Dynamic Demand and Sequential Monopoly: A Model of Endogenous Screening

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Abstract

We analyze a model in which the consumer's flow utility depends on past as well as current consumption, i.e. is written $u(c_t, c_{t-1})$, where the function u is either strictly submodular or strictly supermodular. Suppliers are small, so that each interacts with the consumer only at one date, but search frictions give rise to monopoly power, and firms offer non-linear prices. Consumers have identical preferences so that differences in taste only arise due to differences in past consumption. When firms observe past consumption, they induce excessive (resp. insufficient) consumption when u is submodular (resp. supermodular). Our main focus is on the case where past consumption is unobservable. We show that pure strategy equilibrium fails to exist. In the two-period version of the model, we prove the existence and uniqueness of a mixed strategy equilibrium that gives rise to a distribution of period-one consumptions, and thus an endogenous screening problem in the second period. Consumers are better off when past consumption is unobservable. In the infinite horizon model, we construct a stationary distribution of equilibrium consumption in the supermodular case.

Keywords: dynamic demand, endogenous screening, non-linear pricing.

JEL Codes:

Preliminary and incomplete – Not for general circulation

1 Introduction

Do fast-food restaurants induce over consumption? Do supermarkets encourage wasteful purchases of goods with a limited shelf-life, such as fruit or vegetables?[1] The consumption of restaurant meals and groceries are characterized by inter-temporal substitutability – if a patron has a heavy lunch, she is less hungry at dinner. If a consumer buys a two-for-one deal on salads or ready meals at the supermarket, he is less likely to buy similar goods when stopping at the local store. In these instances, consumptions across different dates are substitutes, so that the flow utility function $u(c_t, c_{t-1})$ is strictly submodular. A different set of examples concerns habit formation or goods where "taste" needs to be developed, and where the concern is that there might be a tendency to under-consumption. A student who frequents jazz concerts is more likely to enjoy them later in life, so that consumptions across dates are complements, and $u(c_t, c_{t-1})$ is supermodular. In both types of examples, purchases at different dates are often from different suppliers. The jazz club at a university town is unlikely to see the student again after graduation. If I go to MacDonald's for lunch, I might prefer Chinese food for dinner.

This paper analyzes a model where flow utility has a Markovian structure, and where a consumer who shops with a supplier today is unlikely to return tomorrow. To focus on the dynamic implications of endogenous choices, we assume that consumers are identical, i.e. any differences in taste only arise due to differences in past consumption. Furthermore, we assume that a supplier at any date has monopoly power – e.g. because of search frictions – so that the market is characterized by sequential monopoly. We allow firms to offer fully non-linear prices and analyze the nature of inter-temporal competition. A benchmark case, is where past consumption is observable by the monopolist today. In this case, our intuitions are confirmed – the first period monopolist induces over-consumption relative to the efficient allocation when u is submodular, and under-consumption when u is supermodular. [2]

Our main focus however, is on the more realistic case where the consumer's past consumption c_{t-1} is unobserved by the monopolist at date t. Our first result is that there cannot be a pure strategy equilibrium where

[1] The "super-size-me" phenomenon and the obesity epidemic is indicative, as is the documented evidence on the waste of perishables after spending several days in the fridge. While intrinsic consumer preferences are surely important, the present paper highlights the role of a different mechanism.

[2] This finding is reminiscent of the literature on long-term contracts, where payoffs and utility are stationary. Here, contracts are short-term, but future utility depends on current consumption.

consumption is deterministic, both when u is submodular and when it is supermodular. We therefore consider mixed strategy equilibria. We consider the simplest version of our model, a two-period one. So all consumers are identical in the first period, and second period utility displays a dependence on past consumption. We show that there exists a unique equilibrium where the first period monopolist offers a large menu, which ranges between the efficient quantity and that chosen in the observable case. The consumer, who is indifferent between all bundles in the menu, chooses each with positive probability. This induces an endogenous screening problem in the second period, since the consumer has private information about his past consumption. We find that the consumer benefits from unobservability, whereas the first period monopolist loses, as compared to the observable consumption benchmark.

Finally, we consider the infinite horizon version of the model, when u is supermodular. We show that there exists a steady state equilibrium where in each period, each supplier offers a menu that ranges from zero to the efficient level, and where the consumer chooses each of these bundles in such a way that the distribution of past consumption is stationary. Here, consumption in the steady is inefficiently low as compared to the efficient benchmark – indeed, some consumers consume below the equilibrium level in the observable consumption case, while others consume more. Work on the infinite horizon model when u is submodular is in progress.

2 Related literature (incomplete)

Our model bears considerable formal similarity with models of common agency (Bernheim and Whinston (1986); Martimort and Stole (2002)) – the principals in these models correspond to our firms, and the agent to the consumer. A key difference is that interaction (and competition between firms) is sequential in our context, whereas in common agency models, the principals compete simultaneously. Thus sequential rationality plays a critical part in our analysis – when a consumer receives an offer from a firm today, she does not have the option of revising her purchases yesterday. Whereas common agency models have a plethora of equilibria and use refinements such as truthfulness to single out a few, we find uniqueness, at least in the two-period setting.

Our paper also relates to the literature on long-term bilateral contracts in a multilateral environment, including Diamond and Maskin (1979) and Aghion and Bolton (1987). In contrast with this literature, our contracts are static, and the dynamics are induced by the agent's preferences.

A key aspect of our analysis is the endogenous screening problem that

arises, due to the fact that the agent's choice is payoff-relevant as well as private information. This is reminiscent of the work on static moral hazard with renegotiation – Fudenberg and Tirole (1990) and Ma (1991). More recent work where types are endogenous includes González (2004), Calzolari and Pavan (2006) and Netzer and Scheuer (2010).

Finally, our work relates to an emerging literature on the pricing of storable goods, theoretical as well as empirical. This includes Hong et al. (2002), Hendel and Nevo (2006a), Hendel and Nevo (2006b), Ariga et al. (2001) and Hendel et al. (2014).

3 The model

The consumer, who lives for two periods, visits seller 1 in the first period and seller 2 in the second period. Her utility is

$$v(x) + u(y, x) - p - q,$$

where x and y is consumption in the first and second period respectively, and p and q are the payments made to sellers 1 and 2 respectively. The value of consumption in the second period depends on the level of consumption in the first period. We assume that both v and u are strictly increasing, strictly concave and twice continuously differentiable. In period 1, utility v is an increasing, differentiable and strictly concave function of first period consumption alone.

We consider two alternative assumptions.

A1: $u(y, x)$ is strictly supermodular.

A2: $u(y, x)$ is strictly submodular.

The sellers maximize their profit by offering a consumer a menu of non-linearly priced bundles. We assume that the sellers produce the good at a constant marginal cost k.

Remark 1 *The socially efficient level of consumption* (x^*, y^*) *is defined as follows. Define* $y^*(x)$ *as the value of* y *that solves* $u_1(y, x) = k$*, if this equation has a positive solution, and zero otherwise. Since* u_1 *is strictly decreasing in* y*, there is a unique value* $y^*(x)$*. Let* x^* *be the value of* x *that solves* $v'(x) + u_2(y^*(x), x) = k$*, and let* $y^* := y^*(x^*)$*. Since* u *is strictly concave, there is unique solution, so that the socially efficient level of consumption* (x^*, y^*) *is unique.*

3.1 Observable consumption

Consider the following benchmark, where first period consumption is observable to the second period seller. Thus seller 2 chooses y that maximizes $u(y,x) - ky$, and sets a price that makes the consumer indifferent between accepting y and her outside option, of non-consumption. Thus firm 2 chooses $y = y^*(x)$, and sets $q = u(y^*(x), x) - u(0, x)$, and the consumer's second period payoff equals $u(y^*(x), x) - q$, which equals $u(0, x)$. The consumer's first period utility from consuming x equals

$$v(x) + u(0, x) - p.$$

If the consumer does not buy from the first period monopolist, her overall payoff equals $v(0) + u(0, 0)$ (since her second period net payoff after x always equals $u(0, x)$). Since the first period firm optimally sets p so that

$$p(x) = v(x) + u(0, x) - [v(0) + u(0, 0)],$$

the bundle that maximizes the first period profits when consumption is observable, x^o, satisfies

$$v'(x^o) + u_2(0, x^o) = k.$$

We shall assume throughout:

A3: $y^o := y^*(x^o) > 0$.

