Failure of Common Knowledge of Language in Common-Interest Communication Games*

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Abstract

This paper explores the fault line that separates communication failure from success when message meaning is compromised because there is mutual but not common knowledge of which messages are available. If the size of the message space is just sufficient to convey all payoff-relevant private information, there may be communication failure at every finite knowledge order. In contrast, using a *language state space* to summarize players' information about the messages available to the sender, for any language state at which a *rich language* condition is satisfied – the sender has second-order knowledge (belief) of an event in the language state space that is small relative to the set of available messages – there exists an equilibrium that is *ex post* optimal at that language state.

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1 Introduction

The tools available to interlocutors matter for their ability to share information. Lack of familiarity with an organization's code is an obvious impediment to communication for a newcomer to the organization, as is inability to speak the local language for a traveler and, more abstractly, having only access to a message space that is small relative to the set of informational states.

On top of limited message availability itself, uncertainty about message availability can reduce the effectiveness of communication. This results from the signal extraction problem a receiver faces who has to contend with messages conveying not only payoff-relevant information but also information about the sender's message availability. Even if the speaker has access to a message space that would be adequate for conveying all payoff-relevant private information, the listener's uncertainty about whether this is the case may lead him to discount the significance of any given message.

In this paper I ask whether these observations extend to higher-order uncertainty. Specifically, can there be a substantial payoff loss from lack of common knowledge of which messages are available to the sender when both sender and receiver either know which set of messages is available to the sender or attach high probability to it?

To shed light on this question I study a class of communication games in which a sender with private payoff-relevant information sends a message to an uninformed receiver before both simultaneously take actions that determine their payoffs as a function of the payoff state. Communication is cheap talk in that messages do not affect payoffs directly. I focus on common-interest games, so that restrictions on message availability and (higher-order) uncertainty about that availability are the only impediments to efficient communication. The set of messages available to the sender is the sender's private information. In addition the receiver has information about which messages may be available to the sender, the sender has information about the receiver's information about message availability and so on. This is captured by an information structure that summarizes players' higher-order knowledge and beliefs about which messages are available to the sender.

I first construct an example in which all equilibria exhibit persistent communication failure at every knowledge order. Second, using a *language state space* to summarize players' information about the messages available to the sender, I show that despite the example, for any language state at which a *rich language* condition is satisfied – the sender has second-order knowledge (belief) of an event in the language state space that is small relative to the set of available messages – there exists an equilibrium that is *ex post* optimal at that

language state.

Finally I show that if one adopts an *ex ante* perspective, first-order knowledge (belief) suffices: if there is high probability of the receiver knowing (believing) which messages are available to the the sender, the expected payoff loss from a failure of common knowledge of which messages are available to the sender is negligible.

There is an extensive literature, starting with the seminal work of Crawford and Sobel [7] on how conflict of interest constrains communication. Uncertainty about whether there is conflict of interest has been studied by Morris [21], Morgan and Stocken [20] and Li and Madarász [17]. Pei [22] considers higher-order uncertainty about incentive alignment. One takeaway from this literature is that full revelation may be impossible, even if interests are aligned. None of these papers considers the case that it is common knowledge from the outset that preferences are fully aligned.

Impediments to communication when there is common knowledge of common interest have only more recently attracted attention: Jäger, Metzger and Riedel [13] investigate the structure of equilibria in common-interest sender-receiver games in which only a finite set of messages is available to the sender. They are interested in the geometry of equilibria in multi-dimensional state spaces and assume that the set of available messages is commonly known. Blume and Board [3] introduce uncertainty about message availability. Optimal equilibria in Blume and Board exhibit indeterminacy of meaning: the receiver faces a signal extraction problem with messages simultaneously conveying, and thus confounding, payoff-relevant information and information about the sender's access to messages. As a result available messages will generally not be used as they would be if message availability were common knowledge. Blume and Board do not consider higher-order uncertainty about message availability.

Lewis [15] introduces common knowledge, when considering language as a convention. He uses equilibria in common-interest communication games to illustrate conventional meaning and emphasizes common knowledge as a necessary attribute for a convention to be sustained. The underlying concern about the role of higher-order knowledge and belief in securing successful communication is also echoed in Grice's [12] mentioning of higher-order knowledge ("he knows (and knows that I know that he knows)") of "conventional meaning of the words," "identity of any references" and "other items of background knowledge" as among the prerequisites for participants in a conversation to make pragmatic inferences (and thus deduce implied meaning from literal meaning), as well as in Clark and Marshall's [5], [6] emphasis on the necessity of "mutual knowledge" for definite reference (the identification

of some individual object).[1] Due to the informal nature of Lewis' discussion, it is not clear what exactly he wants to be the object of (common) knowledge, which would be necessary to investigate the consequences of departures from common knowledge. In the present paper the object of player's knowledge and belief is the set of messages that are available to the sender.

The formal treatment of the robustness of equilibria to common knowledge failures originates with Rubinstein [26]. He shows by example that finite-order knowledge, regardless of the knowledge order, has very different implications than common knowledge, with potentially dramatic efficiency consequences in coordination games. Recently, Weinstein and Yildiz [30] have obtained powerful general results, showing that taking an interim perspective and using small perturbations of payoffs and higher-order beliefs any rationalizable outcome of a game can be transformed into the unique rationalizable outcome of a nearby game. Monderer and Samet [19] put forward a different way of thinking about approximate common knowledge: a player p-believes an event if she believes it with at least probability p; players have common p-belief of an an event if they all p-believe it, they all p-believe that they all p-believe it, and so on. They show that, unlike with finite-knowledge order approximations, when there is high probability that players have common p-belief of the game being played ϵ-optimizers can approximate behavior of players who have common knowledge of the game. Shin and Williamson [28] show that absence of common knowledge may severely constrain equilibrium behavior even if players have common p-belief for large p; specifically they conclude that common knowledge failures rule out complex conventions. Kajii and Morris [14] establish a connection between the *ex ante* probability of an event and the probability that it is common p-belief and use it to give sufficient conditions for the robustness of equilibria to common knowledge failures. My result that concerns the *ex ante* perspective is in the same spirit; the proof is slightly more elementary than that of Kajii and Morris, relying on simple properties of common interest games. None of the above papers directly address strategic communication.

Most closely related to the present paper is Giovannoni and Xiong [11]. They (independently) show that if language types have access to sufficiently many messages, private information about language competence does not degrade efficiency. They look at a considerably larger class of games, with more players and language constraints that affect both the ability to send and the ability to interpret messages. They are not concerned with higher-order uncertainty and require all language types to have access to sufficiently many

[1] Clark and Marshall adopted Schiffer's [27] terminology of referring to common knowledge as "mutual knowledge."

messages.

