (a), each of which could be considered ex-ante unlikely. The first is the terrorist attacks in September 2001. The second is Lehman Brothers' bankruptcy in September 2008. The third is the flash crash in May 2010. The fourth is the downgrade of U.S. treasury debt by the S&P ratings agency in 2011. The fifth is a sharp downturn in China's manufacturing activity in August 2015. In each case, the VIX¹ spikes upon the occurrence of the event and then slowly reverts to its pre-shock levels.

I propose a model to explain three observations: (i) uncertainty spikes upon the realization of rare events; (ii) the spikes persist; and (iii) changes in uncertainty spill over into other financial variables. In this model, agents can invest in information about future states of the world. Upon the occurrence of any state, agents receive signals about the future path of an asset's price. Therefore, the quality of signals received is determined by their informational investment decision. The benefit of investing in information varies positively with the ex-ante likelihood of the state, but the cost is assumed not to vary. In order to preserve informational asymmetries and prevent perfect revelation of information in prices, all trade takes place through a perfectly competitive market making sector.

This model produces three theoretical results. The first result is that agents choose to invest in more information for likely states of the world and less for unlikely states of the world. Consequently, when an agent invests in information about one state of the world, and that state occurs, signals about future states are less noisy (or, of a higher quality). The inverse is also true. Thus, the occurrence of an ex-ante unlikely state results in increased short term and long term uncertainty due to poor signal quality.

Second, the model results in increased long term uncertainty, which persists endogenously. Poor signals result in a more dispersed distribution over possible states in the subsequent period. The average quality of signals purchased under this dispersed distribution will be worse than under a tight distribution, because signal quality is positively correlated with the probability of a state's

1. The VIX is defined as the market's expectation of the annualized percentage standard deviation of the S&P index over a thirty day period.

occurrence. Therefore, an initial unlikely event generates uncertainty, which persists and propagates by preventing agents from concentrating the allocation of their informational resources and prolongs the initial spike in uncertainty.

The third set of theoretical results stem from the interaction of agents' signal quality with the market makers' information set. Because the market making sector prevents perfect price revelation, this interaction causes spillovers into financial variables such as volatility, bid-ask spreads, dispersion of beliefs and asset demand.

The informational investment mechanism I use will deliver uncertainty persistence without long term learning under any circumstances where information is a good. Therefore, there are several environments in which to explore the effects of this mechanism. In this paper, I consider a framework where information gives agents an advantage in trade. Choosing this environment to explore the spillover is helpful because it allows the model to deliver predictions about financial uncertainty, while maintaining portability into a macroeconomic setting. Fundamentally the patterns of uncertainty are universal (that is, not specific to finance), but financial variables have plentiful and frequently observed data. Therefore, taking the theory to the data is more direct in a financial setting.

I provide three sets of empirical results. First, I non-parametrically estimate the ex-ante probabilities of daily price movements in the S&P index. In a regression of the VIX on those probabilities, the coefficients are significantly negative, showing that rare events cause uncertainty. These results hold even when controlling for standard predictors of the VIX such as volume of trade, the size of price changes, and the level of the S&P index.

Second, I test the model's mechanism for persistence - that increased dispersion in an agent's ex-ante beliefs leads to a flatter distribution over future states. I test that increased dispersion in an agent's ex-ante beliefs lowers future probabilities on average, by showing that increases in the levels of the VIX are associated with lower 'highest' probabilities, 'average' probabilities, and dispersion of probabilities, and a smaller negative correlation between volatility and rarity. These results are all consistent with a theory of flatter future ex-ante distributions over price changes, as opposed to the alternative hypothesis of multimodality.

Third, a calibration and simulation tests the model's ability to match the VIX, dispersion of beliefs, and investors' risk appetite. Using one period's change in the S&P index as inputs, the model outputs the next period's opening value of the VIX at a monthly frequency. The model's predicted VIX is highly correlated with the data, both in levels and in first differences, and the

patterns of dispersion and risk match the data as well.

1.1 Alternative Theories

Much of the existing literature on macroeconomic and financial uncertainty employs one of two theoretical underpinnings - rare disasters theory and Bayesian learning. This paper relates to both and contributes to each. Rare disasters theory shows that the realization of disasters (sometimes called black swans) - or usually the increased fear of their occurrence - makes agents more uncertain. This theory (as described by, for example, Orlik and Veldkamp (2014)) achieves a good description of macroeconomic (or, real) uncertainty. The importance of the skew is also documented in Bekaert and Popov (2012). However, if the black swan story were true for financial uncertainty, then the SKEW² (or fatness of the left tail) of the VIX should be positively correlated with the level of the VIX. Fatness of the left tail of the distribution would be a measure of fear of disasters and should cause generalized uncertainty. In fact, as shown by the Chicago Board Options Exchange itself,³ the two are not correlated and at a daily frequency, they are actually negatively correlated - at about -0.13. This indicates that fear of rare events does not constitute a full explanation of changes in financial uncertainty. The model in this paper assumes no skewness in either the underlying process or in beliefs and, as a result, does not have to explain this negative correlation.

The second standard explanation of uncertainty patterns is Bayesian learning, under which a Bayesian agent learns and updates beliefs about the true parameters of the economy through shocks. Bayesian agents learn very quickly, so when using Bayesian learning, there is a tension between making the shocks big enough to match the significant changes in economic dynamics and having the learning process be slow enough that uncertainty is not resolved too quickly. It is evident from Figure 1 panel (a) that because the level of the VIX repeatedly spikes, a purely Bayesian story might be incomplete. Other papers have proposed using anticipated utility equilibria, limited memory, or regime changes (such as is found in Bianchi and Melosi (2013)) to get repeated spikes. The agents in this paper use Bayes rule to update expectations; however, this paper delivers rare events that lead to uncertainty even with an infinite amount of previous data.

This paper's explanation of persistent uncertainty emerges along with a growing literature on the subject. I will discuss the methods of two recent and notable contributions. The first is Kozlowski, Veldkamp, and Venkateswaran (2015), in which the authors posit that as agents use

2. As calculated by the CBOE, it is a non-parametric measure of the skewness of the distribution of the market's expectation of annualized percentage price changes of the S&P index over the next 30 days

3. Report (2010)

standard econometric tools to estimate the distribution of aggregate shocks, the occurrence of an extremely unlikely event changes their beliefs suddenly and increases their uncertainty. The uncertainty persists as the shock lives permanently in the agents’ datasets. Using this setup, they are able to match the downward shift in trend output of the great recession. The second is Fajgelbaum, Schaal, and Taschereau-Dumouchel (2014). In that paper’s framework, uncertainty over underlying parameters of the economy affect firms’ decisions about how much to invest. The results of those decisions inform other firms, which can learn from the distribution of all investment returns. An initial spike in uncertainty causes fewer firms to invest, resulting in lower levels of learning, and therefore uncertainty traps. This paper joins this literature in trying to explain the persistence of uncertainty from a complementary angle: Instead of trying to understand agents’ learning of economic parameters, this model focuses on endogenous information acquisition.

1.2 Literature Review

This paper borrows modeling techniques from two sources. First, it uses a reduced form of the inattention structure built in Woodford (2012), which shows how rare events cause poor conditional identification. Second, in order to preserve uncertainty when trading and prevent price revelation, it uses a simple batch-order version of the market makers designed by Glosten and Milgrom (1985). The initiation and preservation of uncertainty is key, as it allows for the informational and adverse selection dynamics necessary for all of the results of the paper.

There are several strands of the theoretical literature that I interact with. The first is that of uncertainty shocks. As proposed in Bloom (2009), and subsequently expanded upon by many others (including Aghion et al. (2010); Arellano, Bai, and Kehoe (2010); and Gilchrist, Sim, and Zakrajšek (2014)), downturns in output or investment can be thought of as originating from spikes in macroeconomic uncertainty. The causality can either run through the increased option value of waiting or (as in this work) the increased cost of borrowing caused by higher credit spreads. I endogenize the connection between the actual event and the increase in uncertainty, while maintaining flexibility and portability into macro settings.

The second strand is that of uncertainty being tied to rare disasters. Orlik and Veldkamp (2014) shows that tail events (or even fear of tail events) can cause spikes in real uncertainty. Cúrdia, Negro, and Greenwald (2014) show that fattening the tails on the distribution of shocks has better results in matching data on volatility. There is also a sizable literature in finance on the equity premium puzzle started by Rietz (1988), and reinvigorated by Barro (2006), showing that a fear of

tail events can generate excess returns in equities. Learning models, such as Evans and Honkapohja (2001), and Cogley, Matthes, and Sbordone (2011) take anticipated utility approaches. This paper differs in three main respects from the rare events literature: First, there is no skewness in the model - conditional uncertainty can arise from positive shocks as well as negative ones. Second, there is no long-term Bayesian learning about the parameters of the economy. Third, it treats the level of informedness as endogenous.

The third strand is that of market microstructure, informational asymmetries, and imperfect price revelation. Hassan and Mertens (2014) use noise traders to create a DSGE that results in imperfect price aggregation. Fishman and Parker (2015) study a market where adverse selection can occur and lead to multiple equilibria. Bond, Edmans, and Goldstein (2011) show transmission of price information from the financial sector to the real economy. Kurlat (2010) and Guerrieri and Shimer (2012) show that adverse selection can lead to illiquidity even when sellers offer at different prices. Bookstaber and Pomerantz (1989) relates discrete information packet collection to financial volatility. Routledge and Zin (2009) show that adverse selection can lead to illiquidity when the market makers have uncertainty aversion. On the empirical side, Bleaney and Li (2014) and Yip et al. (2002) show that adverse selection is an important component of bid-ask spreads (with the latter finding that it is the single biggest component). This paper draws on all of these different strands, but seeks to answer a fundamentally different type of question: What effects do rare shocks have on a financial market *as communicated* through a particular market microstructure?

This paper also touches on the nature of information collection, as motivated by the notion of inattention, as first brought to light by Sims (2003). Woodford (2012) provides neuro-scientific evidence that traditional formulations of rational inattention are incompatible with actual attentional behavior, and proposes an alternative formulation. This paper takes that alternative as a starting point; although the model does not microfound agents' attentional decisions, the motivation is to stay in step with Woodford (2012). An application of these methods to a macro setting is seen in Maćkowiak and Wiederholt (2009) to explore price stickiness, Paciello and Wiederholt (2013) for optimal monetary policy, and Maćkowiak and Wiederholt (2010) to explore sluggish responses. As in this paper, the only source of adjustment in Maćkowiak and Wiederholt (2010) is attention. Matějka (2010) provides a host of results on price dynamics for inattentive sellers. In finance, Mondria (2010) uses inattention to discuss price comovement, Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014) looks at how fund managers add value to clients, and Van Nieuwerburgh and Veldkamp (2010) match portfolio selection patterns. Rational inattention is used in Maćkowiak

and Wiederholt (2011) along with limited liability and strategic complementarity in actions to get inattention to rare events. I add to this burgeoning literature by using elements of inattentive frameworks to address issues of uncertainty persistence.

The empirical section of this paper interacts with a large literature that wrestles with forecasting the VIX. Additionally, there is a large literature on using the VIX to predict stock returns, such as Bekaert and Hoerova (2014). The VIX forecasting literature focuses in part on parametric estimation of factors that could predict the value of the VIX, and formulates options trading strategies to profit off these predictions. Examples in this literature are Ahoniemi (2008), Harvey and Whaley (1992), Brooks and Oozer (2002), and Konstantinidi, Skiadopoulos, and Tzagkaraki (2008). This paper does not address issues of trading strategy in its notion of forecasting - rather it is interested in understanding movements in the VIX through a micro-founded understanding of the rarity of price movements.