If u is strictly submodular, then notice that $x^o > x^*$ so that the first period monopolist induces excessive consumption relative to the first best. If u is strictly supermodular, then $x^o < x^*$, then the first period seller induces underconsumption relative to the first best. In either case, the reason is that the first period firm takes into account how he affects the consumer's second period outside option, $u(0, x)$, rather than the consumer's utility associated with his actual consumption, $u(y^o, x)$.

4 Unobserved consumption: pure strategies

We are now consider a model when seller 2 can observe neither the offer made by seller 1 nor the consumer's choice in the first period. Our first result is that there does not exist a pure strategy equilibrium in this case, either in the supermodular or the submodular case.

We consider the conditions for the existence of a pure strategy equilibrium where $\hat{x}$ is consumed at $t = 1$. This implies that second period consumption is deterministic, and equals $\hat{y}$, and firm 2 offers charges the bundle price q such that

$$q = u(\hat{y}, \hat{x}) - u(0, \hat{x}). \tag{1}$$

Since the firm chooses y to maximize $q - k$, $\hat{y}$ satisfies

$$u_1(\hat{y}, \hat{x}) = k,$$

or, $\hat{y} = y^*(\hat{x})$. We now examine the consumer's optimal strategy at $t = 2$ as a function of his past consumption, x.When offered the bundle $\hat{y}$ at price q as given in 1, his payoff from the inside option (i.e. consuming from the period 2 monopolist) is given by

$$u(\hat{y}, x) - [u(\hat{y}, \hat{x}) - u(0, \hat{x})].$$

His payoff from the outside option equals $u(0, x)$. Thus it is optimal to choose the inside option if

$$[u(\hat{y}, x) - u(0, x)] - [u(\hat{y}, \hat{x}) - u(0, \hat{x})] \geq 0, \tag{2}$$

and it is profitable to choose the outside option if the inequality is reversed.

Suppose that u is strictly supermodular, i.e. A1 is satisfied. Then the inequality 2 is satisfied if $x \geq \hat{x}$, and is violated if $x < \hat{x}$. Thus the total payoff of the consumer and firm 1 at date 1, $\Pi(x)$, is given by

$$\Pi(x) = \begin{cases} v(x) - kx + u(\hat{y}, x) - [u(\hat{y}, \hat{x}) - u(0, \hat{x})], \text{ if } x \geq \hat{x} \\ v(x) - kx + u(0, x) \text{ if } x < \hat{x}. \end{cases}$$

Thus the right hand derivative at $\hat{x}$ equals

$$\Pi'(\hat{x}^+) = v'(x) - k + u_2(\hat{y}, \hat{x}).$$

While the left hand derivative equals

$$\Pi'(\hat{x}^-) = v'(x) - k + u_2(0, \hat{x}).$$

The first order conditions for optimality require $\Pi'(\hat{x}^+) \leq 0, \Pi'(\hat{x}^-) \geq 0$. This requires

$$\Pi'(\hat{x}^+) - \Pi'(\hat{x}^-) = u_2(\hat{y}, \hat{x}) - u_2(0, \hat{x}) \leq 0.$$

But this is impossible, since strict supermodularity implies that $u_2(y, \hat{x})$ is strictly increasing in y.

The argument is essentially the same when $u(.)$ is strictly submodular, except that directions are reversed. Then the inequality 2 is satisfied if $x \leq \hat{x}$,

and is violated if $x > \hat{x}$. Thus the total payoff of the consumer and firm 1 at date 1, $\Pi(x)$, is given by

$$\Pi(x) = \begin{cases} v(x) - kx + u(\hat{y}, x) - [u(\hat{y}, \hat{x}) - u(0, \hat{x})], \text{ if } x \leq \hat{x} \\ v(x) - kx + u(0, x) \text{ if } x > \hat{x}. \end{cases}$$

Thus the left-hand derivative at $\hat{x}$ equals

$$\Pi'(\hat{x}^-) = v'(x) - k + u_2(\hat{y}, \hat{x}).$$

While the right-hand derivative equals

$$\Pi'(\hat{x}^+) = v'(x) - k + u_2(0, \hat{x}).$$

The first order conditions for optimality require $\Pi'(\hat{x}^+) \leq 0, \Pi'(\hat{x}^-) \geq 0$. This requires

$$\Pi'(\hat{x}^+) - \Pi'(\hat{x}^-) = -u_2(\hat{y}, \hat{x}) + u_2(0, \hat{x}) \leq 0.$$

But this violates submodularity, since $u_2(0, \hat{x}) > u_2(\hat{y}, \hat{x})$.

We conclude therefore:

Proposition 2 *If $u(y, x)$ is strictly submodular or strictly supermodular, there does not exist an equilibrium where first period consumption is deterministic.*

5 Endogenous screening

We now show that there exists an equilibrium where first period consumption is random – the first period firm offers the consumer a menu $(x, p(x))_{x \in I}$, where I is a compact interval $[\underline{x}, \bar{x}]$. The consumer is indifferent between all options in the menu, and chooses option x with a probability density function $f(x)$. Thus the firm at date 2 is confronted with a monopoly screening problem and offers a direct mechanism $(\hat{y}(x), q(x))_{x \in I}$, where each type $x \in I$ finds it optimal to choose $\hat{y}(x)$ at price $q(x)$. This menu in turn induces an informational rent for the consumer at date 2, $\tilde{U}(x) = u(\hat{y}(x), x) - q(x)$, that depends on his first period consumption.

The critical features in this construction are as follows:

1. $v(x) + \tilde{U}(x) - kx$ is constant for every $x \in I$. This ensures that the consumer and firm 1 are indifferent as to which element of I the consumer chooses.

2. The induced distribution over I, $F(.)$, is such that firm 2 finds it optimal to offer $\tilde{U}(x)$ for each $x \in I$.

3. $\hat{y}(x)$ is strictly decreasing in the submodular case, and strictly increasing in the supermodular case.

4. Since x is not observed by firm 2, each $x \in I$ must be optimal from the consumer-firm 1 point of view given that $\hat{y}(x) := \hat{y}$ is a fixed number, i.e. it must maximize $v(x) + u(\hat{y}, x) - kx$.

5. Finally, the endpoints of the interval I are pinned down by the characteristics of the solution to the monopoly screening problem. Since there is "no distortion at the top", the second period consumption of the highest type – e.g. $\underline{x}$ in the submodular case – must be optimal given $\underline{x}$. Combined with point (4) above, this implies that $\underline{x}= x^*$, the first best level of consumption (when utility is supermodular, the highest type corresponds to $\bar{x}$ which must equal x^*). Since there is "no informational rent at the bottom" – type $\bar{x}$ in the submodular case – her consumption level $\bar{x}$ must maximize the joint payoff of the consumer and firm 1 given that she takes the outside option 0 at $t = 2$. This implies $\bar{x} = x^o$. Thus first period consumptions span the range between first best and the observable consumption case, while second period consumptions lie between 0 and the first best consumption, y^*.