In the next section I introduce the model and solution concept. Section 3 provides the example with communication failure at every finite knowledge order. Section 4 considers the circumstances under which there are equilibria that achieve *ex post* optimality at a language state. In Section 5 I give a sufficient condition for equilibria to approximate *ex ante* optimal language use. Section 6 summarizes results, discusses modeling choices and makes a few speculative connections to the philosophy of language.

2 The Setup

I consider common-interest communication games in which a privately informed sender, S, communicates with a receiver, R, by sending a message $m \in M$ from a finite message space M. Following the sender's message both sender and receiver simultaneously choose actions $a_S \in A_S$ and $a_R \in A_R$, where the action space of the sender, A_S, and the receiver, A_R, are both finite. The common payoff $U(a_S, a_R, t)$ depends on the sender's action $a_S \in A_S$, the receiver's action $a_R \in A_R$, and the sender's payoff-relevant private information $t \in T$, her *payoff type*; the sender's message has no direct impact on players' payoffs. The sender's payoff type t is drawn from a commonly known distribution π on the finite payoff state space T, with $\pi(t) > 0$ for all $t \in T$.

To express the idea that the sender may be language constrained and that there may be uncertainty about which constraint exactly the sender is facing, I follow Blume and Board [3] by introducing *language types*. The set of language types is $\Lambda = 2^M \setminus \emptyset$, the collection of all non-empty subsets of the message space M. A sender with language type $\lambda \in \Lambda$ is constrained only to send messages $m \in \lambda \subset M$.

To capture higher-uncertainty about language constraints, I use *information structures* $\mathcal{I} = (\Omega, \mathcal{L}, \mathcal{O}^S, \mathcal{O}^R, q)$, where $\Omega = \{\omega_1, \omega_2, \ldots\}$ is a countable *language state space*; $\mathcal{L} : \Omega \to \Lambda$ specifies the set of messages available to the sender at each language state (her *language type*); $\mathcal{O}^S$ is a partition of Ω, the sender's information partition; $\mathcal{O}^R$ is the receiver's information partition; and, q is the (common) prior on Ω and satisfies $\sum_{\omega' \in \mathcal{O}^i(\omega)} q(\omega') > 0$ for all $\omega \in \Omega$ and $i = R, S$.

It will be convenient to let $\mathcal{L}(\omega) = \lambda_\omega$ and $q(\omega) = q_\omega$. The information partitions express the players' knowledge and beliefs about the sender's language constraints: at language state ω, the sender knows that the true language state is in $\mathcal{O}^S(\omega)$ but no more (where $\mathcal{O}^S(\omega)$ is the element of $\mathcal{O}^S$ containing ω); and similarly for the receiver. Player i's belief conditional on the information $\mathcal{O}^i(\omega), i = S, R$ corresponds to the prior q concentrated on the set $\mathcal{O}^i(\omega)$.

I adopt the consistency condition that the sender cannot make additional inferences from her own language type: if $\omega' \in \mathcal{O}^S(\omega)$, then $\lambda_\omega = \lambda_{\omega'}$.

Information structures encode uncertainty only about the sender's language type, not about the payoff-relevant information t (the sender's *payoff type*). I assume that the distribution π from which t is drawn is independent of q. Given that the sender is fully informed about t, and the receiver knows nothing about t, it would be straightforward to extend the partitions and common prior over the full space of uncertainty, $T \times \Omega$, but to do so would unnecessarily complicate the notation.

A player has 1st-order knowledge of language whenever he knows the sender's language type, and he has nth-order knowledge of language whenever he knows that the other player has $n-1$th-order knowledge of language. The *fundamentals* of the communication game are given by $\mathcal{F} = (\{S, R\}, \{A_S, A_R\}, U, T, \pi, M)$ and a communication game $\mathcal{G} = (\mathcal{F}, \mathcal{I})$ is a pair of fundamentals and an information structure.

A strategy for the receiver,

$$\rho_R : M \times \Omega \to \Delta(A_R),$$

maps pairs of messages and language states to distributions over receiver actions. Receiver strategies must be measurable with respect to the receiver's information, i.e. $\rho_R(m, \omega) = \rho_R(m, \omega')$ for all $m \in M$, all $\omega \in \Omega$ and all $\omega' \in \mathcal{O}^R(\omega)$. A strategy of the sender is a pair (σ_S, ρ_S) consisting of a signaling rule and an action rule. The signaling rule

$$\sigma_S : T \times \Omega \to \Delta(M)$$

maps pairs of payoff states $t \in T$ and language states $\omega \in \Omega$ to distributions over messages. The action rule,

$$\rho_S : T \times M \times \Omega \to \Delta(A_S),$$

maps triples of payoff states, messages and language states to distributions over sender actions. Both rules have to be measurable with respect to the sender's information, i.e. $\sigma_S(t, \omega) = \sigma_S(t, \omega')$ and $\rho_S(t, m, \omega) = \rho_S(t, m, \omega')$ for all $t \in T$, all $m \in M$, all $\omega \in \Omega$ and all $\omega' \in \mathcal{O}^S(\omega)$. The sender's signaling rule must satisfy the sender's language constraint, i.e. $\sigma(t, \omega) \in \Delta(\lambda_\omega)$ for all $\omega \in \Omega$. When dealing with pure strategies, it will be convenient to slightly abuse notation and let strategies be maps into actions rather than distributions over actions, as in $\rho_R : M \times \Omega \to A_R$, $\sigma_S : T \times \Omega \to M$ and $\rho_S : T \times M \times \Omega \to A_S$.

A (perfect Bayesian) equilibrium $(\sigma_S, \rho_S, \rho_R, \beta_S, \beta_R)$ of the communication game consists

of a sender strategy (σ_S, ρ_S), a receiver strategy ρ_R, a belief function

$$\beta_S : \Omega \to \Delta(\Omega)$$

for the sender and a belief function

$$\beta_R : M \times \Omega \to \Delta(T \times \Omega)$$

for the receiver such that if $a_R \in \text{supp}[\rho_R(m, \omega)]$ then

$$a_R \in \arg\max_{a'_R \in A_R} \sum_{a_S \in A_S} \sum_{t \in T} \sum_{\omega' \in \mathcal{O}^R(\omega)} U(a_S, a'_R, t)\rho_S(a_S|t, m, \omega')\beta_R((t, \omega')|m, \mathcal{O}^R(\omega)), \quad (1)$$

for all $m \in M$ and $\omega \in \Omega$; if $(m, a_S) \in \text{supp}[(\sigma_S(t, \omega), \rho_S(t, m, \omega))]$ then

$$(m, a_S) \in \arg\max_{\substack{(m', a'_S) \in \\ \lambda_\omega \times A_S}} \sum_{a_R \in A_R} \sum_{\omega' \in \mathcal{O}^S(\omega)} U(a'_S, a_R, t)\rho_R(a_R|m', \omega')\beta_S(\omega'|\mathcal{O}^S(\omega)) \quad (2)$$