This paper can converse with the literature in self-exciting processes: papers such as Aït-Sahalia, Cacho-Diaz, and Laeven (2015) and Aït-Sahalia, Laeven, and Pelizzon (2014) construct models in which large jumps in one market can make jumps more likely in the same market as well as in other markets. This paper can deliver similar results in beliefs, though with an iid underlying process.

2 Rare Events and Uncertainty

I present a simple model in this section, which formally introduces the underlying mechanism through which rare events increase uncertainty.

2.1 Setup

The simple model has three types of agents, one period with three stages (0, 1, and 2), and one asset. The agents are noise traders, informed traders, and market makers. Noise traders function as hedgers - investors who are not sensitive to prices. Informed traders function as speculators - investors who have an opinion over the future path of an asset's price and seek to profit from it. Market makers are intermediaries through which noise traders and informed traders trade. In this model, the market maker provides a friction that prevent buyers and sellers from interacting directly with each other over the course of the period. The model focuses on how informed traders collect information in an imperfectly informative market, and how they use that information to exploit market prices.

The simple model includes only one period, which allows clear identification of the channel through which an unexpected event can lead to increases in uncertainty. Once the mechanism for uncertainty is established, I will introduce dynamics in later sections to explore uncertainty persistence. The three stages of this simple model correspond to, in order: informational investment, informational payoff and trading, and accounting and conclusion.

2.1.1 Asset

The asset has a stage 2 value that is comprised of two random variables: B , the value of which is revealed in stage 1, and η_B , the value of which is revealed in stage 2. B can be thought of as the ‘event’ - a price change for which investors can prepare. B is realized and publicly revealed in stage 1, when it can take one of two values: H or L . In stage 0, before B is revealed, all agents have a common prior distribution on the values of B : $P(B = H) = \pi_H$ and $P(B = L) = 1 - \pi_H = \pi_L$. If it is ex-ante unlikely for B to take the value L , then π_L will be small, and if B were revealed to equal L in stage 1, that could be considered a rare event.

If B can be thought of as the ‘event,’ η_B can be thought of as the ‘reaction’ - a price change that occurs upon the value of B realizing. Just as B indexes H and L , η_B indexes two random variables: η_H and η_L . These two random variables are independent of one another, and all agents know that each is distributed $P(\eta_B = 1) = P(\eta_B = 0) = 0.5$. When B realizes in stage 1, the relevant variable η_B is determined, but it is not revealed until stage 2. Only one of η_H and η_L will realize, as B will take the value H or L in stage 1. The structure of the asset is best seen in Figure 2.

The possible values of η_H and η_L are independent of one another. Although this assumption is not meant to be a pure reflection of reality in a financial market, it is used here to illustrate that different macroeconomic events may affect asset prices through different and not necessarily dependent channels. It is not important what the relative value of H and L are; rather it is important that η_H and η_L are independent so that collecting information about one does not help identify the value of the other.

2.1.2 Information

There are two types of information in this model: private and public. In stage 0, informed traders can pay to invest in information about each η_B . The investment pays off by giving agents information about η_B before it is revealed in stage 2, allowing them to trade advantageously with respect

to the asset's terminal value: $B + \eta_B$. Investment in a particular value of η_B is advantageous only if that value of B realizes, and provides no benefit if a different value of B realizes. For example, investing in information about η_L will prove useless if $B = H$. Therefore, agents must choose whether to be informed about η_H , η_L , both or neither, knowing that only one informational investment will actually pay off. In stage 1, agents who invested in the correct η_B will receive a perfectly informative private signal about η_B 's value before it is publicly revealed in stage 2. This information allows them to trade advantageously against the market makers, who only have access to public information.

Public information about η_B in this model is exogenously given and constant across values of B . In stage 1, a public signal about whether η_B is 1 or 0 is revealed to all agents and has accuracy $\beta \geq 0.5$. This means that $P(\text{public signal} = 1 | \eta_B = 1) = P(\text{public signal} = 0 | \eta_B = 0) = \beta$. In later sections this assumption will be relaxed, and public information will be allowed to vary across states.

Intuitively, allowing agents to purchase signals about η_B mirrors traders doing research on conditional trading decisions for potential events. Theoretically, this assumption is motivated by the inattention literature. For tractability, the full model of inattention is not treated here. A more rigorous treatment of inattention with the full entropy formulation of Woodford (2012) is provided in Appendix ??, which also allows private agents to be risk averse. The results are quite similar.

2.1.3 Trading Mechanics

Traders inhabit a unit continuum, a fraction $T > 0$ of which are **noise traders**. The noise traders are defined by their behavior: Regardless of prices, half of the noise traders always buy one unit of the asset and half always sell one unit of the asset. Their sole purpose is to provide liquidity, and without them, the results of the simple model would be degenerate. The model's results are not very sensitive to the behavior of the noise traders, and the choice described above is the simplest.

The remaining $1 - T$ traders are risk-neutral **informed traders**. Informed traders want information because information allows them to update their private valuations of the asset and to trade advantageously with the market makers. In order to acquire information, each trader can purchase a signal at cost c in stage 0. A signal gives perfect information about η_B in stage 1 to the trader who bought it. If no signal is purchased, no information is gained. After a trader has received her signal, she can choose to buy one unit of the asset, sell one unit of the asset, or abstain from trade in stage 1. I limit asset purchase decisions to the set $\{-1, 0, 1\}$ and employ risk-neutrality

for simplicity. If the traders were risk-averse and had free volume choices it would not significantly change the results of the model.

The **market makers** broker trades among noise and informed traders. They are perfectly competitive. Market makers observe public information and set a bid and an ask. A bid is the price at which the market maker is willing to buy the asset, and an ask is the price at which the market maker is willing to sell the asset. Conversely, the bid is the price at which traders can sell, and the ask at which traders can buy. All trade must go through market makers - agents cannot trade directly with one another. In reality, most financial trade is carried out through intermediaries, so this assumption is not unrealistic. The particular mathematical formulation of the market making sector I use in this model is based off Glosten and Milgrom (1985). Their structure allows this paper to analyze the effects of changes in public and private information. This type of sector will preserve informational asymmetries and prevent price revelation.

Figure 3 shows the order of events of the simple model. In stage 0, agents decide whether or not to purchase signals. In stage 1, B realizes and is publicly revealed. Agents receive their private signals, and the public signal is revealed. The market making sector sets a bid and an ask, and agents trade. In stage 2, η_B is publicly revealed and agents receive their payoffs.

2.2 Solving the Model

The model is solved backwards. Stage 2 is trivial, as it is merely an accounting exercise. The problems of interest are in stage 1 and stage 0.

2.2.1 Pricing Problem

Consider the market makers' stage 1 problem once the public signal has been revealed and private signals have been seen by informed traders. Market makers see that B has been revealed, so the terminal value of the asset will be B or $B + 1$. The zero-profit conditions, as proven by Glosten and Milgrom (1985) are:

$$\begin{aligned} P(\eta_B = 1 | \text{buy order and MM Info}) + B &= ask \\ P(\eta_B = 1 | \text{sell order and MM Info}) + B &= bid \end{aligned}$$

Since the public signal could be either 1 or 0, there are four potential prices that could be set (two bids and two asks). These prices depend on the fraction s of informed agents that chose to buy the

relevant signal in stage 0:

$$\text{ask}_1 = B + \frac{\left(\frac{T^*(s)}{2} + (1 - T^*(s))\right) \beta}{\frac{T^*(s)}{2} + (1 - T^*(s)) \beta} \quad (1)$$

$$\text{bid}_1 = B + \frac{\frac{T^*(s)}{2} \beta}{\frac{T^*(s)}{2} + (1 - T^*(s))(1 - \beta)} \quad (2)$$

$$\text{ask}_0 = B + \frac{\left(\frac{T^*(s)}{2} + (1 - T^*(s))\right) (1 - \beta)}{\frac{T^*(s)}{2} + (1 - T^*(s))(1 - \beta)} \quad (3)$$

$$\text{bid}_0 = B + \frac{\frac{T^*(s)}{2} (1 - \beta)}{\frac{T^*(s)}{2} + (1 - T^*(s)) \beta} \quad (4)$$

where $T^*(s) = \frac{T}{T+s(1-T)}$. $T^*(s)$ can be thought of as the effective fraction of noise traders because $1 - s$ informed traders *will not* buy a signal and will abstain from trade.

The first two prices - ask_1 and bid_1 - correspond to those the market making sector would set if the public signal were a 1 and the second two correspond to those the market making sector would set if the public signal were a 0.⁴ All four prices will always lie between B and $B + 1$ by definition. Further the ask will always be weakly larger than the bid (otherwise there would be an arbitrage opportunity). The difference between the bid and the ask is called the bid-ask spread, and is an indication of how much the market making sector fears adverse selection.

An increase in $T^*(s)$ due to fewer informed traders buying signals would narrow the bid-ask spread because a higher percentage of noise traders reduces the adverse selection problem faced by market makers. If no traders purchase signals, there would be no adverse selection for market makers to protect against, and spreads would go to zero. Conversely, because $T > 0$, all bids and asks will lie strictly between B and $B + 1$, which means that perfectly informed traders have an incentive to buy or sell.

Knowing how prices will be calculated in stage 1, the signal acquisition problem of stage zero may be discussed.

2.2.2 Traders' Signal Problem

Each trader faces an identical problem in stage 0. Conditional on the strength of β , the trader decides whether purchase a signal for each η_B . The cost of acquiring a signal is fixed at c for all

4. If $\beta = 0.5$, then the public signal is completely uninformative and the two asks and the two bids would equal one another.

values of B . The expected benefit of acquiring a signal for B , given values T and π_B is:

$$\frac{\pi_B}{2} [\beta(1 - \text{ask}_1(s)) + (1 - \beta)(1 - \text{ask}_0(s)) + (1 - \beta)(\text{bid}_1(s)) + \beta(\text{bid}_0(s))] \quad (5)$$

where an individual trader takes s as given in making her decision. Recall that π_B reflects the probability that the state for which the signal is bought will occur and thus the probability that the signal will be useful. β represents the probability that the public signal will be correct. If traders see through their private signal that $\eta_B = 1$, then they will buy at the ask, which is lower than $B + 1$; if they see that $\eta_B = 0$, they will sell at the bid, which is higher than B . Rearranging, and substituting terms, we can see that the trader will choose:

$$\max \left\{ 0, \frac{\pi_B \beta_B (1 - \beta_B) \frac{T^*(s)}{2}}{\left(\frac{T^*(s)}{2} + (1 - T^*(s))(1 - \beta_B) \right) \left(\frac{T^*(s)}{2} + (1 - T^*(s)) \right)} - c \right\} \quad (6)$$

The above expression shows that the agent is more likely to acquire a signal if, all else being equal, $\pi_B \uparrow$, $c \downarrow$, $T^*(s) \uparrow$ or $\beta \downarrow$. As π_B increases, there is a higher likelihood that the agent will benefit from identifying η_B , and as a result, agents will buy signals for higher likelihood states. Although a simple case, this model shows that less likely states are less likely to attract informed agents, as the cost of acquiring a signal in a state is not correlated with the probability of that state occurring. Similarly as the cost of purchasing a signal decreases, more agents will invest in information generally. As $T^*(s)$ increases, bid-ask spreads will decline due to a reduction in the adverse selection problem, but the informativeness of the price will not change. Therefore, the expected profit of being informed will increase, as the prices at which agents can trade are more attractive. As β increases, market makers can still set tighter spreads, as the level of adverse selection has gone down, but now prices are more accurate. Therefore information is less valuable to traders and so a signal is less likely to be purchased. This means that public and private information are strategic substitutes.