5.1 Characterizing equilibrium

We now characterize the conditions that any equilibrium must satisfy. Proposition 1 has established that in any equilibrium, first period consumption must be random. Let X denote the support of the equilibrium distribution of first period consumption – X is a closed set, by definition. We shall also assume that every bundle in X is offered and chosen by the consumer.[3] Recall that $U(x) := u(\hat{y}(x), x) - q(x)$ is the indirect utility of type $x \in X$ at $t = 2$. Finally, note that X cannot contain 0 – in this case, firm 1's profits must equal zero, and this cannot be optimal for firm 1 and the consumer, and hence there exists a open interval that contains X.

First, we extend $U(.)$ so that is defined on an open interval $I \supseteq X$ rather than just the chosen points, X, where $I \subset (0, \infty)$. For $z \in I - X$, let $U(z) := \sup_{x \in X}\{u(\hat{y}(x), z) - q(x\}$. Thus U is specified by prescribing optimal choices for all non-chosen types, and every point in X lies in the interior of I.

[3]That is, we are assuming that the set of chosen bundles is closed, so that every if $x \in X$ has an associated contract $(\hat{y}(x), q(x))$. This assumption is inessential, but simplifies the statement of some results.

Lemma 3 *$U(x)$ is differentiable at every chosen $x \in X$.*

Proof. Fix $x \in X$, and $\hat{y}(x)$. Consider the payoff of the consumer at $t = 2$, $U(x + \delta)$ – this is well defined for δ sufficiently small since U is defined on the open interval I. Since $x + \delta$ can choose the contract for type x,

$$U(x + \delta) \geq u(\hat{y}(x), x + \delta) - q(x)$$

Thus, for $\delta > 0$

$$\frac{U(x + \delta) - U(x)}{\delta} \geq \frac{u(\hat{y}(x), x + \delta) - u(\hat{y}(x), x)}{\delta}.$$

The above inequality implies

$$D_+U(x) := \lim \inf_{\delta \to 0+} \frac{U(x + \delta) - U(x)}{\delta} \geq u_2(\hat{y}(x), x). \tag{3}$$

Since the inequality for $\delta < 0$ has a reversed sign, this yields

$$D^-U(x) := \lim \sup_{\delta \to 0-} \frac{U(x + \delta) - U(x)}{\delta} \leq u_2(\hat{y}(x), x). \tag{4}$$

Now, the total payoff of firm 1 and consumer, $\Pi(x)$, equals

$$\Pi(x) = v(x) + U(x) - kx.$$

Let $D^+U(x) := \lim \sup_{\delta \to 0+} \frac{U(x+\delta)-U(x)}{\delta}$ and $D_-U(x) := \lim \inf_{\delta \to 0-} \frac{U(x+\delta)-U(x)}{\delta}$.

If $x \in X$, then since x is chosen, it must maximize $\Pi(x)$, and necessary conditions are

$$\Pi^+(x) = v'(x) + D^+U(x) - k \leq 0,$$

$$\Pi^-(x) = v'(x) + D_-U(x) - k \geq 0.$$

These inequalities imply $D^+U(x) \leq D_-U(x)$. In conjunction with the inequalities 3 and 4, this implies that for any $x \in X$,

$$D^+U(x) = D_+U(x) = D^-U(x) = D_-U(x) = u_2(\hat{y}(x), x).$$

■

Remark 4 *The result that U is differentiable on X, the set of positive probability types, follows from the endogeneity of types. With exogenous types, it is well known that U need not be everywhere differentiable. Indeed, this result is more general than the specific context of the present model.*

It is standard in mechanism design that single-crossing, incentive compatibility implies weak monotonicity. However, lemma 3 allows a stronger result.

Lemma 5 *Let $(\hat{y}(x), q(x))_{x\in X}$ be incentive compatible for every $x \in X$. $\hat{y}(x)$ must satisfy*

$$v'(x) + u_2(\hat{y}(x), x) = k.$$

Then $\hat{y}(x)$ is strictly decreasing (resp. increasing) in x if $u(.)$ is submodular (resp. supermodular).
Proof. Since U is differentiable at x, if x maximizes $\Pi(.)$, it must satisfy

$$\Pi'(x) = v'(x) + u_2(\hat{y}(x), x) - k = 0.$$

Let $u_{12}(.) < 0$. If $x > x'$ then $\hat{y}(x)$ must be strictly less than $\hat{y}(x')$, or

otherwise the expression for $\Pi'(.)$ above will be strictly negative. Similarly, if $u_{12} > 0$, $\hat{y}(x)$ must be strictly greater than $\hat{y}(x')$. ■

Let $\bar{x}$ denote the minimal element in X and $\bar{x}$ the maximal element. The following lemma shows that if individual rationality is satisfied for type $\bar{x}$ in the submodular case, then it is satisfied for every other type – although familiar, the result is not immediate since the outside option $u(0, x)$ is type dependent.

Lemma 6 *Let $(\hat{y}(x), q(x))_{x\in X}$ be incentive compatible for $x \in X$. Then under submodularity, $U(x) - u(0, x) \geq U(\bar{x}) - u(0, \bar{x}) \forall x \in X$ and any profit maximizing second period contract, $U(\bar{x}) = u(0, \bar{x})$, and the individual rationality constraint binds for type $\bar{x}$. Under supermodularity, the binding individual rationality constraint is for type $\underline{x}$.*

Proof. Incentive compatibility implies if $x < \bar{x}$, since type x can pretend to be $\bar{x}$,

$$U(x) \geq U(\bar{x}) + u(\hat{y}(\bar{x}), x) - u(\hat{y}(\bar{x}), \bar{x}).$$

Since $U(\bar{x}) \geq u(0, \bar{x})$,

$$U(x) - u(0, x) \geq [u(0, \bar{x}) - u(0, x)] - [u(\hat{y}(\bar{x}), \bar{x}) - u(\hat{y}(\bar{x}), x), \tag{5}$$

which is non-negative by submodularity since $\hat{y}(\bar{x}) \geq 0$.

If $U(\bar{x}) > u(0, \bar{x})$, then an IC contract $(\hat{y}(x), q(x))_{x\in X}$ cannot be profit maximizing, since a uniform reduction in payoffs $U(x)$ by $U(\bar{x}) - u(0, \bar{x})$,

by raising $q(x)$ by the same amount, preserves incentive compatibility and increases profits. For the supermodular case, replace $\bar{x}$ by $\underline{x}$ in the above argument, and the right hand side of the inequality corresponding to 5 is non-negative under supermoduarity. ■

Lemma 7 *Under sub-modularity, $\overline{x}$ equals the value of x that maximizes $v(x) + u(0, x) - kx$, and so $\bar{x} = x^o$, and $\hat{y}(\overline{x}) = 0$. Under superodularity, $\underline{x}$ equals the value of x that maximizes $v(x) + u(0, x) - kx$, and so $\underline{x}= x^o$, and $\hat{y}(\underline{x}) = 0$.*

Proof. Under submodularity, since the second period participation constraint binds for the highest value of x that is offered by seller 1 and accepted by the consumer, the second period payoff of the consumer equals $u(0, \bar{x})$. Thus $\bar{x}$ must equal the value of x that maximizes

$$v(x) + u(0, x) - kx,$$

or $\bar{x} = x^o$. Since $x^0 \in X$, the second part of lemma 5 implies that $\hat{y}(\overline{x}) = 0$. The proof for supermodularity is identical. ■