for all $m \in M$ and $\omega \in \Omega$;

$$\beta_S(\tilde{\omega}|\mathcal{O}^S(\omega)) = \frac{q_{\tilde{\omega}}}{\sum_{\omega' \in \mathcal{O}^S(\omega)} q_{\omega'}} \quad (3)$$

for all $\omega \in \Omega$ and all $\tilde{\omega} \in \mathcal{O}^S(\omega)$; and

$$\beta_R((\tilde{t}, \tilde{\omega})|m, \mathcal{O}^R(\omega)) = \frac{\sigma_S(m|\tilde{t}, \tilde{\omega})\pi(\tilde{t})q(\tilde{\omega})}{\sum_{t' \in T, \omega' \in \mathcal{O}^R(\omega)} \sigma_S(m|t', \omega')\pi(t')q(\omega')} \quad (4)$$

for all $\omega \in \Omega$, all $\tilde{\omega} \in \mathcal{O}^R(\omega)$, all $t \in T$ and all $m \in M$ whenever the denominator on the right-hand side is positive.

A strategy profile $(\sigma_S, \rho_S, \rho_R)$ is *optimal* if it maximizes the *ex ante* expected payoff $\mathbb{E}_{t,\omega}[U|\sigma'_S, \rho'_S, \rho'_R] :=$

$$\sum_{a_S \in A_S} \sum_{a_R \in A_R} \sum_{t \in T} \sum_{m \in \lambda_\omega} \sum_{\omega \in \Omega} U(a_S, a_R, t)\rho'_S(a_S|t, m, \omega)\rho'_R(a_R|m, \omega)\sigma'_S(m|t, \omega)\pi(t)q(\omega),$$

among all strategy profiles $(\sigma'_S, \rho'_S, \rho'_R)$ that satisfy the sender's language constraints at each ω and the measurability restrictions on sender and receiver strategies. Existence of an optimal strategy follows from standard compactness arguments. Evidently, for any strategy profile $(\sigma'_S, \rho'_S, \rho'_R)$ there exists a profile $(\sigma'_S, \rho'_S, \hat{\rho}_R)$ with at least the same *ex ante* expected payoff and in which $\hat{\rho}_R$ is a pure strategy of the receiver that satisfies the receiver's measurability

restrictions, and similarly for the sender. Therefore there exists an optimal strategy profile that is in pure strategies. Since players have identical payoffs, any deviation that raises a player's *ex ante* expected payoff automatically raises their joint expected payoff. This implies that an optimal strategy profile is a Bayesian Nash equilibrium strategy. If necessary, such a profile can be transformed into a Perfect Bayesian Equilibrium profile by simply replacing all receiver responses after messages that he expects to receive with probability zero given his information by responses to (arbitrary) messages that he expects to receive with positive probability given his information; evidently the altered strategy yields the same *ex ante* expected payoff for the receiver and therefore for both players, and the sender cannot have a profitable deviation because this would violate optimality. This is summarized in the following observation.

Observation 1 *There exists an* ex ante *optimal strategy profile that is in pure strategies and is a Perfect Bayesian Equilibrium profile.*

For each t let $a^t = (a^t_S, a^t_R) \in \arg\max_{(a_S,a_R)} U(a_S, a_R, t)$. A strategy profile is *ex post optimal at language state* ω if at that language state it achieves the payoff $U(a^t, t)$ for every payoff state $t \in T$.

Without the measurability constraints, an *ex ante* optimal profile would have to be *ex post* optimal at all language states ω that have positive probability under the prior. With the measurability constraints, this is no longer the case; if the receiver is uncertain about the sender's language type he may not be able to best respond to all language types in the support of his beliefs and may have to compromise. As noted in Blume and Board [3] indeterminacies of meaning may arise due to the fact that the receiver cannot distinguish messages sent by an articulate sender, who had other options, from messages sent by an inarticulate sender who had no (or few) alternatives. One of the objectives of this paper is to identify sufficient conditions on the information structure that make it possible that for a given language state one can find a PBE that is *ex post* optimal at that language state.

3 An example - *Opportunity*

Consider a common-interest communication game, which may be called *Opportunity*: In *Opportunity* one player, the receiver, has a safe action and the other, the sender, has a unique best reply to that action. There is an opportunity to improve on this action profile through a coordinated move by both players. While the existence of this opportunity is common knowledge between the players, only the sender knows the exact nature of the

opportunity. Absent language constraints (or uncertainty about such constraints), the sender can inform the receiver of the opportunity and they can both take advantage of it. With language constraints the pursuit of the opportunity is risky and I will show that higher-order uncertainty about message availability can be an impediment to a successful pursuit of the opportunity.

Formally, there are two equally likely payoff states t_1 and t_2 with the common payoffs given in the two tables below. The receiver actions are a_R, b_R and c_R and the sender actions a_S, b_S and c_S. The receiver's safe action is c_R, independent of the payoff state, and the sender's unique best reply to the receiver's safe action is c_S, also independent of the payoff state. The common payoff from that action profile is 5, which is dominated by the opportunity to achieve a common payoff of 6 via the action profile (a_S, a_R) in payoff state t_1 and via the action profile (b_S, b_R) in payoff state t_2. The message space is $M = \{m_1, \ldots, m_{|M|}\}$. Let $|M| \geq 2$, so that

	a_S	b_S	c_S
a_R	6	0	0
b_R	0	0	0
c_R	4	4	5

t_1

	a_S	b_S	c_S
a_R	0	0	0
b_R	0	6	0
c_R	4	4	5

t_2

Figure 1: Payoff States; $\text{Prob}(t_1) = \text{Prob}(t_2) = \frac{1}{2}$

in principle it is possible to communicate the payoff state. In this example I want to explore the consequences of allowing for the possibility that the sender is constrained to send only one of the messages in the message space. Whenever the sender has only access to one of the messages, I say that the message is 'compromised'. All messages will be compromised with positive probability. I am interested in situations where both parties know that messages are not compromised but this fact is not common knowledge. To this end, consider the following information structure: Players' knowledge and beliefs about the language constraints the sender is facing are encoded in the language state space $\Omega = \{\omega_1, \omega_2, \ldots\}$. For each message in M there is exactly one state at which that message is compromised. In addition, there are K states at which all message in M are available, where K may be infinite. The information structure is chosen so that whenever a message is compromised the receiver does not know that this is the case. Whenever all messages are available, there is at most finite-order

knowledge of this fact. The players' information partitions are given by

$$\begin{aligned}
&\text{Sender}: \mathcal{O}^S = \{\{\omega_1\}, \dots, \{\omega_{|M|}\}, \{\omega_{|M|+1}, \omega_{|M|+2}\}, \dots, \{\omega_{|M|+K-1}, \omega_{|M|+K}\}\} \\
&\text{Receiver}: \mathcal{O}^R = \{\{\omega_1, \dots, \omega_{|M|+1}\}, \{\omega_{|M|+2}, \omega_{|M|+3}\}, \dots, \{\omega_{|M|+K-2}, \omega_{|M|+K-1}\}, \{\omega_{|M|+K}\}\}.
\end{aligned}$$