An **equilibrium**, given a value of β , is defined as a set of prices $\{\text{ask}, \text{bid}\}$, and a purchase decision $Pur \in \{1, 0, -1\}$ that satisfy equations (1)-(4), and a set of purchase decisions $\{\text{buy}, \text{don't buy}\}$ by the measure $(1 - T)$ informed traders that satisfy equation (6) in stage 0.

2.3 Key Predictions

Given the equilibrium definitions, there are two key propositions:

Proposition 1. *s^* , the equilibrium fraction of informed traders who purchase a signal, is non-decreasing in π_B .*

Proposition 2. *For any given c , there is a sufficiently low $\pi_B \in (0, 1)$ such that all traders will not buy a signal for state B .*

All proofs of propositions and corollaries are in Appendix A). These results follow directly from equation (6), and illustrate the key mechanism of this section: as the probability of a state goes down, agents are less informed on average upon its occurrence. Further, there is always a sufficiently unlikely state such that all agents will be uninformed when it occurs.

This model was set up to uncover the mechanism under which low probability events can trigger spikes in uncertainty. As is evident, private signals are more likely to be purchased when the probability of the state increases. These propositions show that when rare events occur, uncertainty increases through an informational investment mechanism. The structure that drives this result is that the value of investment varies with the ex-ante probability of the state, but the cost of investment does not. As a result, rare events result in poor signal quality, lower levels of information, and uncertainty.

3 Rare Events and Persistence

I will now extend the simple model to multiple periods to show how uncertainty can persist endogenously. Informational investment is the inter-temporal choice variable, and will be the only inter-temporal aspect of the model. I will not consider consumption smoothing, investment adjustment costs, or other such methods. As seen in the previous section, investment in information can affect trading decisions today. As we will see in this section, information investment today can also affect informational investment decisions tomorrow. This multi-period model will test how much an informational investment mechanism can cause uncertainty to persist.

3.1 Dynamic Model

Suppose that the model described in the previous section runs for many periods, each of which has three stages to match the initial setup. I will assume that informational investment in time t impacts not only agents' beliefs over η_B in time t , but also agents' beliefs over B in time $t + 1$. In order to find the distribution of B_{t+1} given B_t , $\eta_{B,t}$ and choices by private traders, assume that $B_{t+1} = B_t + \eta_B + u_{t+1}$. That is, the value of B in period $t + 1$ is simply the terminal value of

the asset in period t , plus a noisy variable u . This variable exists so that, at the outset of each period, agents do not know which value B will take, and the structure of informational investment can be used again. A trader's information investment decision in one period is affected by the informational investment decision of the period before. As before, the underlying process of the asset is not dependent on decisions by agents, but the distributions agents have over the variables that define the asset are dependent on the agents' decisions.

For the rest of the theoretical section, I will simplify the signal acquisition process. Instead of heterogenous agents making individual decisions to acquire information, I will use a representative agent who either acquires a signal or does not. This agent could be thought of as a single trader purchasing information and then commanding power in the market equal to a share $1 - T$ of trade, or as a way to enforce a symmetric decision-making among a continuum of traders. As shown in Appendix B, this simplification has no effect on the utility of traders in the static or dynamic version of the model, and it eliminates additional unstable equilibria in the dynamic version. Since characterizing the nature of these equilibria is not a priority of this paper, introducing a representative agent allows focus on the result of changes in signal quality across states rather than the result of changes across agents within a state.

As with η_B , the agent can receive a signal about u with an accuracy determined by the signal acquisition decision. $\pi_{t+1} = 0.5$ if the agent does not buy a signal and $\pi_{t+1} = k > 0.5$ if the agent does buy a signal. To simplify notation, I will define $\pi_t \equiv \max\{\pi_H, \pi_L\}$. The representative trader now needs to solve the following problem given a value of β :

$$V(\pi, 1 - \pi) = \sum_{\pi} \max \left\{ \frac{\pi\beta(1 - \beta)\frac{T}{2}}{\left(\frac{T}{2} + (1 - T)(1 - \beta)\right)\left(\frac{T}{2} + (1 - T)\right)} - c + \delta\pi V(k, 1 - k), \delta\pi V(0.5, 0.5) \right\} \quad (7)$$

In the multi-period case, there are two effects: Selecting a signal impacts trade of the asset in t , but it also impacts information collection in $t + 1$. Now, a signal is more valuable, as it has long term informativeness as well as short. I have deliberately separated the long term and short term effects of signal acquisition, although they could have been combined without much trouble. By separating them, we can see that reduction of noise in η is useful to agents because it allows them to trade advantageously against market makers, and reduction of noise in u is useful because it allows agents to collect information more efficiently going forward.

The definition of a **dynamic equilibrium** for a given β and k in stage 0 is the representative trader's choice to buy a signal or not to satisfy equation (7) and a choice of prices by market makers to satisfy equation (1)-(4).

3.1.1 Information Over Time

The multi-period model produces uncertainty persistence because a trader's lack of information in one period carries forward into future periods. This is best shown through the step-wise incentive structure that informed traders face for different costs of information. At the extremes, if information is cheap and c is close to 0, traders will always purchase a signal, as it costlessly provides information. Conversely, if information is exorbitantly expensive and c goes to infinity, traders will never purchase a signal, and $V(\pi, 1 - \pi) = 0$ for all π . In between, there are two intermediate thresholds:

1. Purchase signals for state B if $\pi_B = k$ and if $\pi_B = 0.5$ but not if $\pi_B = 1 - k$. In this case, the agent will be perfectly informed until the unlikely state occurs, at which point they will be uninformed. Recall that when no signal is purchased, $\pi_{t+1} = 0.5$. Once they reach this state, they will purchase signals for both (now equiprobable states in $t + 1$) and uncertainty will last for only one period.
2. Purchase signals for state B if $\pi_B = k$ but not if $\pi_B = 0.5$ or $\pi_B = 1 - k$. In this case the agent will be perfectly informed until the unlikely state occurs, at which point they will be *permanently* uninformed. This type of persistence is a stark version of what will be developed in the next section.

Proposition 3. *For any β and any $\pi_t \in (0, 1)$, there is a sufficiently high value of c , such that the private trader will not purchase a signal and $\pi_{t+1} = 0.5$.*

Corollary 1. *There is a threshold \bar{c} such that if $\pi_t = 0.5$ and $c > \bar{c}$, $\pi_{t+1} = 0.5$.*

Proposition 3 shows that for sufficiently high costs, or equivalently, for sufficiently rare events, signal quality is poor, and the ex-ante distribution of B_{t+1} will be disperse. The corollary shows that there is a cost such that the agent will not choose to buy a signal for $\pi = 0.5$. Thus, for certain threshold cost, if an event occurs with a very low probability ($1 - k$), the agent will be permanently uninformed thereafter.

Figure 4 illustrates the function describe in this section. The x -axis corresponds to periods of time. The y -axis is the degree of uncertainty faced by agents over η_B . The red vertical lines correspond to events that occur with a low ex-ante probability. In the first panel, the agent purchases signals when $\pi_t = 0.5$ and when $\pi_t = k$, but not when $\pi_t = 1 - k$. When rare events

occur, the agent is uncertain, faces a uniform distribution in the next period, and is able to purchase signals to guarantee certainty in the following period. In the right panel, the agent only purchases signals when $\pi_t = 1 - k$. An initial rare event makes the agent uncertain, and when she faces a uniform distribution, she can no longer buy signals, and is trapped in uncertainty forever.

Intuitively, the nature of the model’s trap can be related to the real world as follows: one could imagine an investor trying to make informational investment decisions about sporting events in a tournament in order to place bets. Sporting events have binary outcomes, so it is a natural point of comparison to the simple model. Suppose that early in the tournament, a low quality team plays a high quality team. The high quality team is expected to win, so an investor spends more time researching and simulating the high quality team’s chances in potential future games. If the low quality team wins, the investor will not have as much information about how the low quality team will perform in future games and may not have the time or energy to do additional research. As a result, the investor will not have strong enough opinions to place bets for the subsequent round and as a result, will not do research about either team to predict the outcomes of the rounds that follow. The investor might choose, after an initial upset, not to place any additional bets for the rest of the tournament - the effort it would take become informed about future outcomes outweighs the potential benefit of making informed bets.

The starkness of the result of this simple model - that uncertainty can be a trap that lasts forever - can be attributed to the binary nature of the simple model. In an all-or-nothing investment framework, it is possible to get permanent non-investment. The simple model shows that agents will choose to be uninformed if their distributions are sufficiently uniform - if there are too many events that could potentially occur, it is not worth paying attention to any single one. As we will see in the following section, once informational investment decisions are made freely, uncertainty will persist, but it will not become permanent.

4 General Setup

The simple model generates spikes and persistence in uncertainty, but it has stark predictions, due to the all-or-nothing form of informational investment. This section loosens the assumptions of the model to allow for more nuanced predictions. The analytical results of Section 2 continue to hold. Additionally, the assumptions of Section 3 are loosened, and instead of uncertainty traps that last forever, the general model delivers uncertainty persistence. Relaxing the structure of

informational investment, the state space, and the nature of public information allows for a richer set of predictions about dispersion of beliefs and asset-demand, as well as implications for volatility and bid-ask spreads, which are illustrated by some non-calibrated simulations.

4.1 Structure

The structure of the economy is the same as the previous section. As before, there are many periods. Within each period there are again three stages. In stage 0 the representative agent decides how accurate future signals should be. In stage 1, the public signal of the asset's value is revealed, prices are set, and financial trade occurs. In stage 2, the final value of the asset is revealed and agents purchase and consume.

4.2 Asset

There are, as before, two components of the asset, B_t and $\eta_{B,t}$, and the terminal value of the asset in stage 2 is $V_t = B_t + \eta_{B,t}$. $B_t \sim \mathcal{N}(\mu_{B,t}, \sigma_{B,t}^2)$ and $\eta_{B,t} \sim \mathcal{N}(0, \sigma_\eta^2)$ for each B . In stage 0, neither B_t nor $\eta_{B,t}$ is known, but $\mu_{B,t}$, $\sigma_{B,t}^2$, and σ_η^2 are all known. In stage 1, B is publicly revealed, and private and public signals about the realization of $\eta_{B,t}$, with accuracy $\sigma_{\gamma,t}^2$ and $\sigma_{\beta,t}^2$, respectively, are gathered as well. In stage 2, $\eta_{B,t}$ is revealed and the gains and losses are realized. The data-generating process is a random walk: $\mu_{B,t+1} = V_t + \epsilon_{t+1}$, where $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma_\epsilon^2)$. At the beginning of stage 0 of period $t+1$ a signal u_{t+1} is revealed that is distributed $\mathcal{N}(\epsilon_{t+1}, \sigma_{u,t+1}^2)$ where $\sigma_{u,t+1} \propto \sigma_{\gamma,t}$. One advantage of generalizing the asset's structure is that the model can view forecasts of the asset's value as continuous distributions, which allows a quick relation, empirically, to the VIX.

The model will again be solved backwards, and stage 2 is again trivial.

4.3 Stage 1 Problems

$\sigma_{\beta,t}(B_t)$ is the accuracy of the public signal about $\eta_{B,t}$ conditional on a realization of B_t , and $\sigma_{\gamma,t}(B_t)$ is the accuracy of private signals about $\eta_{B,t}$ for every B_t . Then the ex-post public mean and variance over $\eta_{B,t}$ after seeing a public signal x are $\mu_{p,t} = \frac{\frac{1}{\sigma_{\beta,t}^2}x}{\frac{1}{\sigma_{\eta,t}^2} + \frac{1}{\sigma_{\beta,t}^2}}$, and $\sigma_{p,t}^2 = \frac{1}{\frac{1}{\sigma_{\eta,t}^2} + \frac{1}{\sigma_{\beta,t}^2}}$.