Lemma 8 *$\underline{x} = x^*$ under submodularity and $\bar{x} = x^*$ under supermodularity. In either case, $\hat{y}(x^*) = y^*$, where (x^*, y^*) is the efficient profile.*

Proof. Recall that $y^*(x)$ denotes the first best second period quantity conditional on any level of x of first period consumption. Since there is no distortion at the top in the second period screening problem, seller 2 must offer $y^*(\underline{x})$ to the consumer who consumed $\underline{x}$ in the first period in the submodular case. On the other hand, lemma 5 establishes that any $x \in X$,including $\underline{x}$ must satisfy

$$v'(\underline{x}) + u_2(\hat{y}(\underline{x}), \underline{x}) = k.$$

These two conditions imply

$$v'(\underline{x}) + u_2(y^*(\underline{x}), \underline{x}) = k,$$

which implies that $(\underline{x}, y^*(\underline{x}))$ satisfy the conditions for the first best allocation. Since the first best allocation is unique, this implies $\underline{x} = x^*$. When u is supermodular, the "top" corresponds to $\bar{x}$, and the rest of the argument is identical. ■

To summarize, the characterization in the above lemmata imply that $\bar{x} = x^o$ and $\underline{x}= x^*$ in the submodular case. When u is supermodular, $\bar{x} = x^*$ and $\underline{x}= x^*$. We now construct an equilibrium where the equilibrium distribution has support $[\underline{x}, \bar{x}]$.

5.2 Existence of an equilibrium

We now explicitly construct an equilibrium.

Theorem 9 *There exists an equilibrium in which*

1. *Seller 1 offers a two-part tariff. The entree fee equals to the seller 1's value added in the socially efficient consumption stream:*

$$v(x^*) + u(y^*, x^*) - kx^* - v(0) - u(y^*, 0).$$

The per-unit price equals to the marginal cost k.

2. *Seller 2 offers a menu that includes every bundle in $[0, y^*]$. The bundles in this menu are indexed by the first period consumption x. The price of a bundle $\hat{y}(x)$ is*

$$q(x) = v(x) + u(\hat{y}(x), x) - kx - [v(\overline{x}) + u(0, \overline{x}) - k\overline{x}]$$

3. *In the first period, the consumer randomly chooses the bundle according to a distribution F. In the second period, he chooses a consumption $\hat{y}(x)$ where x is his first period consumption.*

If u is submodular, the support of the distribution F is $[x^*, x^o]$ and

$$F(x) = \exp\left[\int_x^{\overline{x}} \frac{u_{21}(\hat{y}(z), z)}{u_1(\hat{y}(z), z) - k} dz\right].$$

If u is supermodular, the support of the distribution F is $[x^o, x^*]$ and

$$F(x) = 1 - \exp\left[\int_{\underline{x}}^{x} \frac{u_{21}(\hat{y}(z), z)}{k - u_1(\hat{y}(z), z)} dz\right].$$

Proof. We verify that the proposed strategy profile is an equilibrium. For simplicity we focus on the submodular case, since the argument is very similar when u is supermodular. Suppose that the consumer is mixing in the first period: $x \sim F$. Seller 2 is facing the consumer with private valuation. Consider a direct mechanism in which the consumer reports his first period consumption and seller 2 provides a bundle $\hat{y}(x)$ at a price $q(x)$ based on this report. Notice that if $\hat{y}(x)$ is implementable, then it must be decreasing.

The outside option of the consumer in the second period is $u(0,x)$. In the submodular case, the binding IR constraint is for type $\bar{x}$, and thus the payoff of any consumer is given by

$$u(0,\overline{x}) - \int_{x}^{\overline{x}} u_2(\hat{y}(z), z)dz.$$

Thus the consumer's utility from first period consumption y is given by

$$v(x) + u(0,\overline{x}) - \int_{x}^{\overline{x}} u_2(\hat{y}(z), z)dz.$$

The price charged by the seller 2 for the bundle $\hat{y}(x)$ is

$$q(x) = u(\hat{y}(x), x) - u(0,\overline{x}) + \int_{x}^{\overline{x}} u_2(\hat{y}(z), z)dz \tag{6}$$

The expected profit for this seller is

$$\mathbb{E}\left[u(\hat{y}(x), x) + u_2(\hat{y}(x), x)\frac{F(x)}{f(x)} - k\hat{y}(x)\right] - u(0,\overline{x}).$$

Maximizing pointwise, we see that the allocation rule $\hat{y}(x)$ must satisfy the first order condition

$$u_1(\hat{y}(x), x)f(x) + u_{21}(\hat{y}(x), x)F(x) - kf(x) = 0. \tag{7}$$

Lemma 10 *If $\hat{y}(x)$ is decreasing and u is submodular (or If $\hat{y}(x)$ is increasing and u is supermodular), equation (6) implies IC in the second period, which is*

$$u(\hat{y}(t), x) - q(t) \leq u(\hat{y}(x), x) - \hat{b}(x)$$

for all x, t.

Proof. Consider $x > t$. Since $\hat{y}(x)$ is decreasing and $u_{21} < 0$ (or, alternatively, $\hat{y}(x)$ is increasing and $u_{21} > 0$)

$$\begin{aligned}
&q(t) - q(x) = u(\hat{y}(t), t) - u(\hat{y}(x), x) + \int_{t}^{x} u_2(\hat{y}(z), z)dz \geq \\
&u(\hat{y}(t), t) - u(\hat{y}(x), x) + \int_{t}^{x} u_2(\hat{y}(t), z)dz = \\
&u(\hat{y}(t), t) - u(\hat{y}(x), x) + u(\hat{y}(t), x) - u(\hat{y}(t), t) = \\
&u(\hat{y}(t), x) - u(\hat{y}(x), x).
\end{aligned}$$

The case with $x < t$ is identical. ■

If the consumer decides not to buy in the first period, he consumes 0 and gets $v(0)$. In the second period this consumer reports $\underline{x}$ and gets a payoff equal to

$$v(0) + u(\hat{y}(\underline{x}), 0) - u(\hat{y}(\underline{x}), \underline{x}) + u(0, \overline{x}) - \int_{\underline{x}}^{\overline{x}} u_2(\hat{y}(z), z) dz.$$

Thus seller 1 can charge a price for amount x that equals

$$p(x) = v(x) - v(0) + u(\hat{y}(\underline{x}), \underline{x}) - u(\hat{y}(\underline{x}), 0) + \int_{\underline{x}}^{x} u_2(\hat{y}(z), z) dz.$$

In particular, the price for the bundle $\underline{x}$ equals

$$p(\underline{x}) = v(\underline{x}) - v(0) + u(\hat{y}(\underline{x}), \underline{x}) - u(\hat{y}(\underline{x}), 0).$$

Seller 1's profit from selling a bundle x has to be independent of x, hence

$$v'(x) + u_2(\hat{y}(x), x) - k = 0. \tag{8}$$

Equations (8) and (7) pin down unknown functions F and $\hat{y}$.

Lemma 11 *Let functions F and $\hat{y}$ be a solution to equations (8) and (7). Then F is a c.d.f. and $\hat{y}$ is decreasing (increasing) whenever u is submodular (supermodular).*

Proof. By taking a derivative of equation (8) with respect to x we get

$$\hat{y}'(x) = -\frac{u_{22}(\hat{y}(x), x) + v''(x)}{u_{21}(\hat{y}(x), x)}$$

therefore $sign(\hat{y}'(x)) = sign(u_{21}(\hat{y}(x), x))$.