Whenever it is not a singleton, the typical information set of the sender is of the form $\{\omega_{|M|+n}, \omega_{|M|+n+1}\}$, where n is odd. For the receiver, the typical information set is of the form $\{\omega_{|M|+n}, \omega_{|M|+n+1}\}$, with n even, with the exceptions of $\{\omega_1, \dots, \omega_{|M|+1}\}$ and $\{\omega_{|M|+K}\}$. Assume that at ω_n, $n = 1, \dots, |M|$, the sender's language type is $\lambda_{\omega_n} = \{m_n\}$, (i.e. the sender has only message m_n available) and that at language states ω_n with $n > |M|$ the sender's language type is $\lambda_n = M$, (i.e. the sender has all messages in M available). The common prior, q, is given by $q(\omega_n) = \frac{1-\eta}{|M|(1-\eta^{K+1})}$ for $n = 1, \dots, |M|$ and $q(\omega_{|M|+k}) = \frac{(1-\eta)\eta^k}{1-\eta^{K+1}}$ for $k = 1, \dots, K$, with $\eta \in (0, 1)$.

I will consider three versions of the example, one to explore the effect of having a large message space, $|M|$, one to examine the effect of only having a high (finite) knowledge order, K, and one where both of these are combined.

Version 1, $K = 2, |M| \geq 2, \eta < \min\left\{\frac{1}{6}, \frac{1}{|M|}\right\}$: In this version of *Opportunity* at state $\omega_{|M|+2}$ the receiver has first-order knowledge of the sender having all messages available, and thus it is mutual knowledge at $\omega_{|M|+2}$ that all messages are available. Nevertheless, there is no equilibrium with efficient message use at $\omega_{|M|+2}$.

In order to be willing to take either action a_R or b_R the receiver must assign at least probability 2/3 to one of the two payoff states t_1 or t_2. The sender's strategy at $\mathcal{O}^S(\omega_{|M|+1})$ is maximally informative about the payoff state if there are messages, $m_1, m_2 \in M$ such that $\sigma_S(m_1|t_1, \mathcal{O}^S(\omega_{|M|+1})) = 1$ and $\sigma_S(m_2|t_2, \mathcal{O}^S(\omega_{|M|+1})) = 1$. Hence, if the sender uses a maximally informative strategy at $\mathcal{O}^S(\omega_{|M|+1})$ the receiver's posterior probability of the sender's payoff type being t_1 at $\mathcal{O}^R(\omega_1)$ following message m_1 is

$$\frac{(q(\omega_1) + q(\omega_{|M|+1}))\pi(t_1)}{(q(\omega_1) + q(\omega_{|M|+1}))\pi(t_1) + q(\omega_1)\pi(t_2)} = \frac{\frac{1}{|M|} + \eta}{2\frac{1}{|M|} + \eta}.$$

Therefore, as long as $\eta < \frac{1}{|M|}$, even if the sender uses a maximally informative strategy at $\mathcal{O}^S(\omega_{|M|+1})$ the receiver is unwilling to take actions a_R or b_R at $\mathcal{O}^R(\omega_1)$ after any sender message. For $\eta < \frac{1}{|M|}$, irrespective of the sender's strategy, it is uniquely optimal for the receiver to respond to every message received at $\mathcal{O}^R(\omega_1)$ with the action c_R. This in turn implies that at her information set $\mathcal{O}^S(\omega_{|M|+1})$, regardless of the message she sends, the

sender expects the receiver to take action c_R with at least probability $\frac{1}{1+\eta}$. Hence, the sender's payoff from sending any message and then taking action c_S at $\mathcal{O}^S(\omega_{|M|+1})$ is at least $\frac{1}{1+\eta}5$ whereas her payoff from any other pure strategy is at most $\frac{1}{1+\eta}4 + \frac{\eta}{1+\eta}6$. The former exceeds the latter as long as $\eta < \frac{1}{6}$. Hence at $\mathcal{O}^S(\omega_{|M|+1})$, after every message that the sender sends with positive probability, it is uniquely optimal for her to take action c_S. Therefore, at $\mathcal{O}^R(\omega_{|M|+2})$ and after any message that is sent with positive probability in equilibrium the receiver knows that the sender will take action c_S and it is uniquely optimal to respond with action c_R. Thus, for $\eta < \min\{\frac{1}{6}, \frac{1}{|M|}\}$, in every equilibrium, the unique common payoff at language state $\omega_{|M|+2}$ is 5, whereas if it were common knowledge that the receiver has all messages available, there is an equilibrium with common payoff 6 at that language state. There is no equilibrium in which communication would achieve *ex post* optimality at $\omega_{|M|+2}$, despite the fact that players have mutual knowledge of the availability of all messages at that language state.[2]

Version 2, $K = \infty, |M| = 2, \eta < \frac{1}{6}$: In this version of *Opportunity* the size of the message space is just enough to communicate the payoff state when all messages are available. Nevertheless, in every equilibrium payoffs are bounded away from the maximum for every finite order of knowledge that all messages are available.

Proposition 1 *In every equilibrium of Opportunity with $K = \infty$, $|M| = 2$, and $\eta < \frac{1}{6}$ at each of his information sets the receiver uses pure responses following messages that he receives with positive probability at that information set and at each of the sender's information sets either*

1. *t_1 sends a distinct message and takes action a_S and t_2 sends a distinct message and takes action c_S (Case 1), or*

2. *t_1 sends a distinct message and takes action c_S and t_2 sends a distinct message and takes action b_S (Case 2), or*

3. *the sender sends both messages and takes action c_S after each message (Case 3), or*

4. *the sender sends only one message and takes action c_S after that message (Case 4).*

[2]For the sake of simplicity in this example language constraints take a stark form; each message sometimes is the only available message, in which case the sender must send that message. Silence is not an option. Is show in the appendix that the conclusion of Version 1 of *Opportunity*, that it may be impossible to achieve *ex post* optimality even with mutual knowledge that all message are available, continues to hold when the sender has the option to remain silent, where silence is interpreted as a message that is always available.

Proof: The claim trivially holds for the sender at $\mathcal{O}^S(\omega_1)$ and $\mathcal{O}^S(\omega_2)$. I will now establish the claim for the receiver at $\mathcal{O}^R(\omega_3)$ and the sender at $\mathcal{O}^S(\omega_3)$.