4.3.1 Traders

Traders have access to private and public information and, as they are risk neutral, will choose to buy one unit of the asset if their private mean lies above the ask and will sell one unit of the asset if their private mean lies below the bid. One can think of this problem in two different ways: first, that the representative trader receives one signal and is able to make a choice between buying $1 - T$ units, selling $1 - T$ units, or abstaining; or second, that the a continuum of $1 - T$ traders each make individual signals based off of idiosyncratic signals purchased on their behalf by the representative trader in stage 0. The two cases are mathematically equivalent ex-ante to risk-neutral agents.

$$U(\text{public signal, private signal}) = \max \{(\text{bid}_t - x), (x - \text{ask}_t), 0\} \quad (8)$$

Where $x = E[V|\text{public signal, private signal}] = \frac{\frac{\mu_{p,t}}{\sigma_{p,t}^2} + \frac{\text{private signal}}{\sigma_{\gamma,t}^2}}{\frac{1}{\sigma_{p,t}^2} + \frac{1}{\sigma_{\gamma,t}^2}}$. Traders' decisions in stage 1 are independent of those in stage 2, and are not inter-temporal.

4.3.2 Market Makers

Market makers have access to all public information about the asset and also know the function $\sigma_\gamma(B)$. The dispersion of private beliefs (assuming the signal is purchased on behalf of a continuum of investors) can be calculated as $\sigma_{\text{disp},t}^2 = \frac{\frac{1}{\sigma_{\gamma,t}^2}}{\left(\frac{1}{\sigma_{\gamma,t}^2} + \frac{1}{\sigma_{p,t}^2}\right)^2}$. Conditional on a realization of B , a public mean μ_p , a public variance σ_p^2 , and a dispersion of beliefs $\sigma_{\text{disp},t}^2$, the expected profits for a choice of a bid and an ask by the market making sector are:

$$\Pi_{\text{ask}} = \int \phi_{p,t}(x) \left((1 - \Phi_{\text{disp},x}(\text{ask}))(1 - T) + \frac{T}{2} \right) (\text{ask} - x) dx \quad (9)$$

$$\Pi_{\text{bid}} = \int \phi_{p,t}(x) \left(\Phi_{\text{disp},x}(\text{bid})(1 - T) + \frac{T}{2} \right) (x - \text{bid}) dx \quad (10)$$

where $\phi_{p,t} \sim \mathcal{N}(\mu_{p,t}, \sigma_{p,t}^2)$ and $\phi_{\text{disp},x} \sim \mathcal{N}\left(\frac{\frac{\mu_{p,t}}{\sigma_{p,t}^2} + \frac{x}{\sigma_{\gamma,t}^2}}{\frac{1}{\sigma_{p,t}^2} + \frac{1}{\sigma_{\gamma,t}^2}}, \sigma_{\text{disp},t}^2\right)$. Perfect competition means that the bid and the ask must be selected to set the profits above to zero.

4.4 Stage 0 Problems

In stage 0, conditional private signals are selected.

4.4.1 Traders

The asset's value follows a random walk process both within and between periods. Within a period, the increment is η_t and between periods it is ϵ_t . The noise with which the public signals about η and ϵ are observed are proportional to one another. Traders take $\sigma_\beta(B)$ as given, and pick $\sigma_\gamma(B)$ to maximize expected future profits subject to a proportional cost c :

$$V(\mu_{B,t}, \sigma_{B,t}^2, \sigma_{\beta,t}^2) = \int \max_{\sigma_{\gamma,t}^2(B_t)} \phi(B_t) \int \int p(\text{public signal} = x) (p(\text{private signal} = y) U(x, y) dy) dx \\ + \delta \left[\int \int \phi(B_t) V(\mu_{B,t+1}(u_{t+1}, \eta_t), \sigma_{B,t+1}^2, \sigma_{\beta,t+1}^2) du d\eta \right] - c\nu(\sigma_{\gamma,t}^2(B_t)) dB_t$$

where $\phi \sim \mathcal{N}(\mu_{B,t}, \sigma_{B,t}^2)$, u_t is the signal about ϵ , $\mu_{B,t+1}(u_{t+1}, \eta_t) = \frac{\sigma_{\epsilon,t+1}^2 u_{t+1} + \sigma_{u,t+1}^2 (B_t + \eta_t)}{\sigma_{\epsilon,t+1}^2 + \sigma_{u,t+1}^2}$, $\sigma_{B,t+1}^2 = \frac{\sigma_{\epsilon,t+1}^2 \sigma_{u,t+1}^2}{\sigma_{\epsilon,t+1}^2 + \sigma_{u,t+1}^2}$, and ν is a convex decreasing function such that $\frac{\partial^2 \nu}{\partial \sigma_\beta^2} > \frac{\partial^2 U}{\partial \sigma_\beta^2}$ everywhere.

4.5 Propositions

Proposition 4. $\frac{\partial U}{\partial \sigma_\gamma} < 0$.

Corollary 2. For any given c , $\forall \delta > 0$, if $\sigma'_\beta(B) = 0$, then $\frac{\partial \sigma_\gamma^*(B)}{\partial \phi(B)} \leq 0$.

This result confirms the intuition from Proposition 2 holds: attention is a good, albeit a costly one. As such, more attention is paid to higher probability states than to low probability states. Therefore, lower probability events reduce identification and increase uncertainty.

Corollary 3. For any given c , $\forall \delta > 0$, if $\sigma'_\beta(B) = 0$, then for any two points B_1 and B_2 , such that wlog, $\phi_{B,t}(B_1) > \phi_{B,t}(B_2)$. Then at time $t + 1$, $E_{B_1}[\phi_{B,t+1}(B)] > E_{B_2}[\phi_{B,t}(B)]$.

The rarer the event that occurs, the longer the subsequent uncertainty is likely to last. There is an upper bound on how uncertain the agents can be, as $\sigma_\eta < \infty$ and $\sigma_\epsilon < \infty$, so even though σ_γ and σ_β are unbounded, the variance of the posterior distributions are bounded.

4.5.1 Public Entity

It was shown in the simple model that public and private information are substitutes. Therefore, it would be of interest to find out whether deteriorating public information changes private traders' incentives. To analyze this, I introduce a public entity who chooses the quality of public information to maximize a given objective function (the particular function is not very important, so long public information is treated as a good).

The public entity works to select conditional signal quality, σ_β for each potential value of B_t to maximize expected accuracy:

$$\int \min_{\sigma_\beta(B)} \phi_{B,p,t}(B) \left(\int p(\eta\text{-signal} = x) \sigma_p^2 dx \right) - c_p \nu_p(\sigma_\beta^2(B)) d\phi(B) \quad (12)$$

where ν_p is a convex decreasing function that satisfies the condition that $\frac{\partial^2 \nu_p}{\partial \sigma_\beta^2} > \frac{\partial^2 \sigma_p^2}{\partial \sigma_\beta^2}$ everywhere.

The public entity seeks to be accurate, but the particular objective function is not overly important. One could think of the public information as being reports or actions from public institutions like the Federal Reserve or the government, or research reports published by financial institutions. An alternative asymmetric objective is considered in the appendix, to show that the particular choice of objective is not driving the results. These changes in the structure of public information no longer allow us to describe patterns analytically, but simulations still permit insight into the dynamics.

4.5.2 Dynamic Equilibrium

Given values of $\{\mu_{B,t}, \sigma_{B,t}^2, c, c_p, \sigma_{\eta,t}^2, \sigma_{\epsilon,t}^2\}$, a dynamic equilibrium is defined by a choice of σ_β by the public entity that solves equation 12 a choice of a policy function $\sigma_\gamma(\mu_B, \sigma_B^2, \sigma_\beta^2)$ by the traders that satisfies equation 11, individual decisions to buy, sell, or abstain by traders to solve equation 8, and a choice of a bid and an ask by Market Makers to solve equations 9, given observed values of $\{\sigma_{\beta,t}, \sigma_{\gamma,t}, \sigma_{\eta,t}\}$, a public signal, and a set of private signals distributed $\mathcal{N}(\eta_t, \sigma_{\eta,t}^2)$.

4.6 Predictions and Spillovers

Figure 5 shows the results of the previous propositions. In this snapshot,⁵ one can see how agents (both public and private) choose to invest in information. The blue dashed line is the ex-ante distribution of B . Then, conditional on these selections, one can see what the expectations for credit spreads, levels of borrowing, demand for financial assets, bid-ask spreads, uncertainty, volatility, and dispersion of beliefs are for each potential value of B .

As is expected by the propositions above, both σ_β and σ_γ are U-shaped, with better signal quality for high probability states and worse signal quality for low probability states.

5. The parameter values here are $\mu_{B,t} = 10$, $\sigma_\eta^2 = 4$, $\sigma_\epsilon^2 = 4$, $c_p = 0.0002$, $\delta = 0.95$, $c = 1$, and $\nu_p(x) = \nu(x) = \frac{1}{x}$. α and ρ are selected with the calibration of section 5 in mind, while σ_η and σ_ϵ are set equal to indicate that the uncertainty from period to period is not more or less than stage to stage. The rest of the parameters will deliver similar results.

The results of these attentional choices are shown in panel (b). Most importantly here, uncertainty and volatility both spike for tail events. But it is important to note that bid-ask spreads, and dispersion of beliefs also are higher for tail events, and are comparatively lower for higher probability events. This bears out the first hypothesis in the introduction - that attentional choices, or investment in information ex-ante can lead to spikes in second moments that are reflected in a financial setting. Similarly panel (c) shows that volume of trade, or balance sheet size, or leverage, plummets for tail events as investors are less well informed in those states and therefore less likely to want to take on risk.

4.7 Persistence

The above was an illustration of the comparative statics. Consider now the effect of the persistence mechanism. In Figure 6 the asset's value takes a three standard deviation drop in period 2 and then immediately rebounds in period 3 to its prior level, where it then remains for several periods.

There are several aspects of this figure of note: first, the large drop in the asset's value corresponds to a low-probability event ex-ante, and therefore large increases in all the financial variables (drop in asset demand); second the rebound is just as large in size as the drop, but due to the flattening of the distribution in period 2, it actually lowers uncertainty somewhat - as uncertainty increases agents' ex-ante beliefs about the likelihood of large price movements; third, as the asset's value does not move at all after period 3, the only source of persistence is the time it takes for agent's confidence in their forecasts to regain shape (for the variance of their beliefs to decline).

5 Taking the Model to Data

The model of this paper addresses the nature of informational investment. It has delivered three key predictions that I will now attempt to evaluate empirically.

First, propositions (2) and (4) both state low probability events lead to lower levels of information. The effects of these lower levels of information manifest in higher levels of uncertainty. Therefore, I first document that low probability events do lead to spikes in uncertainty.

Next, and perhaps most illustratively, I use the model and actual S&P price changes to see how well the model can forecast values of the VIX, dispersion, and risk taking. In Appendix C I provide some evidence that the channels through which the above effects take place is preparedness, as measured by a function of how well a representative consumer is hedged against different price

movements.

5.1 Rare Events Trigger Uncertainty

First, I show that lower probability events cause higher levels of uncertainty. These are the predictions of Propositions (2) and (4).