The solution to equation (7) is

$$\ln F(x) = \int_{x}^{\overline{x}} \frac{u_{21}(\hat{y}(z), z)}{u_1(\hat{y}(z), z) - k} dz \tag{9}$$

This solution is increasing in x if $u_1(\hat{y}(x), x) \geq k$ for all $x \in [\underline{x}, \overline{x}]$. Lower bound $\underline{x}$ solves $u_1(\hat{y}(\underline{x}), \underline{x}) = k$ since $F(\underline{x}) = 0$. Moreover, since both u and v are strictly concave

$$\begin{aligned}
\frac{d}{dx}u_1(\hat{y}(x), x) &= u_{21}(\hat{y}(x), x) + u_{11}(\hat{y}(x), x)\hat{y}'(x) \\
&= u_{21}(\hat{y}(x), x) - \frac{u_{22}(\hat{y}(x), x) + v''(x)}{u_{21}(\hat{y}(x), x)}u_{11}(\hat{y}(x), x) \\
&= \frac{u_{11}(\hat{y}(x), x)u_{22}(\hat{y}(x), x) - (u_{21}(\hat{y}(x), x))^2 + u_{11}(\hat{y}(x), x)v''(x)}{-u_{21}(\hat{y}(x), x)} > 0.
\end{aligned}$$

Therefore $u_1(\hat{y}(x), x)$ is strictly increasing in x and $u_1(\hat{y}(x), x) \geq k$ for all $x \in [\underline{x}, \overline{x}]$. To summarize, the solution $F(x)$ is an increasing function, $F(\underline{x}) = 0$ and $F(\overline{x}) = 1$, therefore $F(x)$ is a c.d.f.[4] ■ ■

5.3 Uniqueness

We now show that equilibrium is unique. For simplicity, we focus on the submodular case – the results below apply equally to both cases. As a first step, we show that our characterization results in subsection 5.1 imply that equilibrium payoffs are unique, for the consumer and for firm 1. Fix an equilibrium σ of the game associated with the contracting problem. Let $V(\sigma)$ denote the consumer's ex ante utility in this equilibrium – it equals the payoff $v(x) + U(x) - p_1(x)$, where $x \in X$, the set of offered bundles at $t = 1$. Let $\pi_1(\sigma)$ denote the expected profit of firm 1 in this equilibrium, and let $\Pi(\sigma) = V(\sigma) + \pi_1(\sigma)$ denote the sum of payoffs of the consumer and firm 1. Let $\tilde{\sigma}$ denote the specific equilibrium constructed in the previous section, the support of which is the largest possible set, $[x^*, x^o]$.

Proposition 12 *$V(\sigma)$ and $\pi(\sigma)$ are invariant across equilibria, and do not depend on σ. For any $x \in X, U(x, \sigma) = U(x, \tilde{\sigma})$.*

Proof. By lemma 7 the maximal quantity offered and chosen in period one equals x^o in any equilibrium, and the informational rent that accrues to the consumer is zero in this case. Since x^o is in the support of every equilibrium,

$$\Pi(\sigma) = v(x^o) + u(0, x^o) - kx^o \forall \sigma,$$

establishing that $\Pi(\sigma)$ does not depend on σ. Since x^* is in the support of every equilibrium,

$$\Pi(\sigma) = v(x^*) + U(x^*; \sigma) - kx^* \forall \sigma.$$

[4] Strictly speaking, the ODE (7) has a solution (9) on $(\underline{x}, \overline{x}]$, but we can continuously extend it to $\underline{x}$ with value of zero.

Since $\Pi(\sigma)$ is invariant, the above equation implies that $U(x^*;\sigma)$ is independent of σ, and can be written as $U(x^*)$.

Lemma 8 shows that the minimal first period quantity chosen in any equilibrium equals x^*, and thus the maximal second period quantity equals y^*. This defines the outside option of the consumer at $t = 1$ – her payoff from not consuming at $t = 1$ equals

$$V(\sigma) = v(0) + U(x^*) + u(0, y^*) - u(x^*, y^*),$$

where the equality follows from the fact that under the firm optimal contract, the outside option must bind. Since $U(x^*)$ is invariant across equilibria, this establishes that so is $V(\sigma)$, and therefore $\pi(\sigma)$.

The last statement in the proposition is proved as follows: since $v(x) + U(x, \sigma) = V$ for any $x \in X$, and $v(x) + U(x, \tilde{\sigma}) = V$ for any $x \in [x^*, x^o]$, this implies $U(x, \sigma) = U(x, \tilde{\sigma})$. ■

In the light of this proposition, we write Π, V and π_1 for the payoffs that arise in *any* equilibrium. Let $\tilde{F}$ denote the c.d.f. associated with $\tilde{\sigma}$, as defined in the previous section, and let $\tilde{f}$ denote the associated density.

Theorem 13 *Equilibrium is essentially unique; any equilibrium differs from the constructed equilibrium only on a set of first period consumptions that are negligible.*

Proof. Our proof hinges on two facts that have been established. First, for any equilibrium σ with support $X, U(x;\sigma) = U(x;\sigma)$, and second, $\hat{y}(x)$ is uniquely determined and coincides with that under $\tilde{\sigma}$. In other words, the payoff and the allocation for any chosen type is uniquely determined. Let F denote the c.d.f. corresponding to σ, and let $M \subseteq X$ denote mass points. Let $\mathcal{G}$ be a collection of maximal open intervals in $[x^*, x^o] - X$ that denote the gaps in X. Hellwig (2010) characterizes the optimal allocation rule in a screening problem when the distribution of types includes mass points as well as intervals where the distribution is absolutely continuous with respect to Lebesgue measure, and his arguments also apply here. Let (a, b) be the "largest interval" in $\mathcal{G}$: i.e. for any other interval in the collection (c, d), $a > d$. Let m denote the largest value in M. Suppose $m < a$, so that the first gap is larger than the first mass point. Now for every $x \in [b, x^o]$, $F(x) = \tilde{F}(x)$, since the differential equation is identical and so is the initial condition. In the equilibrium $\tilde{\sigma}$, the payoff for type a continues to be given by the differential equation, i.e.

$$U(a;\tilde{\sigma}) = U(b) - \int_a^b [u_2(\hat{y}(x), x)dx.$$

In equilibrium σ, the binding incentive constraint is more slack, since it is given by the payoff that type b can get by pretending to be type a, and therefore

$$U(a;\sigma) = U(b) + [u(\hat{y}(b), a) - u(\hat{y}(b), b)]$$

Since $\hat{y}(x)$ is strictly decreasing, $U(a;\tilde{\sigma}) > U(a;\sigma)$. Since $\hat{y}(a)$ is identical in the two equilibria, this implies that the payoff $U(a,\sigma) < U$, and thus σ cannot be an equilibrium.

Now suppose that $m \geq b$ so that the largest mass point is greater than the the largest gap. $\hat{y}(m)$ must equal the same value in σ as it does in $\tilde{\sigma}$, and must therefore satisfy the first order condition for optimality. Since $F(x) = \tilde{F}(x) \forall x > m$, $\hat{y}(m)$ will not be optimal for firm 2 if $f(m) > \tilde{f}(m)$, where f is the density function corresponding to F. Hence $\hat{y}(m)$ cannot be optimal if there is a mass point at m. ■

5.4 How does unobservability affect payoffs?

How does the privateness of transactions affect the payoffs of firm 1 and the consumer, relative to the case where the consumer's past consumption is observed by firm 2? Note that the total payoff of the two parties, Π, can be evaluated at any point in the support. In particular, at x^o, the consumer takes his outside option at $t = 2$, and so the total payoff equals

$$\Pi = v(x^o) + u(0, x^o).$$

This is *identical* with the total payoff of the two parties when consumption is observable – although the consumer purchases $y^o > 0$, firm 2 appropriates the entire surplus, and hence her second period payoff is $u(0, x^o)$. We turn to the distribution of the total payoff between the two parties in the two cases. In the observable case, the consumer's payoff equals

$$v(0) + u(0, 0) := V^o.$$

In the unobservable case, the results now differ depending on whether u is supermodular or submodular. So we consider these in turn.