In order to be willing to take either action a_R or b_R the receiver must assign at least probability 2/3 to one of the two payoff states t_1 or t_2. The sender's strategy at $\mathcal{O}^S(\omega_3)$ is maximally informative about the payoff state if, without loss of generality, $\sigma_S(m_1|t_1, \mathcal{O}^S(\omega_3)) = 1$ and $\sigma_S(m_2|t_2, \mathcal{O}^S(\omega_3)) = 1$. If the sender uses a maximally informative strategy at $\mathcal{O}^S(\omega_3)$ the receiver's posterior at $\mathcal{O}^R(\omega_3)$ following message m_1 is

$$\frac{(q(\omega_1) + q(\omega_3))\pi(t_1)}{(q(\omega_1) + q(\omega_3))\pi(t_1) + q(\omega_1)\pi(t_2)} = \frac{\frac{1}{2} + \eta}{1 + \eta}.$$

Therefore, as long as $\eta < \frac{1}{2}$, even if the sender uses a maximally informative strategy at $\mathcal{O}^S(\omega_3)$ the receiver is unwilling to take actions a_R or b_R at $\mathcal{O}^R(\omega_3)$ after any sender message: for $\eta < \frac{1}{2}$, irrespective of the sender's strategy, it is uniquely optimal for the receiver at $\mathcal{O}^R(\omega_3)$ to respond to every message received with the action c_R. This in turn implies that at her information set $\mathcal{O}^S(\omega_3)$, regardless of the message she sends, the sender expects the receiver to take action c_R with at least probability $\frac{1}{1+\eta}$. Hence, the sender's payoff from sending either message and then taking action c_S at $\mathcal{O}^S(\omega_3)$ is at least $\frac{1}{1+\eta}5$ whereas her payoff from any other pure strategy is at most $\frac{1}{1+\eta}4 + \frac{\eta}{1+\eta}6$. The former exceeds the latter as long as $\eta < \frac{1}{6}$. Hence at $\mathcal{O}^S(\omega_3)$, after every message that the sender sends with positive probability, it is uniquely optimal for her to take action c_S. This shows that the claim holds for all $\mathcal{O}^R(\omega_j)$ and $\mathcal{O}^S(\omega_j)$ with $j \leq 3$. I will prove by induction that the claim holds for all $j \geq 1$.

Assuming that the claim holds for all $j = 0, \ldots, k$ at information sets $\mathcal{O}^S(\omega_{3+2j})$ of the sender and information sets $\mathcal{O}^R(\omega_{3+2j})$ of the receiver, I will show that it also holds at $\mathcal{O}^S(\omega_{3+2(k+1)})$ for the sender and at $\mathcal{O}^R(\omega_{3+2(k+1)})$ for the receiver. More specifically, I will argue that if behavior at information set $\mathcal{O}^S(\omega_{3+2k})$ of the sender and information set $\mathcal{O}^R(\omega_{3+2k})$ of the receiver conforms with one of the cases in the statement of the proposition, then it follows that behavior at $\mathcal{O}^S(\omega_{3+2(k+1)})$ and $\mathcal{O}^R(\omega_{3+2(k+1)})$ also conforms with a (possibly different) case in the statement of the proposition:

First consider Case 1 (and Case 2, modulo changes in notation), i.e. suppose that at $\mathcal{O}^S(\omega_{3+2k})$, without loss of generality, t_1 sends message m_1 and takes action a_S and t_2 sends message m_2 and takes action c_S and at $\mathcal{O}^R(\omega_{3+2k})$ the receiver does not mix in response to messages he receives with positive probability. Then at $\mathcal{O}^R(\omega_{3+2(k+1)})$ the posterior proba-

bility of state $\omega_{2+2(k+1)}$ after receiving message m_1 is

$$P(\omega_{2+2(k+1)}|m_1) = \frac{P(m_1|\omega_{2+2(k+1)})}{P(m_1|\omega_{2+2(k+1)}) + P(m_1|\omega_{3+2(k+1)})\eta} \geq \frac{1}{1+2\eta}.^3$$

Hence the expected payoff to the receiver from taking action a_R in response to message m_1 at $\mathcal{O}^R(\omega_{3+2(k+1)})$ is at least $\frac{1}{1+2\eta}6$, the expected payoff from b_R is no more than $\frac{2\eta}{1+2\eta}6$ and the expected payoff from c_R is no more than $\frac{1}{1+2\eta}4 + \frac{2\eta}{1+2\eta}6$. This implies that a_R is the unique optimal response to m_1 as long as $\eta < \frac{1}{6}$. Similarly, at $\mathcal{O}^R(\omega_{3+2(k+1)})$ the posterior probability of state $\omega_{2+2(k+1)}$ after receiving message m_2 is also greater or equal to $\frac{1}{1+2\eta}$. Hence the expected payoff to the receiver from taking action c_R in response to message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$ is at least 4 and the expected payoff from either b_R or a_R is no more than $\frac{2\eta}{1+2\eta}6$. This implies that c_R is the unique optimal response to m_2 for all $\eta \in (0,1)$. At her information set $\mathcal{O}^S(\omega_{3+2(k+1)})$ the sender assigns probability $\frac{1}{1+\eta}$ to state $\omega_{3+2(k+1)}$ and probability $\frac{\eta}{1+\eta}$ to state $\omega_{4+2(k+1)}$. Therefore if her payoff type is t_1 her payoff from sending m_1 and taking action a_S is at least $\frac{1}{1+\eta}6$, which is greater than 5 as long as $\eta < \frac{1}{5}$. Hence at $\mathcal{O}^S(\omega_{3+2(k+1)})$ it is uniquely optimal for payoff type t_1 of the sender to send message m_1. If instead the sender's payoff type is t_2 her payoff from sending message m_2 and taking action c_S is at least $\frac{1}{1+\eta}5$ whereas her payoff from any other pure strategy is at most $\frac{1}{1+\eta}4 + \frac{\eta}{1+\eta}6$. Hence, at $\mathcal{O}^S(\omega_{3+2(k+1)})$ it is uniquely optimal for payoff type t_2 to send message m_2 and take action c_S, as long as $\eta < \frac{1}{6}$. Up to changes in notation, the same reasoning applies in *Case 2.* In summary, if players behave according to *Case 1 (Case 2)* at their information sets $\mathcal{O}^S(\omega_{3+2k})$ and $\mathcal{O}^R(\omega_{3+2k})$ respectively, then they also do so at their information sets $\mathcal{O}^S(\omega_{3+2(k+1)})$ and $\mathcal{O}^R(\omega_{3+2(k+1)})$.