5.1.1 Strategy

The first hurdle is deciding how to measure the probabilities of events. The seminal work of Breeden and Litzenberger (1978) allows recovery the ex-ante probabilities of price movements from options data. They show that the double derivative of call option prices on a range of strike prices gives the prices of Arrow-Debreu securities along the same range.⁶ Taking the shortest horizon and the full set of strikes of call options, I construct a vector of implied volatilities, which I use to price one-day options. The double-difference of those one-day options yield a probability distribution over price movements in the following trading day.

A concern might arise from this methodology. Due to risk-aversion present in the market, the options data may be overestimating the probability of negative events and underestimating the probability of positive events. This would bias coefficients in any potential regression. To account for this, I also calculate the probabilities using the physical distribution based on fitting a nonparametric kernel density estimator to a physical distribution based on a moving window of historical largest intraday price movements using the novel method of Botev, Grotowski, Kroese, et al. (2010). Effectively, using a density of data acquired in a historical window (the results of this exercise are not very sensitive to the size of the window), the kernel density estimator is formed by assuming a linear-diffusion process (such as, in a simple case, the heat equation). There is then an estimator for each value of t , as the process gradually diffuses completely to its limiting distribution. The algorithm is a two-state process, which first models the dissipation of heat into a space of uniform diffusivity. This diffusion process depends on the nature of the data, and so the second stage again models dissipation but with nonuniform diffusivity, learned from the first stage.

6. Of course, given the relative illiquidity of options markets, the data needs to be cleaned and prepared in order to apply this method. Starting with daily options prices for the S&P index from January 2nd, 1996 to March 15th, 2015, which totals about 5 million data points, I then eliminate, in order, the prices that allow the following arbitrage opportunities: (1) If the bid-ask spread on a particular security was negative. (2) If it was possible to buy and immediate exercise an option for profit. (3) If the value of options were not monotonic in the strikes. (4) If the double difference (price of Arrow-Debreu securities) were negative. After these adjustments there are about 2.5 million prices left. These are then arranged each into implied volatility surfaces, and I use a spline to interpolate between strikes and maturities.

See Ramsey (2014) for an excellent summary.

A second hurdle is deciding how to measure financial uncertainty. The VIX is a measure constructed by the Chicago Board of Exchange to measure, based off of options prices, the market's expectation of volatility of the S&P index in the next 30 days, reported in annualized percentage terms. This metric reflects the market's uncertainty over future price paths, and so will serve as a useful proxy for *uncertainty*. Therefore, the regression that we're interested in will take the following form:

$$\Delta \log(VIX_t) = \alpha_0 + \alpha_1 \Delta \log(Prob_t) + \alpha_2 \Delta \log(X_t) + \epsilon_t \quad (13)$$

$Prob_t$ is the minimum of a vector with two values: the probabilities of the highest and lowest intraday values of the *S&P* during day t as calculated in the two methods discussed above. VIX_t is the closing value of the VIX index on day t . The hypothesis is that α_1 should be negative. Furthermore since the VIX and other control variables in this regression are serially correlated⁷, the regression will be run in first differences of logs and will report Newey-West standard errors. X_t is a vector of control variables, such as the value of the S&P index, the volume of trade in the index, and the intraday volatility of the index, which should alleviate concerns of omitted variable bias. Additionally, I include the Skewness of the VIX in the control group, just to add a more rigorous test of the claim in the introduction, that the Skewness and level of the VIX are not positively correlated.

I use a standard set of regressors augmented with the probabilities. The value of controls such as the underlying and the volume of trade has been used to explain implied volatility by such papers as Simon (2003), Harvey and Whaley (1992), Franks and Schwartz (1991), and Fleming, Ostdiek, Whaley, et al. (1995). This regression is novel in that previous work have not tested the effect of the ex-ante probabilities on ex-post implied volatility.

5.1.2 Data

All options price data comes from WRDS. From 1996 to 2015 it includes daily closing bids, closing asks, implied volatilities, for all listed options on the *S&P* index by strike and maturity. I also have daily opening and closing values of the VIX from Yahoo!. All summary statistics are reported in table 1.

7. As decided by Dickey-Fuller tests on each variable.

5.1.3 Results

Before we turn to the regression results, Figure 7 is a useful way of looking at the data. For each day, I have a probability of the intraday high point and a probability of the intraday low point. I take the minimum of those two probabilities, and arrange those along the x -axis as follows: the left half of the x -axis are all the days on which the intraday low was less likely than the intraday high. The low point probabilities are arranged from least likely to most likely. On the right half are all the days on which the intraday high was less likely than the intraday low. The high point probabilities are arranged from most likely to least likely. This is to mimic the x -axis seen in the graphs in section 4.6. Then corresponding to each probability I take the closing value of the VIX index for each day. As is visible from the fitted quadratic curve, the basic shape of the VIX seems to mimic that of σ_γ .

As is evident from the table, and as was expected, the coefficients on the minimum probability and its lags are all negative in all columns. The results of the table are robust to many other control variables, and seem to confirm the finding of the model that ex-ante low probability events are followed by spikes in the VIX. Importantly, the spikes in uncertainty cannot solely be caused by the magnitude of the change in the S&P itself, nor in changes in volatility, as they are being controlled for. In terms of magnitude, a one percent increase in the size of intraday volatility increases the VIX by about 0.01 percent. On the other hand a one percent increase in the size of the probability of an event decreases the VIX by about 0.002 percent. This may seem small, but probabilities can fluctuate wildly, and can drop or increase many-fold in a day. This non-linearity is precisely why it is beneficial to look at this regression in log terms. Very rare events will be orders of magnitude smaller than the norm, and so will significantly increase the VIX. It is also important to note that the Skew coefficient is negative and significant, indicating that a fear of large negative events is not positively correlated with overall uncertainty.

Touching on the issue of causality here, an alternative hypothesis to the one posited here is that a rare event may in fact spike uncertainty, and that the spike in uncertainty is what causes large price fluctuations. The lagged terms should allay this concern somewhat, as an unlikely price movement not only impacts today's VIX but tomorrow's as well. However, the idea that an increase in uncertainty makes large price fluctuations more likely, is very much in step with the predictions of the model. The flattening of a distribution, referred to often here, means that not only does the center of a distribution get depressed, but the tails fatten somewhat as well, thus harking back to

the ‘self-exciting’ process mentioned earlier. The model posits that both directions of causality are valid, but that timing is crucial - one rare event causes a flattening, *which in turn* may cause more events to be ill-identified.

5.2 Simulation and Forecasting

This section of the paper is devoted to testing how well the mechanisms described can actually match empirical patterns. To that end, all of the variables in the model have some real world parallel, that they can be calibrated to, with the exception of the costs of informational investment. Therefore, this section will calibrate all of the variables possible, and then estimate values of c and c_p that allow predicted uncertainty to come closest to the VIX.

5.2.1 Strategy

The list of variables that need to be calibrated are:

Variable	Value	Source/Justification
$\mu_{B,t}$	S&P Index	—
σ_η^2	23%	—
σ_u^2	23%	—
δ	0.95	—
T	0.38	Wang (2002)

The VIX is directly tied to the S&P index. Therefore the asset in the model will track the value of the index. Both σ_η and σ_u will be measured in percentage terms, in order to match the units of the VIX. However, unlike the VIX, the uncertainty will not be ‘annualized’ but will be period-by-period - and therefore the reported VIX will be divided by the square root of 12. The discount factor is set at 0.95, and the ratio of noise traders is taken from Wang (2002) who measures the volume and number of noise traders in foreign exchange markets. Data on this variable is difficult to come by, but thankfully its value not crucial in estimation.

Then the model takes the parameters above as inputs, and the previous period’s maximum price change, and the previous period’s VIX (either estimated or actual) and outputs uncertainty, calculated as $Var[\eta|\text{private and public information}]$. There are two simulations, in the first, the VIX each period is updated to its correct value in order to predict the next period’s value (PbP); in the second, the actual value of the VIX is only input in the first period, and only price changes are input thereafter (PC).

Forecast	Mean	Var
VIX	6.12	5.19
VIX PbP	5.29	4.21
VIX PC	5.57	5.98

In order to construct a baseline for comparison, I also consider a naive forecaster, who uses the previous period’s largest percentage deviation as the prediction for the next period’s VIX. Additionally, given the inputs, I am also able to construct simulations for the path of risk-taking and the path of dispersion of beliefs. These can also be compared to actual time-series from the data.

5.2.2 Results

As is evident from Figure 8, the paths of the simulated VIXs and the actual VIX seem relatively close to one another, all spiking in the same places. Some summary statistics for the forecasts are below, the two simulations have a lower mean squared error and a higher R^2 than the naive forecast.

Forecast	MSE	R^2
Monthly PbP	2.88	0.476
Monthly PC	2.46	0.525
Monthly Naive	13.27	0.245

Further using a Diebold-Mariano test on the vectors of squared errors of each of the forecasts, we see that the best forecast is PC, followed by PbP, followed by the naive forecast. The intuition of the two sets of simulations from the last section should be evident here - the spikes after periods of large price movements, and the subsequent slow return to previous levels merely build on earlier intuition. Additionally, we can see in Figure 9 the forecasts (in blue and red) for dispersion of beliefs and risk-taking, plotted against the same time series as in the first figure of the paper (in yellow).

Forecast	MSE	R^2
Dispersion PbP	87.02	0.138
Dispersion PC	39.06	0.155
Debt Margin PbP	4.64	0.445
Debt Margin PC	4.77	0.394

Using only price changes in the S&P index, and an initial value of the VIX, the model is able to give predictions about these variables that, while not identical, still match the general shapes of each, and explain significant fractions of their variance.

6 Conclusion

I have presented, solved, and simulated a model that describes the process of informational investment by private investors. The solution to the model leads to contingent information sets that are most accurate near the mean of a distribution and most inaccurate in the tails. Therefore, upon the realization of a tail event, uncertainty spikes. The interaction of the information sets lead to concurrent spikes in bid-ask spreads and dispersion of beliefs, as well as drops in asset demand. However, an initial increase in uncertainty results in a flatter distribution, which in turn, leads to lower levels of information investment on average. As a result, initial spikes in uncertainty can persist for several periods before resolving.

Further, I present a variety of empirical evidence that is consistent with the model, including that low probability events are a significant predictor of increases in the VIX, that such events have persistent effects on the VIX and lower the probabilities of subsequent price movements, and that forecasting simulations of the model are able to mimic the actual path of expected volatility and match patterns in credit spreads, dispersion of beliefs and debt margin.