When u is submodular, the consumer who chooses the outside option at $t = 1$ chooses the contract for type x^* at $t = 2$, and therefore gets a total payoff

$$v(0) + [u(y^*, 0) - u(y^*, x^*)] + u(0, x^*) = V.$$

The difference in payoffs is

$$V - V^o = [u(0, x^*) - u(0, 0)] - [u(y^*, x^*) - u(y^*, 0)] = \pi_1^o - \pi_1 > 0,$$

where π_1^o denotes firm 1's profits in the observable case. The second equality in the above follows since the total payoff Π is equal in the two cases. The strict inequality arises since u is strictly submodular. We conclude that the consumer is strictly better off when consumption is unobservable, and firm 1 is strictly worse off to exactly the same extent.

When u is supermodular, the consumer who chooses the outside option at $t = 1$ chooses the contract for type x^o at $t = 2$, which is the null contract, and therefore gets a total payoff

$$v(0) + u(0, 0) = V.$$

This is exactly equal to V^o, and hence unobservability has no distributional effect on first period payoffs when u is supermodular

Turning to firm 2, its profits in the observable case equal

$$\pi_2^o = u(y^o, x^o) - u(0, x^o) - ky^o.$$

Alternatively, we may write its profits as

$$\pi_2^o = [v(x^o) + u(y^o, x^o) - k(x^o + y^o)] - \Pi.$$

The above expression arises since we may write firm two's profits as the total social surplus from the consumption profile (x^o, y^o) – the term in square bracket – minus the total payoff to the other two parties, Π.

Its maximal profit in the unobservable case is when first period consumption is optimal, at x^*, and equals

$$u(y^*, x^*) - U(x^*) - ky^*.$$

Since Π, the total payoff of firm 1 and the consumer is the same at x^* and at x^o, implying

$$v(x^*) + u(y^*, x^*) + U(x^*) - kx^* = v(x^o) + u(0, x^o) - kx^o.$$

Thus the profit of firm 2 when x^* is chosen at $t = 1$ equals

$$\pi_2(x^*) = [v(x^*) + u(y^*, x^*) - k(x^* + y^*)] - \Pi.$$

The above expression decomposes firm 2's profits at x^* into the total social surplus at the profile (x^*, y^*) – the term in square brackets – minus the

payoff to the other two parties, Π. Since social surplus is maximal at (x^*, y^*), $\pi_2(x^*) > \pi_2^o$, and the entire increase in social surplus accrues entirely to firm 2.

On the other hand, when first period consumption is x^o, second period consumption is zero and so is firm 2's profit. More generally, as first period consumption decreases, so does social efficiency monotonically, for two reasons – first period overconsumption falls, and so does the efficiency of the second period allocation, since distortion decreases. The entire benefit of this efficiency gain accrues to firm 2.

Lemma 14 *The gain (loss) of seller 2 from unobservability of past consumption equals to the increase (decrease) of the social welfare:*

$$\mathbf{E}[\pi_2(x)] - \pi_2^o = \mathbf{E}[W(x)] - W^o$$

Proof. According to Theorem 9, if seller 2 sells the amount $\hat{y}(x)$ he obtains a profit

$$\pi_2(x) = v(x) + u(\hat{y}(x), x) - u(0, x^o) - v(x^o) + k(x^o - x - \hat{y}(x))$$

In the benchmark case with observable consumption, the seller 2 obtains a profit

$$\pi_2^o = u(y^o, x^o) - u(0, x^o) - ky^o.$$

Subtracting one profit from the other and taking an expectation we show that

$$\begin{aligned}\mathbf{E}[\pi_2(x)] - \pi_2^o &= \mathbf{E}[v(x) + u(\hat{y}(x), x) - k(x + \hat{y}(x))] \\ &\quad - [v(x^o) + u(y^o, x^o) - k(x^o + y^o)] \\ &= \mathbf{E}[W(x)] - W^o.\end{aligned}$$

■

Remark 15 *Equilibrium social welfare conditional on first period consumption x is decreasing in x when u is submodular, and increasing when u is supermodular.*

Proof. Equilibrium social welfare conditional on first period consumption x is

$$W(x) = v(x) + u(\hat{y}(x), x) - k(x + \hat{y}(x))$$

Taking a derivative, we obtain

$$\begin{aligned}W'(x) &= v'(x) + u_2(\hat{y}(x), x) - k + \hat{y}'(x)u_1(\hat{y}(x), x) - k\hat{y}'(x) \\ &= \hat{y}'(x)\left[u_1(\hat{y}(x), x) - k\right],\end{aligned}$$

where the second line follows from the first period first order condition, equation (8). Since the second period consumption is always distorted, the term in square brackets is always positive, and hence W is increasing in x when $\hat{y}$ is (the supermodular case), and decreasing in x when $\hat{y}$ is decreasing in x, as in the submodular case. ■

5.5 An example

In this section we consider a numerical example of our model. Suppose that the utility of the agent is given by the following expressions

$$\begin{aligned} v(x) &= -2x^2 + 6x \\ u(y, x) &= -x^2 - 3y^2 + axy + 2x + 8y \end{aligned}$$

The parameter $a = u_{21}(y, x)$ is a measure of substitutability of the past and current consumption.

The variables of interest for this example are presented in Table 1. The equilibrium distributions of first period consumption for the two cases, $a = -1$ and $a = -3$ are given on Figures 1 and 2.

In these two cases, the equilibrium distribution is skewed to the left: most of the consumers consume an amount close to the socially efficient one. This is also reflected in the fact that efficiency loss in equilibrium, $W^* - W$, is small compared to the one in benchmark model with observable consumption, $W^* - W^o$.

The distribution of the social welfare is also interesting: consumers and the second period seller obtain higher payoffs when past consumption is unobservable compared to the case when past consumption is observable. The profit comparison for the first period seller is the opposite of that: this seller's profit is reduced by unobservable past consumption.

According to the results in Section 5.4, the first seller's loss is consumer's gain and social welfare's increase is fully absorbed by the second seller. This result is reflected in the numbers obtained for this example. Notice that the gain from unobservability of past consumption for seller 2 is much smaller then the gain for the consumer.

6 Infinite horizon

6.1 Supermodular u

Consider a model with an infinite time horizon. The consumer visits a new seller in each period. Each seller maximizes profit by offering a menu of

	$a=-1$	$a=-3$
x^*	1	0.78
x^o	1.17	1.17
y^*	1	0.78
y^o	0.97	0.58
$W^\star$	7	5.44
W^o	6.92	5.10
W	6.99	5.39
(π_1^o, π_2^o)	(4.08, 2.84)	(4.08, 1.02)
(π_1, π_2)	(3.00, 2.92)	(1.81, 1.31)
Consumer's utility, obs. consumption	0	0
Consumer's utility, unobs. consumption	1.08	2.27

Note: Superscript * denotes variables for the first best case and superscript ^{o} denotes variables for the benchmark model with observable consumption.