Next, consider Case 3, i.e. at her information set $\mathcal{O}^S(\omega_{3+2k})$ the sender sends both messages with positive probability and takes action c_S after each message and at his information set $\mathcal{O}^R(\omega_{3+2k})$ the receiver does no mix after either message. To save space, it will be useful to introduce the following notation: $p_{m_1|a_S,k+1}$ stands for the probability that the sender sends message m_1 conditional on taking action a_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$; $p_{a_S|k+1}$ is the probability of the sender taking action a_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$; $p_{m_1|\tilde{a}_S,k+1}$ stands for the probability that the sender sends message m_1 conditional on taking an action that is not a_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$; $p_{\tilde{a}_S|k+1}$ stands for the probability that the sender takes an action that is not a_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$; and, $p_{m_1|k}$ stands for the probability that the sender sends message m_1 at $\mathcal{O}^S(\omega_{3+2k})$.

Without loss of generality, let m_1 be sent with probability at least 1/2 at $\mathcal{O}^S(\omega_{3+2k})$, i.e.

[3] Note that each these probabilities is conditioning on $\mathcal{O}^R(\omega_{3+2(k+1)})$, which has been suppressed in the notation to save space.

$p_{m_1|k} \geq \frac{1}{2}$. The probability $P(a_S|m_1, \mathcal{O}^R(\omega_{3+2(k+1)}))$ that the receiver assigns to the sender taking action a_S after receiving message m_1 at his information set $\mathcal{O}^R(\omega_{3+2(k+1)})$ equals

$$\frac{p_{m_1|a_S,k+1}p_{a_S|k+1}q(\omega_{3+2(k+1)})}{p_{m_1|a_S,k+1}p_{a_S|k+1}q(\omega_{3+2(k+1)}) + p_{m_1|\tilde{a}_S,k+1}p_{\tilde{a}_S|k+1}q(\omega_{3+2(k+1)}) + p_{m_1|k}q(\omega_{2+2(k+1)})},$$

which is less or equal to $\frac{\eta}{\eta+\frac{1}{2}} < \frac{1}{2}$ for all $\eta < 1/2$. The same argument implies that it is also the case that $P(b_S|m_1, \mathcal{O}^R(\omega_{3+2(k+1)})) < \frac{1}{2}$. Since this is less than $\frac{2}{3}$, the probability that would be required for the receiver to consider an action other than c_R, the receiver will respond to m_1 with c_R at $\mathcal{O}^R(\omega_{3+2(k+1)})$.

By the induction hypothesis, the receiver at $\mathcal{O}^R(\omega_{3+2k})$ does not randomize. Since the sender at $\mathcal{O}^S(\omega_{3+2k})$ assigns probability $\frac{1}{1+\eta} > \frac{6}{7}$ (since $\eta < \frac{1}{6}$) to state ω_{3+2k}, it cannot be the case that the receiver takes one of the actions a_R or b_R in response to either message at $\mathcal{O}^R(\omega_{3+2k})$, since otherwise the sender's expected payoff for one of her payoff types from taking an action other than c_S would be at least $\frac{1}{1+\eta}6 > \frac{36}{7}$, which strictly exceeds the highest feasible payoff, 5, from action c_S. This implies that the receiver's response to message m_1 is c_R at both $\mathcal{O}^R(\omega_{3+2k})$ and $\mathcal{O}^R(\omega_{3+2(k+1})$ and hence the sender expects a payoff of 5 from sending m_1 at $\mathcal{O}^S(\omega_{3+2k})$. Combining that with the fact that at $\mathcal{O}^S(\omega_{3+2k})$ by the assumption of the case considered the sender takes action c_S after each message, it follows that since the sender also sends m_2 with positive probability at $\mathcal{O}^S(\omega_{3+2k})$ the receiver must respond with c_R to m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which also implies that the receiver is not randomzing after either message at $\mathcal{O}^R(\omega_{3+2(k+1)})$.

Then at her information set $\mathcal{O}^S(\omega_{3+2(k+1)})$, regardless of the message she sends, the sender expects the receiver to take action c_R with at least probability $\frac{1}{1+\eta}$. Hence, the sender's payoff from sending either message and then taking action c_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$ is at least $\frac{1}{1+\eta}5$ whereas her payoff from any other pure strategy is at most $\frac{1}{1+\eta}4 + \frac{\eta}{1+\eta}6$. The former exceeds the latter as long as $\eta < \frac{1}{6}$. Hence at $\mathcal{O}^S(\omega_{3+2(k+1)})$, after every message that the sender sends with positive probability, it is uniquely optimal for her to take action c_S. This, however, implies that at information sets $\mathcal{O}^S(\omega_{3+2(k+1)})$ of the sender and $\mathcal{O}^R(\omega_{3+2(k+1)})$ of the receiver players behave in accordance with either *Case 3* or *Case 4*.

The final case to consider is Case 4, i.e. at $\mathcal{O}^S(\omega_{3+2k})$ the sender sends a single message regardless of her payoff type t and takes action c_S after that message; and, at $\mathcal{O}^R(\omega_{3+2k})$ the receiver uses pure responses after messages he receives with positive probability. Without loss of generality let the single message sent at $\mathcal{O}^S(\omega_{3+2k})$ be m_1. When considering the previous case, I showed that following a message that is sent with probability at least $\frac{1}{2}$ at

$\mathcal{O}^S(\omega_{3+2k})$ and after which the sender takes action c_S at that information set, it is uniquely optimal for the receiver to respond with c_R at $\mathcal{O}^R(\omega_{3+2(k+1)})$ for $\eta < \frac{1}{2}$. This argument applies here to m_1.

I will consider subcases that are distinguished by the sender's strategy at $\mathcal{O}^S(\omega_{3+2(k+1)})$.

Subcase 4.1: If the sender exclusively sends message m_2 at $\mathcal{O}^S(\omega_{3+2(k+1)})$, then message m_2 is not informative about t at $\mathcal{O}^R(\omega_{3+2(k+1)})$ and therefore at that information set the receiver responds to m_2 with c_R. Since the sender assigns probability $\frac{1}{1+\eta}$ to state $\omega_{3+2(k+1)}$ at $\mathcal{O}^S(\omega_{3+2(k+1)})$, she expects the receiver to take action c_R with at least probability $\frac{1}{1+\eta}$ after sending message m_2 and therefore action c_S is uniquely optimal for the sender at $\mathcal{O}^S(\omega_{3+2(k+1)})$ after having sent message m_2. This implies that the sender behaves according to *Case 4* at $\mathcal{O}^S(\omega_{3+2(k+1)})$.

Subcase 4.2: If the sender exclusively sends message m_1 at $\mathcal{O}^S(\omega_{3+2(k+1)})$, she expects the receiver to take action c_R with probability at least $\frac{1}{1+\eta}$ following that message and since $\frac{1}{1+\eta}5 > \frac{1}{1+\eta}4 + \frac{\eta}{1+\eta}6$ for $\eta < \frac{1}{6}$, it is uniquely optimal for her to take action c_S following m_1 at her information set $\mathcal{O}^S(\omega_{3+2(k+1)})$. Hence, if the sender only sends message m_1 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ then her behavior at that information set is described by *Case 4*.