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Figures and Tables

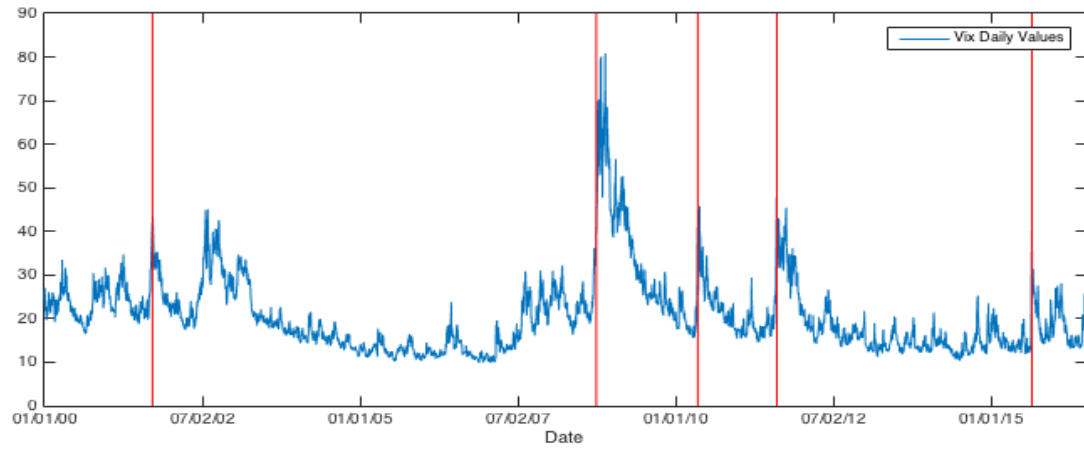


Figure 1: Daily values of VIX with markers for key rare events

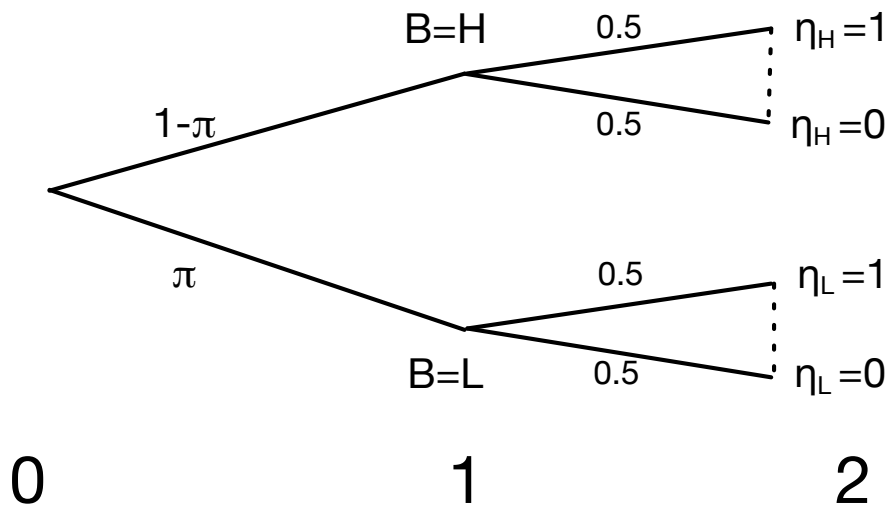


Figure 2: Structure of the Asset

0

1

2

Traders choose which
signals to buy.

B observed.

η_B publicly observed.

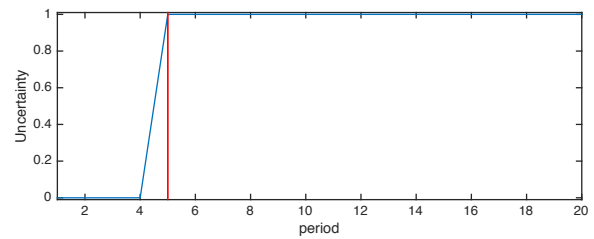
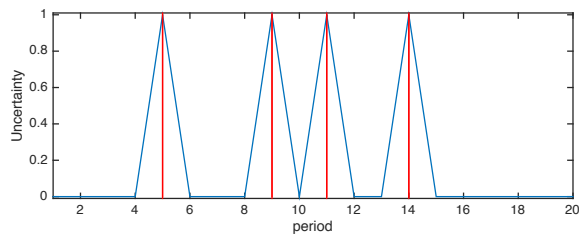
Public signal about η_B
revealed.

Agents receive profit
or loss.

Market Making sector
sets bid and ask.

Traders trade on private
signals about η_B .

Figure 3: Timeline of within-period actions and decisions



(a) Temporary Uncertainty for Low Information Costs (b) Uncertainty Persists for High Information Costs

Figure 4: Uncertainty Reactions in Dynamic Model

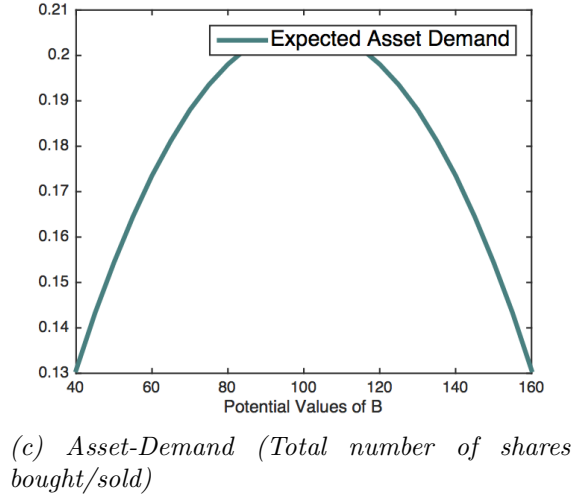
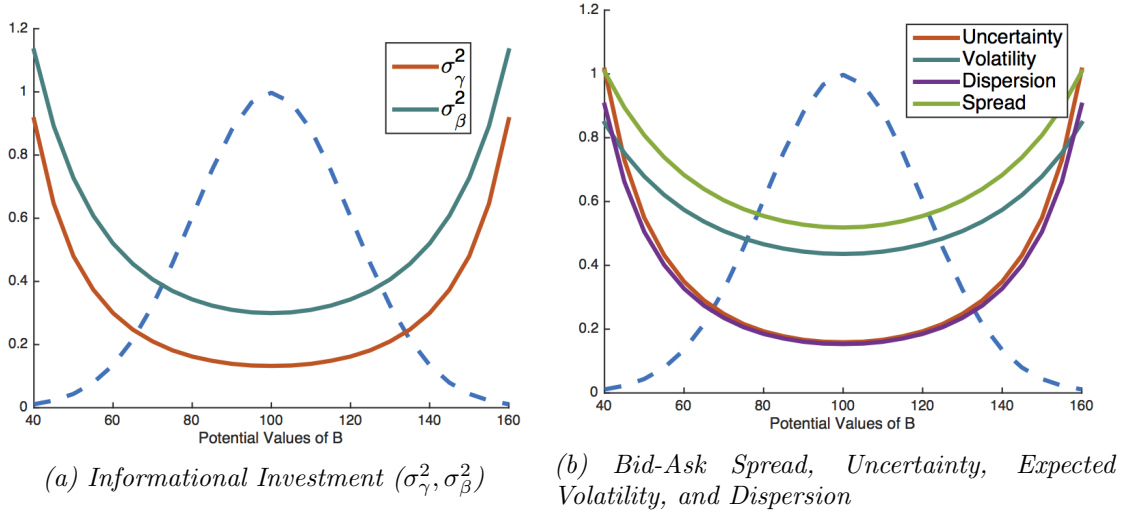


Figure 5: Simulation of Continuous State Model

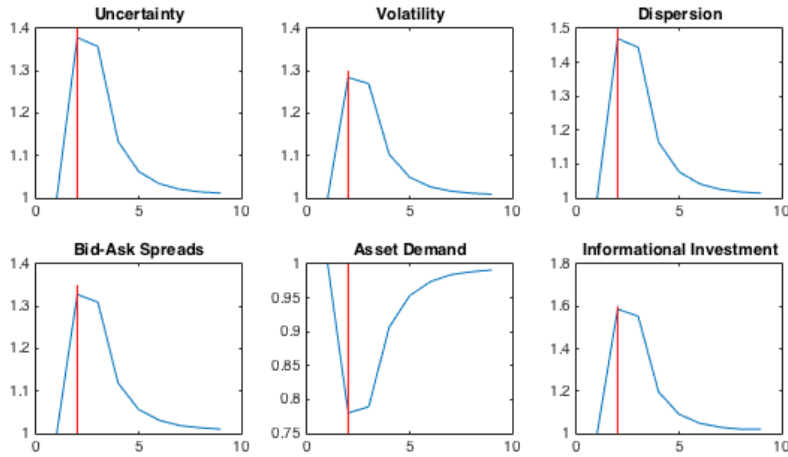


Figure 6: Impulse Responses of Continuous State Model to a three standard deviation drop and rebound in the asset's value.

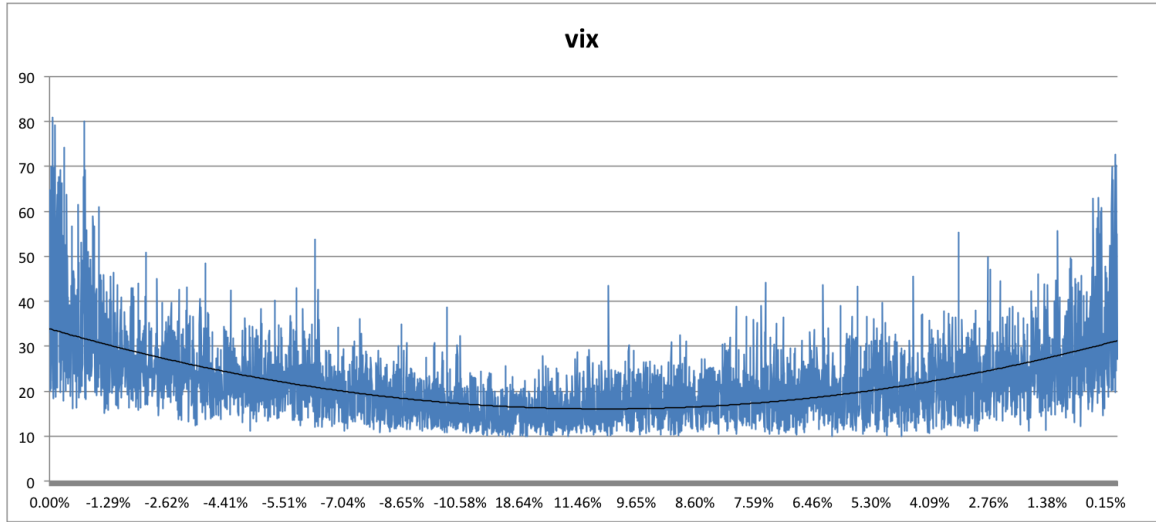


Figure 7: Closing levels of VIX as a function of the ex-ante probabilities of price changes. Probabilities are arranged such that most likely events are in the center, and least likely are in the tails.