Table 1: Numerical example

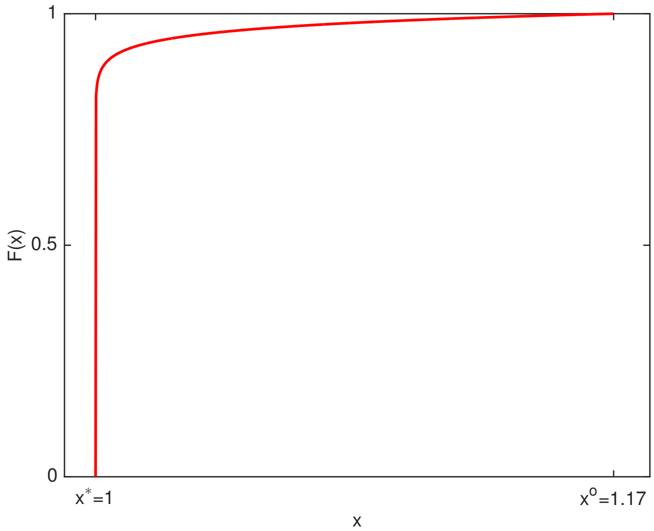

Figure 1: Distribution of first period consumption for $a=-1$.

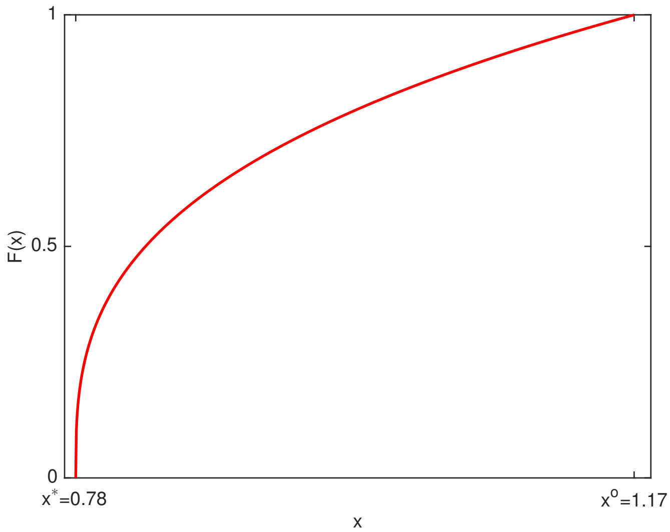

Figure 2: Distribution of first period consumption for $a = -3$.

choices that consists of nonlinearly priced bundles. A seller observes neither offers made by other sellers nor consumer's past choices.

The consumer maximizes the discounted sum of period payoffs:

$$\sum_{t=0}^{\infty} \beta^t [u(c_t, c_{t-1}) - p_t].$$

For this problem to be well defined we assume that c_{-1} is exogenously given. We also assume that the good is not storable and that the consumer must consume what he bought immediately. The function u is assumed to be continuously differentiable, strictly concave, supermodular and bounded from above. We also assume that function $S(x) := u_1(x, x) + \beta u_2(x, x)$ is decreasing.[5]

We consider stationary equilibria in which all sellers offer identical menus. Let $p(x)$ be a price of a bundle x. In this case, consumer's maximization problem can be rewritten in the recursive form in the following way:

$$V(c_{t-1}) = \max_{c_t}\{u(c_t, c_{t-1}) - p(c_t) + \beta V(c_t)\}.$$

Using the envelope theorem we obtain that

$$V'(c_t) = u_2(c_{t+1}, c_t).$$

[5]For β sufficiently close to 1, this assumption is implied by strict concavity of u: $u_{11} + \beta u_{22} + (1 + \beta)u_{21} \leq 0$ when β is close to 1 and $u_{21} < \sqrt{u_{11}u_{22}}$.

Assuming that p is differentiable, the first order condition for consumer's maximization problem is

$$u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t) = p'(c_t).$$

Before we analyze the equilibrium of the model, let us consider two benchmarks: the first best allocation of consumption and the equilibrium of the model with observable past consumption.

Remark 16 *The first best allocation* $\{c_t^*\}_{t=0}^{\infty}$ *solves*

$$u_1(c_t^*, c_{t-1}^*) + \beta u_2(c_{t+1}^*, c_t^*) = k.$$

Consider the benchmark in which the past consumption is observable. In this case, the offer of the monopolist is conditional on past consumption. Moreover, the monopolist extracts all the consumer's surplus net of the outside option (which is consuming zero in the current period).

Remark 17 *In the stationary equilibrium of the benchmark model with observable past consumption, the consumption* c^o *solves*

$$u_1(c^o, c^o) + \beta u_2(0, c^o) = k \tag{10}$$

Moreover, $c^o < c^*$.

Proof. Suppose a seller serves the consumer with past consumption equal to x. The seller's profit is

$$\pi = u(c, x) + \beta V(c) - u(0, x) - \beta V(0) - kc.$$

The first order condition for this problem is

$$u_1(c, x) + \beta V'(c) = kc$$

Also notice that

$$V(c) = u(0, c) + \beta V(0),$$

hence

$$V'(c) = u_2(0, c).$$

Combining these two equations with the condition for stationary equilibrium we obtain equation (10).

Since u is supermodular,

$$u_1(c^o, c^o) + \beta u_2(c^o, c^o) > k,$$

therefore $c^o < c^*$. ■

Let us consider a candidate for an equilibrium in which each seller faces a continuous distribution of past consumption for its buyer. We denote this distribution and its support by F and $[\underline{x}, \overline{x}]$ respectively. Each seller offers a menu $(x, p(x)), x \in [\underline{x}, \overline{x}]$.

Consider a consumer with a consumption c_{t-1} in the previous period. This consumer's equilibrium value must satisfy

$$V'(c_{t-1}) = u_2(c_t(c_{t-1}), c_{t-1}) \tag{11}$$

where $c_t(c_{t-1})$ is the consumers optimal choice of consumption. Integrating this equation we obtain that

$$u(c_t, c_{t-1}) - p(c_t) + \beta V(c_t) - V(\underline{x}) = \int_{\underline{x}}^{c_{t-1}} u_2(c_t(x), x)dx \tag{12}$$

Lemma 18 *If $c_t(c_{t-1})$ is decreasing and u is submodular (or If $c_t(c_{t-1})$ is increasing and u is supermodular), equation (12) implies IC, which is*

$$u(c_t(z), x) + \beta V(c_t(z)) - p(c_t(z)) \leq u(c_t(x), x) + \beta V(c_t(x)) - p(c_t(x))$$

for all x, z.

Proof. By subtracting $u(c_t(z), z) + \beta V(c_t(z)) - p(c_t(z))$ from both sides of the IC inequality and using equation (12) we obtain that

$$u(c_t(z), x) - u(c_t(z), z) \leq \int_{\underline{z}}^{x} u_2(c_t(x), x)dx$$

The rest of the proof is identical to the proof of Lemma 10. ■

We can solve for $p(c_t)$ and write seller's expected profit in the following form

$$\mathbf{E}\left[u(c_t, c_{t-1}) + \beta V(c_t) - V(\underline{x}) - \int_{\underline{x}}^{c_{t-1}} u_2(\hat{y}(x), x)dx - kc_t\right]$$

The seller's optimality condition is

$$u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t) - u_{21}(c_t, c_{t-1})\frac{1 - F_{t-1}(c_{t-1})}{f_{t-1}(c_{t-1})} = k.$$

In a stationary equilibrium, the distribution of consumption is F is independent of time. We combine this observation with the fact that current

consumption is increasing in the past one to obtain that the consumption is constant across time: $c_t = c_{t-1}$. Therefore

$$-\frac{f(x)}{1-F(x)} = \frac{u_{21}(x,x)}{k - u_1(x,x) - \beta u_2(x,x)}.$$

The solution to this ODE is

$$F(x) = 1 - \exp\left[\int_{\underline{x}}^{x} \frac{u_{21}(t,t)}{k - u_1(t,t) - \beta u_2(t,t)} dt\right].$$

Notice that $\underline{x} = 0$. Indeed, since $\underline{x}$ is the worst type, the IR constraint for this type should bind and therefore

$$V(\underline{x}) = u(0, \underline{x}) + \frac{\beta u(0,0)}{1-\beta}.$$

which means that $V'(\underline{x}) = u_2(0, \underline{x})$. Together with an envelope condition (11) and stationarity this implies that $\underline{x} = c_t(\underline{x}) = 0$.