Subcase 4.3: Suppose that at $\mathcal{O}^S(\omega_{3+2(k+1)})$ the sender sends both messages with positive probability and that for both payoff types t the probability of sending message m_2 is at least $\alpha > \frac{1+\eta}{2}$. Then the highest value that the receiver's posterior probability of facing payoff type t_i after having observed message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$,

$$p_{t_i|m_2,k+1} = \frac{p_{m_2|t_i,k+1}q(\omega_{3+2(k+1)})}{p_{m_2|t_i,k+1}q(\omega_{3+2(k+1)}) + p_{m_2|t_{-i},k+1}q(\omega_{3+2(k+1)})}$$

can take is $\frac{1}{1+\alpha}$. Recall that for the receiver to willing to take an action other than c_R, he must assign at least probability $\frac{2}{3}$ to one of the payoff types of the sender. Hence, in order for the receiver to be willing to take an action other than c_R in response to m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$ it must be the case that $\frac{1}{1+\alpha} \geq \frac{2}{3}$. It follows that as long as $\alpha > \frac{1}{2}$, it is uniquely optimal for the receiver to take action c_R in response to receiving message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$. Similarly, the highest value that the receiver's posterior probability of facing payoff type t_i after having observed message m_2 at $\mathcal{O}^R(\omega_{3+2(k+2)})$,

$$p_{t_i|m_2,k+2} = \frac{p_{m_2|t_i,k+1}q(\omega_{4+2(k+1)})+p_{m_2|t_i,k+2}q(\omega_{3+2(k+2)})}{p_{m_2|t_i,k+1}q(\omega_{4+2(k+1)})+p_{m_2|t_i,k+2}q(\omega_{3+2(k+2)}))+p_{m_2|t_{-i},k+1}q(\omega_{4+2(k+1)})+p_{m_2|t_{-i},k+2}q(\omega_{3+2(k+2)})}$$

can take is $\frac{1+\eta}{1+\eta+\alpha}$. Therefore, for the receiver to be willing to take an action other than c_R at $\mathcal{O}^R(\omega_{3+2(k+2)})$ in response to message m_2 it must be the case that $\frac{1+\eta}{1+\eta+\alpha} \geq \frac{2}{3}$. It follows that as long as $\alpha > \frac{1+\eta}{2}$, it is uniquely optimal for the receiver to take action c_R in response

to receiving message m_2 at $\mathcal{O}^R(\omega_{3+2(k+2)})$. Thus, as long as at $\mathcal{O}^S(\omega_{3+2(k+1)})$ for both payoff types t the probability of sending message m_2 is greater than $\frac{1+\eta}{2}$, it is uniquely optimal for the sender to take action c_S after having sent message m_2 at her information set $\mathcal{O}^S(\omega_{3+2(k+1)})$ and therefore the behavior at information sets $\mathcal{O}^S(\omega_{3+2(k+1)})$ and $\mathcal{O}^R(\omega_{3+2(k+1)})$ conforms with *Case 3*.

Subcase 4.4: Suppose that at $\mathcal{O}^S(\omega_{3+2(k+1)})$ the sender sends both messages with positive probability and that at $\mathcal{O}^S(\omega_{3+2(k+1)})$ there is a payoff type t for whom the probability of sending message m_2 is less or equal to $\frac{1+\eta}{2}$. Then that payoff type will send message m_1 with at least probability $\frac{1-\eta}{2}$. I showed above that for $\eta < \frac{1}{2}$ the receiver has a unique best reply to m_1 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which is to take action c_R, and that given this behavior it is uniquely optimal for the sender to take action c_S following message m_1 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ as long as $\eta < \frac{1}{6}$. The receiver's posterior probability of the sender taking action c_S following message m_1 at $\mathcal{O}^R(\omega_{3+2(k+2)})$ satisfies $P(c_S|m_1, \mathcal{O}^R(\omega_{3+2(k+2)})) \geq P(\omega_{2+2(k+2)}|m_1, \mathcal{O}^R(\omega_{3+2(k+2)}))$, since at $\omega_{2+2(k+2)}$ the sender always takes action c_S when sending m_1. The probability $P(\omega_{2+2(k+2)}|m_1, \mathcal{O}^R(\omega_{3+2(k+2)}))$ equals

$$\frac{P(m_1|\omega_{2+2(k+2)})q(\omega_{2+2(k+2)})}{P(m_1|\omega_{2+2(k+2)})q(\omega_{2+2(k+2)}) + P(m_1|\omega_{3+2(k+2)})q(\omega_{3+2(k+2)})},$$

which is greater or equal to

$$\frac{\frac{1}{2}\frac{1-\eta}{2}}{\frac{1}{2}\frac{1-\eta}{2} + \eta} = \frac{1-\eta}{1+3\eta}$$

since each payoff type has probability $\frac{1}{2}$ and at least one payoff type sends message m_1 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ with at least probability $\frac{1-\eta}{2}$. Therefore, as long as $\frac{1-\eta}{1+3\eta} > \frac{1}{3}$ it is uniquely optimal for the receiver to respond with action c_R to m_1 at $\mathcal{O}^R(\omega_{2+2(k+2)})$; this is the case with $\eta < \frac{1}{3}$. This implies that the sender's payoff from sending message m_1 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ is 5.

If there is a payoff type of the sender who sends both messages with positive probability at $\mathcal{O}^S(\omega_{3+2(k+1)})$, that payoff type must receive the same payoff, 5, from sending message m_2. Without loss of generality, let that payoff type be t_1.

Then if the receiver randomizes in response to message m_2 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ payoff type t_1 of the sender cannot be using action c_S or b_S after sending message m_2, and hence expects the receiver to take the actions a_R and c_R with equal probability in response to receiving message m_2. Hence,

$$\frac{1}{1+\eta}\rho_R(a_R|m_2, \omega_{3+2(k+1)}) + \frac{\eta}{1+\eta}\rho_R(a_R|m_2, \omega_{4+2(k+1)}) = \frac{1}{2},$$

from which it follows that

$$\rho_R(a_R|m_2, \omega_{3+2(k+1)}) \geq \frac{1}{2}(1-\eta).$$

Therefore, if payoff type t_2 also sends message m_2 at information set $\mathcal{O}^S(\omega_{3+2(k+1)})$, her payoff from doing so would be at most $\left(1 - \frac{1}{2}\frac{1-\eta}{1+\eta}\right)6 \leq \left(1 - \frac{1}{2}\frac{1-\frac{1}{6}}{1+\frac{1}{6}}\right)6 = \frac{27}{7} < 5$, the payoff t_2 could achieve at $\mathcal{O}^S(\omega_{3+2(k+1)})$ by sending message m_1 and taking action c_S. This rules out t_2 also sending message m_2 at information set $\mathcal{O}^S(\omega_{3+2(k+1)})$. This implies that it is uniquely optimal for the receiver to take action a_R after receiving message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which contradicts the assumption that he is randomizing at that information set. But if the receiver is not randomizing following m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, he must be taking action c_R (if he took a different action, one of the payoff types of the sender could achieve a payoff of at least $\frac{1}{1+\eta}6 > \frac{1}{1+\frac{1}{6}}6 > 5$). Hence, the sender must be using action c_S both messages at $\mathcal{O}^S(\omega_{3+2(k+1)})$, and thus behave as in *Case 3*.