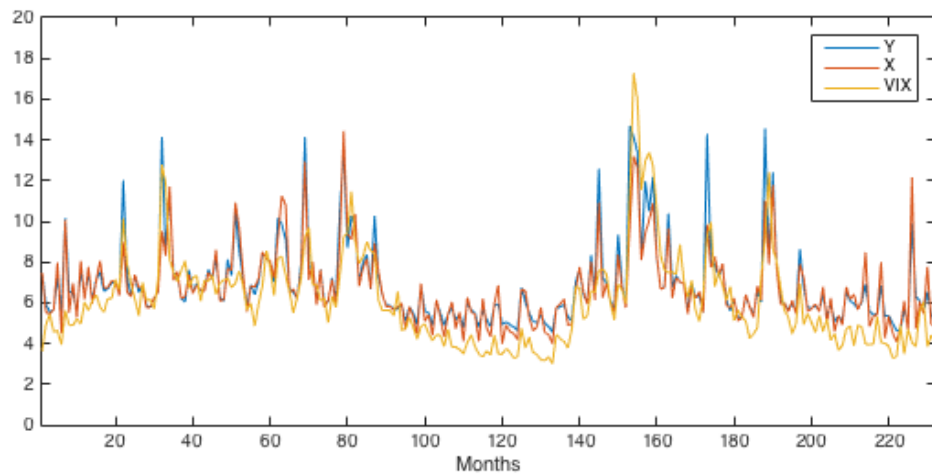
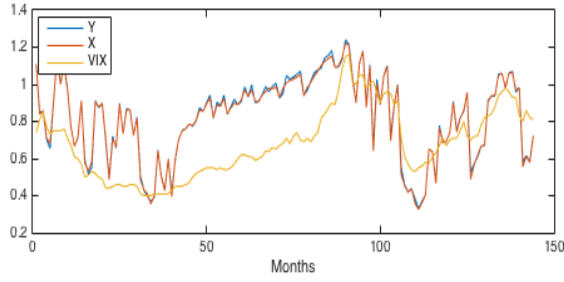
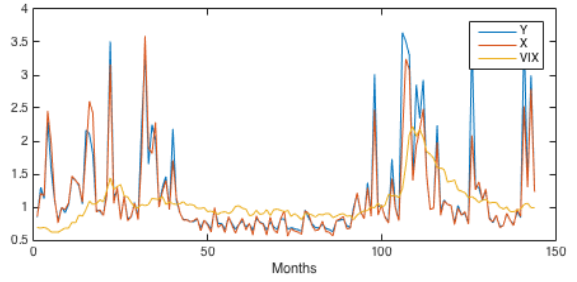


Figure 8: Monthly simulation of the VIX using probabilities as input

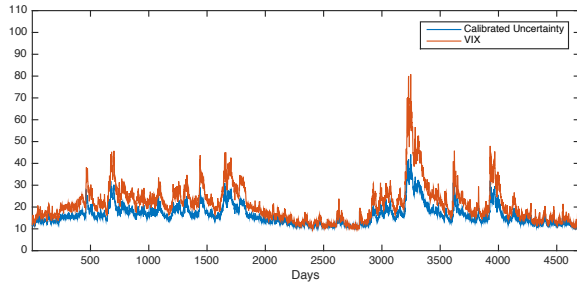


(a) *Simulation of Risk Appetite*

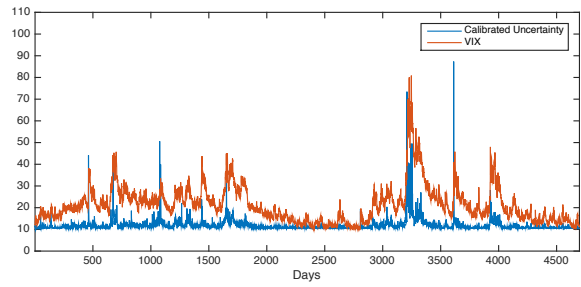


(b) *Simulation of Dispersion of Beliefs*

Figure 9: Additional Simulations

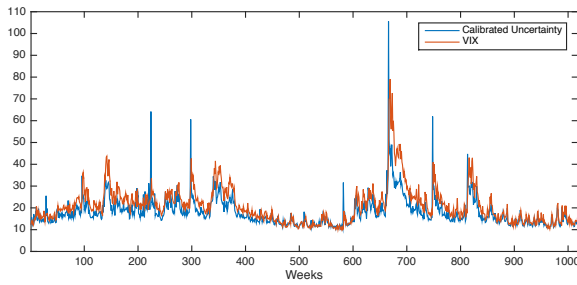


(a) *Daily using probabilities as input*

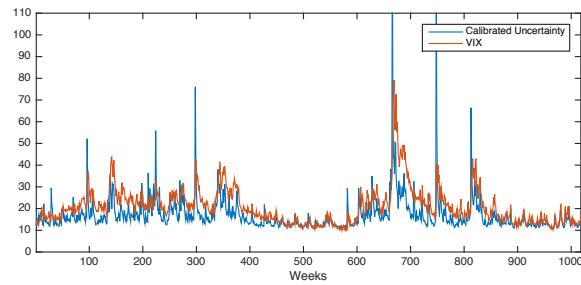


(b) *Daily using price changes as input*

Figure 10: Forecasting with Daily Data



(a) *Weekly using probabilities as input*



(b) *Forecasting with Weekly Data*

Figure 11: Weekly using price changes as input

Variable	Mean	StDev	Min	Max
Vix	21.07	8.38	9.89	80.86
Volatility	16.67	10.65	1.79	110.04
Volume	244,924	171,056	1,499	1,145,623
Physical Prob of Intraday Low	10.26%	5.59%	3.20e-7%	23.17%
Physical Prob of Intraday High	9.27%	5.09%	3.02e-5%	24.44%
Risk-Neutral Prob of Intraday Low	8.56%	4.55%	6.14e-5%	31.12%
Risk-Neutral Prob of Intraday High	8.15%	4.54%	2.15e-4%	35.08%

Table 1: Summary Statistics

VARIABLES	(1) Vix	(2) Vix	(3) Vix	(4) Vix	(5) Vix	(6) Vix
Min	-0.00416*** (0.00108)	-0.00556*** (0.00132)	-0.00398*** (0.000648)	-0.00361*** (0.000644)	-0.00188*** (0.000669)	-0.00188*** (0.000667)
L.Min		-0.00278*** (0.00103)	-0.00242*** (0.000636)	-0.00226*** (0.000629)	-0.00240*** (0.000635)	-0.00241*** (0.000636)
Close			-3.670*** (0.101)	-3.668*** (0.101)	-3.640*** (0.0993)	-3.614*** (0.0999)
Volume				0.0134*** (0.00350)	0.00505 (0.00331)	0.00534 (0.00332)
Volatility					0.0115*** (0.00154)	0.0114*** (0.00154)
Skew						-0.000495** (0.000242)
Constant	-0.000144 (0.000870)	-0.000120 (0.000866)	0.000755 (0.000578)	0.000750 (0.000577)	0.000753 (0.000572)	0.000769 (0.000573)
Observations	4,535	4,527	4,527	4,527	4,527	4,509

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Notes: The regression is run in first differences, and Newey-West standard errors are shown in parentheses. Min is the lower of the ex-ante probability of the intraday high and the ex-ante probability of the intraday low. Volume is the volume of trade in the S&P index; Volatility is the size of the intraday price movement; Close is the closing value of the S&P index; Skew is the skewness of the VIX as calculated by the CBOE Data is at a daily frequency, and probabilities are risk-neutral, extracted from options data.

Table 2: Effect of price changes on the VIX

VARIABLES	(1) Vix	(2) Vix	(3) Vix	(4) Vix	(5) Vix	(6) Vix
Min	-0.00575*** (0.00118)	-0.00670*** (0.00145)	-0.00576*** (0.000885)	-0.00529*** (0.000893)	-0.00168 (0.00117)	-0.00177 (0.00117)
L.Min		-0.00176 (0.00119)	-0.00293*** (0.000733)	-0.00277*** (0.000730)	-0.00297*** (0.000729)	-0.00300*** (0.000729)
Close			-3.695*** (0.100)	-3.693*** (0.100)	-3.664*** (0.0992)	-3.640*** (0.0996)
Volume				0.00956*** (0.00329)	0.00356 (0.00327)	0.00383 (0.00328)
Volatility					0.0126*** (0.00196)	0.0124*** (0.00196)
Skew						-0.000452** (0.000229)
Constant	0.000104 (0.000865)	0.000133 (0.000863)	0.000860 (0.000574)	0.000853 (0.000573)	0.000854 (0.000568)	0.000872 (0.000570)
Observations	4,541	4,536	4,536	4,536	4,536	4,519

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Notes: The regression is run in first differences, and Newey-West standard errors are shown in parentheses. Min is the lower of the ex-ante probability of the intraday high and the ex-ante probability of the intraday low. Volume is the volume of trade in the S&P index; Volatility is the size of the intraday price movement; Close is the closing value of the S&P index; Skew is the skewness of the VIX as calculated by the CBOE;

Data is at a daily frequency, and probabilities are physical, extracted from a kernel density estimation of historical distributions.

Table 3: Effect of price changes on the VIX

VARIABLES	(1) vix	(2) vix	(3) vix
volume	2.13e-05*** (1.92e-06)	1.88e-05*** (2.10e-06)	2.13e-05*** (2.41e-06)
Open1	-5.16e-06** (2.08e-06)	-5.55e-06** (2.26e-06)	-2.68e-06 (2.60e-06)
Open2	2.65e-06 (2.87e-06)	5.07e-06 (3.13e-06)	2.52e-06 (3.52e-06)
Open3	-3.45e-05*** (1.63e-06)	-3.31e-05*** (1.87e-06)	-3.34e-05*** (1.87e-06)
Open4	-9.26e-06*** (9.07e-07)	-8.39e-06*** (9.91e-07)	-7.90e-06*** (1.16e-06)
Close1	-4.44e-05*** (2.89e-06)	-4.10e-05*** (3.20e-06)	-4.28e-05*** (3.49e-06)
Close2	-2.88e-05*** (3.28e-06)	-2.49e-05*** (3.52e-06)	-2.70e-05*** (4.07e-06)
Close3	-4.21e-05*** (2.26e-06)	-4.06e-05*** (2.59e-06)	-4.02e-05*** (2.59e-06)
Close4	-2.69e-05*** (1.47e-06)	-2.43e-05*** (1.57e-06)	-2.62e-05*** (1.97e-06)
SPdiff	-0.0193*** (0.00715)	0.115*** (0.0151)	-0.114*** (0.0159)
volatility	0.207*** (0.0201)	0.198*** (0.0234)	0.178*** (0.0255)
plow	-97.51*** (9.049)	-90.42*** (10.37)	-101.2*** (12.58)
phigh	-37.64*** (9.293)	-21.14** (10.09)	-53.01*** (14.08)
Constant	17.89*** (0.866)	16.50*** (0.908)	17.71*** (1.116)
Observations	4,166	2,225	1,940

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Notes: This table reports the results of three regressions. Each column corresponds to a different subset of the data - these first is for the whole sample, the second is for days on which the S&P index closed up, the second is for days on which it closed down. The dependent variable is the estimated stochastic discount factor, the time frequency is daily. The dependent variable is the change in the level of the VIX - the regression is run in first differences and Newey West Standard Errors are reported. Volume is the volume of trade in the S&P index, SPdiff is the first difference of the S&P index's value, volatility is the intraday price change, plow and phigh are the ex-ante probabilities of the intraday low and high respectively, calculated with the risk neutral distribution. The Open variables are the net positions of the market in different ranges of strikes for positions that have taken on risk. The Close variables are the net levels of positions that were closed out in different ranges of strikes.

*** indicates statistical significance at the 1% level; ** at the 5% level; and * at the 10% level.

Table 4: Effect of price changes on the VIX

A Proofs

Proof. Proposition (2)

It will actually be easier to start with the intuition for this proposition and then move to Proposition 1. It is quick to see that since the benefit of acquiring a signal in equation 6 is increasing in $T^*(s)$, that agents' informational choices are strategic substitutes. Therefore, an agent receives the most benefit if she is the only one purchasing a signal (i.e. $s = 0$). Therefore, consider the maximal benefit possible: $T^*(s) = 1$:

$$\pi_B \beta_B (1 - \beta_B) 2 - c$$

We want to show that for any c , there is a π_B such that agents will not purchase a signal. The condition for that π_B is:

$$\begin{aligned} \pi \beta_B (1 - \beta_B) 2 - c &= 0 \\ \pi \beta_B (1 - \beta_B) 2 &= c \\ \pi &= \frac{c}{2\beta_B(1 - \beta_B)} \end{aligned}$$

This condition is necessary and sufficient for no agent to purchase a signal. Note that it is not necessarily the case that there is a sufficiently large value of π for which all agents will buy a signal. Such a result depends on the value of c . \square

Proof. Proposition (1)

Let us first find the conditions under which all agents would purchase a signal. If all agents purchase a signal then $T^* = T$. The necessary and sufficient condition for all agents purchasing a signal is:

$$\begin{aligned} \frac{\bar{\pi} \beta_B (1 - \beta_B) \frac{T}{2}}{\left(\frac{T}{2} + (1 - T)(1 - \beta_B)\right) \left(\frac{T}{2} + (1 - T)\right)} &> c \\ \frac{c \left(\frac{T}{2} + (1 - T)(1 - \beta_B)\right) \left(\frac{T}{2} + (1 - T)\right)}{\beta_B (1 - \beta_B) \frac{T}{2}} &> \bar{\pi} \end{aligned}$$

Therefore if $\underline{\pi} < \pi < \bar{\pi}$, some but not all agents will purchase a signal, and the solution will be interior. Define the benefit of purchasing as signal as $P(s, \pi, \beta_B) \equiv \frac{\pi \beta_B (1 - \beta_B) \frac{T}{2}}{\left(\frac{T}{2} + (1 - T)(1 - \beta_B)\right) \left(\frac{T}{2} + (1 - T)\right)}$. At interior solutions, it must be the case that the marginal investor is indifferent between purchasing a signal and not, so $P = c$. Since c is fixed, and since $P_s < 0$ and $P_\pi > 0$, we have that $s_\pi^* > 0$, and

so we are done. \square

Proof. Proposition (3)

Define $P(\pi) \equiv \frac{\beta_B(1-\beta_B)^{\frac{T}{2}}}{(\frac{T}{2}+(1-T)(1-\beta_B))(\frac{T}{2}+(1-T))}$ to be the benefit of purchasing a signal if the state for which information was purchased realizes. It must be the case that the lower bound on $V(\pi, 1-\pi)$ is 0, as a negative payoff could be avoided by abstention from purchase.