It remains to show that the solution is a c.d.f. Clearly, when $x = c^*$ $F(x) = 1$. Therefore $\overline{x} = c^*$. A necessary condition for a solution to be a c.d.f. is for all $x \in [\underline{x}, \overline{x}]$ to satisfy the following inequality

$$u_1(x,x) + \beta u_2(x,x) \geq k$$

At $x = \overline{x}$, the inequality is satisfied as equality. For this inequality to be satisfied everywhere, decreasing $S(x)$ is sufficient.

Consider the numerical example from Section 5.5 extended to the infinite time horizon. Assuming that $\beta = 0.95$ we calculate equilibrium distribution of consumption for three cases: $a = 0$, $a = 1$ and $a = 3$. The results are shown on Figure 3.

6.2 Submodular u

6.2.1 Continuous distribution of past consumption

Similar to supermodular case, the seller's optimality condition is

$$u_1(c_t, c_{t-1}) + \beta u_2(c_{t+1}, c_t) + u_{21}(c_t, c_{t-1}) \frac{F_{t-1}(c_{t-1})}{f_{t-1}(c_{t-1})} = k.$$

Consider a change of variables $z_t = F_t(c_t)$, and $\phi_t(z_t) = F_t^{-1}(z_t)$. The equation 13 can be rewritten as

$$\phi'_{t-1}(z_{t-1}) = \frac{k - u_1(\phi_t(z_t), \phi_{t-1}(z_{t-1})) - \beta u_2(\phi_{t+1}(z_{t+1}), \phi_t(z_t))}{u_{21}(\phi_t(z_t), \phi_{t-1}(z_{t-1})) z_{t-1}}.$$

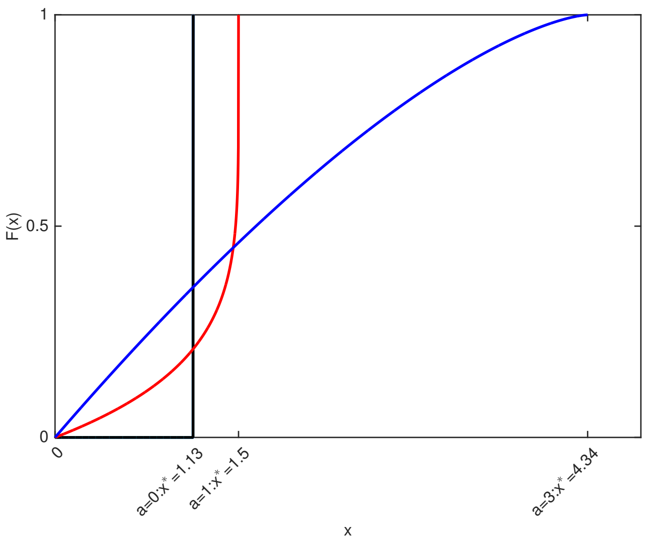

Figure 3: Distribution of consumption in equilibrium for $a = 0$ (black), $a = 1$ (red) and $a = 3$ (blue).

Incentive compatibility constraint implies that $\forall t : a_{t+1} = 1 - a_t$, therefore

$$\phi'_{t-1}(z) = \frac{k - u_1(\phi_t(1-z), \phi_{t-1}(z)) - \beta u_2(\phi_{t+1}(z), \phi_t(1-z))}{u_{21}(\phi_t(1-z), \phi_{t-1}(z))z}. \tag{13}$$

Example 19 *Consider an example with quadratic utility function*

$$u(y, x) = \frac{a}{2}x^2 + \frac{b}{2}y^2 + gxy + dx + ey.$$

Equation (13) for this example can be rewritten as

$$\phi'_{t-1}(z) = \frac{k - (b + \beta a)\phi_t(1-z) - g\phi_{t-1}(z) - \beta g\phi_{t+1}(z) - e - \beta d}{gz}.$$

In steady state, $\phi_t = \phi_{t+1}$. Let us look for a steady state solution in a linear form: $\phi(z) = \gamma z$ (recall that the lower bound on the distribution of consumption is zero). Plugging this function into the differential equation we get

$$\gamma(g - b - \beta a + g + \beta g)z = k - (b + \beta a)\gamma - e - \beta d.$$

Since this equation has to hold for all $z \in [0, 1]$ and $\gamma > 0$, we get

$$g - b - \beta a + g + \beta g = 0$$
$$k - (b + \beta a)\gamma - e - \beta d = 0$$

or

$$(2+\beta)g = b+\beta a$$
$$\gamma = \frac{k-e-\beta d}{b+\beta a}$$

If the first condition is satisfied, the steady state distribution exists. It is uniform with the support $\left[0, \frac{k-e-\beta d}{b+\beta a}\right]$.

Proposition 20 *If* $(2+\beta)g \neq b+\beta a$*, there exists no equilibrium with a continuous distribution of consumption.*

Proof. *Consider equation (13) which is a necessary condition for an equilibrium*

$$gz\phi'_{t-1}(z) = k-(b+\beta a)\phi_t(1-z) - g\phi_{t-1}(z) - \beta g\phi_{t+1}(z) - e - \beta d.$$

Notice that $\phi_t(0) = 0$ *and* $\phi_t(1) = \frac{k-e-\beta d}{b+\beta a}$*. Consider a change of variables* $\Phi_t(z) = \frac{b+\beta a}{k-e-\beta d}\phi_t(z)$*. The function* Φ_t *must be increasing, and must satisfy* $\Phi_t(0) = 0$ *and* $\Phi_t(1) = 1$ *which makes it a c.d.f.*

$$z\Phi'_{t-1}(z) = \frac{b+\beta a}{g} - \frac{b+\beta a}{g}\Phi_t(1-z) - \Phi_{t-1}(z) - \beta\Phi_{t+1}(z). \tag{14}$$

Let

$$\mu_t = \int_0^1 z\Phi'_t(z)dz$$

and notice that

$$\int_0^1 \Phi_t(z)dz = 1-\mu_t.$$

Integrating both sides of the equation (14) we get

$$\mu_{t-1} = \frac{b+\beta a}{g} - \frac{b+\beta a}{g}(1-\mu_t) - (1-\mu_{t-1}) - \beta(1-\mu_{t+1})$$

or

$$\mu_t = \frac{1+\beta}{\beta} - \frac{b+\beta a}{\beta g}\mu_{t-1}$$

The solution to this equation is

$$\mu_t = \frac{g(1+\beta)}{b+\beta a+\beta g} + C\left(-\frac{b+\beta a}{\beta g}\right)^t$$

where C is a constant that is determined by the initial conditions. Note that if $C = 0$ we have a stationary solution and we already know that it exists only when $(2 + \beta)g = b + \beta a$. Therefore we need to consider the case when $C \neq 0$. Notice that concavity of the utility function implies that $\frac{b+\beta a}{\beta g} > 1$, hence μ_t is unbounded for any $C \neq 0$, which is impossible because μ_t is an expectation of a random variable with the support $[0, 1]$. ■

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