Further, the upper bound on $V(\pi, 1-\pi)$ is achieved when $c = 0$, and equals $Y \equiv \frac{P}{1-\delta}$. Since V is bounded between 0 and Y , there must exist a c such that $\frac{\pi_B\beta_B(1-\beta_B)^{\frac{T}{2}}}{(\frac{T}{2}+(1-T)(1-\beta_B))(\frac{T}{2}+(1-T))} - c + \delta\pi_B V(k, 1-k) < 0 \leq \pi_B V(0.5, 0.5)$. \square

Proof. Proposition (4)

First, we can rewrite equations 9 as zero-profit conditions as follows:

$$\begin{aligned} \frac{T}{2}(\text{ask} - \mu_p) &= \int \phi_p(x)(\text{ask} - x)(1 - \Phi_{\text{disp},x}(\text{ask}))dx(T-1) \\ \frac{T}{2}(\mu_p - \text{bid}) &= \int \phi_p(x)(x - \text{bid})\Phi_{\text{disp},x}(\text{bid})dx(T-1) \end{aligned}$$

The left-hand side of these expressions is the profit earned from noise traders - $\frac{T}{2}$ purchases or sales, which in expectation are equal to μ_p . On the right hand side is the expected loss from adverse selection to informed traders. Market Maker profit is increasing in the ask and decreasing in the bid, both by extracting more from noise traders, and giving away less to informed traders.

Focusing just on the bid equation (the ask follows similarly):

$$\begin{aligned}
& \frac{\partial}{\partial \sigma_\gamma^2} \int \phi_p(x)(x - \text{bid}) \Phi_{\text{disp},x}(\text{bid}) dx (1 - T) \\
& \propto \frac{\partial}{\partial \sigma_\gamma^2} \int \phi_p(x)(x - \text{bid}) \Phi_{\text{disp},x}(\text{bid}) dx \\
& = \frac{\partial}{\partial \sigma_\gamma^2} \int_{-\infty}^{\text{bid}} \phi_p(x)(x - \text{bid}) \Phi_{\text{disp},x}(\text{bid}) dx + \frac{\partial}{\partial \sigma_\gamma^2} \int_{\text{bid}}^{\infty} \phi_p(x)(x - \text{bid}) \Phi_{\text{disp},x}(\text{bid}) dx \\
& = \int_{-\infty}^{\text{bid}} \underbrace{\phi_p(x)(x - \text{bid})}_{< 0} \underbrace{\int_{-\infty}^{\text{bid}} \frac{\partial}{\partial \sigma_\gamma^2} \phi_{\text{disp},x}(y) dy}_{> 0} dx \\
& \quad + \int_{\text{bid}}^{\infty} \underbrace{\phi_p(x)(x - \text{bid})}_{> 0} \underbrace{\int_{\text{bid}}^{\infty} \frac{\partial}{\partial \sigma_\gamma^2} \phi_{\text{disp},x}(y) dy}_{< 0} dx \\
& < 0
\end{aligned}$$

Therefore, by similar logic we have that an decrease in σ_γ^2 , holding all other variables fixed, results in lower profits for the market maker.

In order to increase profits again, the market maker will need to lower the bid and raise the ask, to make up the difference from noise traders. Thus, an decrease in σ_γ^2 increases the transfer from noise traders to informed traders in expectation. So U depends negatively on σ_γ^2 . \square

Proof. Corollary (2)

The necessary condition is that $\frac{dU}{d\phi(B_t)} \Rightarrow 0$. This is trivially true. \square

Proof. Corollary (3)

If $\phi_{B,t}(B_1) > \phi_{B,t}(B_2)$, then $\sigma_{\gamma,t}(B_1) < \sigma_{\gamma,t}(B_2)$. Then $V_{B_1}[\phi_{B,t+1}(B)] < V_{B_2}[\phi_{B,t}(B)]$. So $E_{B_1}[\phi_{B,t+1}(B)] > E_{B_2}[\phi_{B,t}(B)]$. \square

B Dynamic Heterogeneous Signal Acquisition

In this section, I will consider the dynamic formulation of the simple model, while still allowing agents to behave heterogeneously with regards to purchasing signals. For simplicity I will set $\beta = 0.5$.

Assume that $\pi_{t+1} = 0.5 + (s/2)\rho$ where $\rho < 1$. Unlike the representative agent, heterogeneous agents do not internalize the long-term consequences of their individual decisions. Then an individual

agent faces the following problem in terms of signal acquisition:

$$V(\pi, 1 - \pi) = \sum_{\pi} \max \left\{ \frac{\pi_B \beta (1 - \beta) \frac{T}{2}}{\left(\frac{T}{2} + (1 - T)(1 - \beta_B) \right) \left(\frac{T}{2} + (1 - T) \right)} - c \right. \\ \left. + \delta V(\pi_{t+1}, 1 - \pi_{t+1}), \delta V(\pi_{t+1}, 1 - \pi_{t+1}) \right\}$$

So the condition for buying a signal in the dynamic game is the same as in the static game:

$$\begin{aligned} \frac{\pi_t \beta (1 - \beta) \frac{T^*}{2}}{\left(\frac{T^*}{2} + (1 - T^*)(1 - \beta_B) \right) \left(\frac{T^*}{2} + (1 - T^*) \right)} - c &> 0 \\ \frac{\pi_t \frac{T^*}{8}}{\left(\frac{T^*}{2} + \frac{(1 - T^*)}{2} \right) \left(\frac{T^*}{2} + (1 - T^*) \right)} &> c \\ \frac{\pi_t \frac{T^*}{4}}{\left(1 - \frac{T^*}{2} \right)} &> c \\ \frac{\pi_t T^*}{4 - 2T^*} &> c \end{aligned}$$

Therefore the equilibrium level of s for interior solutions is defined as:

$$\begin{aligned} \pi_t T^* &= c(4 - 2T^*) \\ T^*(\pi_t + 2c) &= 4c \\ \frac{T}{T + s(1 - T)} &= \frac{4c}{\pi_t + 2c} \\ \frac{T(\pi_t + 2c)}{4c(1 - T)} - \frac{T}{1 - T} &= s \end{aligned}$$

This is linear in π_t . Therefore, the shape of π_{t+1} as a function of π_t is as follows. For very low levels of π_t , $\pi_{t+1} = 0.5$, for intermediate levels, $\pi_{t+1} = 0.5 + \frac{s\rho}{2}$, and depending on parameter values, for high values of π_t , $\pi_{t+1} = 0.5(1 + \rho)$. Visually, it would look like a straight line at 0.5 kinked into a positively sloped line at $\pi = 2c$, and rising until $\bar{\pi} = \frac{(4 - 2T)c}{T}$, at which point it remains at $\pi_{t+1} = 0.5(1 + \rho)$ until $\pi_t = 1$.

Intersections of this ‘policy function’ with the 45-degree line determine steady states, of a sort. There are four types of intersections:

1. $\pi_t = \pi_{t+1} = 0.5$. For sufficiently high values of c , agents will be uncertain forever, caught in an uncertainty trap. This steady state is absorbing and inescapable.

2. $0.5 < \pi_t < \frac{1+\rho}{2}$, and $\frac{T\rho}{8c(1-T)} < 1$. This steady state is absorbing, but not inescapable.
3. $0.5 < \pi_t < \frac{1+\rho}{2}$, and $\frac{T\rho}{8c(1-T)} > 1$. This steady state not absorbing.
4. $\pi = \frac{1+\rho}{2}$. This steady state is absorbing but not inescapable.

The first and the last steady state are what remain when we switch from heterogeneous decision making to a representative agent.

C Options Volume

The key mechanism in the model of this paper is that of inattention, or informational investment. Unfortunately, there is no direct measure I am aware of for such a variable. However, in this appendix, I consider a proxy: hedging.

C.1 Interpretation

One could think of purchasing informational investment or paying attention, as a form of reducing conditional volatility or uncertainty. Therefore, purchasing securities that reduce exposure to volatility could be seen to have a direct correlation with the mechanism utilized in the model. After speaking with several investors, including some portfolio managers and a hedge fund manager, a common pattern that emerged was that the way in which they prepared for the unexpected was to utilize options to develop a long position in volatility. Therefore, we will turn to options data, to answer these questions.

C.2 Dataset

The dataset used here is from MarketDataExpress, and provides information on the types and sizes of trades in the SPX options index at a daily frequency. Specifically, for each day, the dataset provides volumes for all *non market-making entities* four different types of trade: open buy, open sell, close buy, and close sell for puts and calls at every strike. An open trade is one where the entity is taking on risk, while a close trade is one where the entity is closing out an existing position.

C.3 Strategy

Given the data above, a useful way to proxy how prepared agents are for different events would be to construct net long positions in options (either puts or calls). Given four different buckets (strikes

less than 90% of the value of the underlying, strikes between 90% and 100% of the underlying, strikes between 100% and 110% of the underlying, and strikes over 110% of the underlying) one could create two variables per bucket - subtracting the number of open sells from open buys in each bucket to create open interest and subtracting the number of close sells from close buys in each bucket to assess reduction of risk. Then for each date, the sum of those buckets for all options traded before that date that expire after that date would be a measure of preparedness. I drop the first 500 days to allow for the buildup of a portfolio. Then the following regression could be estimated:

$$vix_t = \alpha_0 + \alpha_i \{volume_{it}\} + controls + \epsilon_t$$

The desire is to understand, controlling for the ex-ante probabilities of events, changes in volatility, and the underlying, etc. whether an extra unit of preparedness - an extra unit of volatility protection negatively correlated with the level of the vix. So the null hypothesis is that $\alpha_i = 0$ against the alternative that it is significantly negative.

C.4 Results

The three columns of table 4 correspond to days on which the S&P moved down, the days on which it moved up, and all days. With the exception of open long positions in options that have strikes between 100% and 110% of the underlying, every single other type of long position in options mitigates - is negatively correlated with - the level of the VIX. Controlling for this, we still see that the underlying is negatively correlated with the VIX, and that the probabilities of the intraday low and intraday high are negatively correlated as well - with the negative event having a larger impact than the positive event - maintaining the central result that ex-ante unlikely events spike the VIX.