Subcase 4.5: If both payoff types send distinct messages at information set $\mathcal{O}^S(\omega_{3+2(k+1)})$ and (without loss of generality) payoff type t_1 sends message m_1, then t_1 takes action c_S following m_1. If the receiver is randomizing after message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, then payoff type t_2 is not taking action c_S after sending message m_2 at $\mathcal{O}^S(\omega_{3+2(k+1)})$, because doing so would result in a payoff less than 5, whereas sending message m_1 and taking action c_S would result in a payoff of 5. But if t_2 takes action b_S at $\mathcal{O}^S(\omega_{3+2(k+1)})$, then the receiver does not randomize are after m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which leads to a contradiction. Therefore, either the receiver takes action c_R after observing message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which implies that the sender takes action c_S after sending message m_2 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ and corresponds to *Case 3* behavior, or the receiver takes action b_R after observing message m_2 at $\mathcal{O}^R(\omega_{3+2(k+1)})$, which implies that the sender takes action b_S after sending message m_2 at $\mathcal{O}^S(\omega_{3+2(k+1)})$ and corresponds to *Case 1* behavior. □

In this version of *Opportunity* there are two types of equilibria. In one the sender always takes action c_S and the receiver always takes action c_R. In the other there is a language state at which the sender sends one message at one payoff state and takes action c_S and sends the other message at the other payoff state and takes action a_R if this payoff state is t_1 and action b_R otherwise. In the latter, the separation pattern prevails for all higher language states. In both types of equilibria there is at least one payoff state at which the action pair is (c_R, c_S) at all language states. Hence, regardless of the order of knowledge with which players know that the sender is unconstrained, play remains inefficient.

Version 3 ($K > 2$, $|M| > 2$): As in the other two versions of *Opportunity*, for sufficiently small $\eta > 0$, the receiver responds to all messages with action c_R at his information set $\mathcal{O}^R(\omega_1)$ and the sender takes action c_S at $\mathcal{O}^S(\omega_{|M|+1})$, after every message she sends with positive probability. Now, however, it is possible to construct equilibria that achieve efficiency at all language states $\omega_{|M|+k}$ for $k > 2$. To see this, let the sender use a strategy in which she exclusively sends message m_1 at $\mathcal{O}^S(\omega_{|M|+1})$ regardless of the payoff state and in which at language states $\omega_{|M|+k}$ with $k > 2$ she sends message m_2 and takes action a_S if the payoff state is t_1 and sends message m_3 and takes action b_S if the payoff state is t_2. Then it is a best reply for the receiver at each of his information sets $\mathcal{O}^R(\omega_{|M|+k})$ with $k \geq 2$ to respond to message m_2 with a_R and to message m_3 with b_R. Also, for any such strategy of the receiver the postulated sender strategy is a best reply. Hence, if the message space is sufficiency large and the sender knows that the receiver knows that all messages are available, i.e. the sender has second-order knowledge of all messages being available to the sender, it is possible in equilibrium for the sender to communicate the payoff state and for players to coordinate on the payoff maximizing actions in each payoff state.

4 Language states that admit ex post optimality

In this section I show that the observation made in Version 3 of the *Opportunity* example is general: access to a large set of messages can mitigate the consequences of higher-order uncertainty about message availability. The key is a *rich language* condition: the sender has second-order knowledge (belief) of an event in the language state space that is small in relation to the set of messages available to her. When this condition is satisfied at state $\hat{\omega} \in \Omega$, there is an equilibrium that is *ex post* optimal at $\hat{\omega}$.

The reason is intuitive. Roughly, and focussing for now on knowledge rather than belief: if the receiver knows an event in the language state space that is small in relation to the set of available messages, there will be messages he does not expect to receive; we can change his responses to ones consistent with *ex post* optimality after those messages without affecting payoffs; and, we can then have the sender start using those messages and likewise take actions consistent with *ex post* optimality.

More precisely, since there exists an *ex ante* optimal equilibrium, it suffices to show that when the *rich language* condition is satisfied at a language state then an *ex ante* optimal equilibrium must be *ex post* optimal at that language state. To this end, note that for every *ex ante* optimal equilibrium there is an equilibrium in pure strategies with the same payoff.

If a large number of messages are available to the sender, not all of them will be used in a pure-strategy equilibrium at any given language state. If the receiver knows that this is the case, one can adjust his responses to these messages without risk of lowering payoffs. If this set of unused messages is large, we can find one for each payoff state that would be part of an *ex post* optimal strategy profile for that payoff state. Finally, if the sender knows that this is the case, she can switch to the formerly unused messages and take actions that match the receiver actions optimally at each payoff state. Therefore, if the *ex ante* optimal equilibrium were not also *ex post* optimal at a language state at which the rich language condition is satisfied we could raise *ex ante* expected payoffs, which would violate *ex ante* optimality. Recall that for any event $F \subseteq \Omega$ the event $K_i(F) := \{\omega \in \Omega | \mathcal{O}^i(\omega) \subseteq F\}$ is the set of language states at which player i knows F.

Definition 1 *The rich language condition is satisfied at language state $\hat{\omega} \in \Omega$ if*

$$\exists E \subseteq \Omega \ s.t. \ \hat{\omega} \in K_S K_R(E) \ and \ |\lambda_{\hat{\omega}}| \geq (|E| + 1)|T|.$$

The following result establishes that satisfaction of the rich language condition at a language state is sufficient for the existence of an *ex post* optimal equilibrium at that language state.

Proposition 2 *In any common-interest communication game, for any language state $\hat{\omega}$ at which the rich language condition is satisfied there exists an equilibrium that is* ex post *optimal at $\hat{\omega}$.*

Proof: In any communication game with common interests there exists an *ex ante* optimal strategy profile $e^* = (\sigma_S^*, \rho_S^*, \rho_R^*)$ that is in pure strategies. Evidently, e^* is an equilibrium. Suppose that e^* is not *ex post* optimal at $\hat{\omega}$. Since $|\lambda_{\hat{\omega}}| \geq (|E| + 1)|T|$, in the equilibrium e^* there are at least $|T|$ messages, $m_1, m_2, \ldots, m_{|T|} \in \lambda_{\hat{\omega}}$, that are not used at any $\omega \in E$. Therefore the receiver never observes these messages at any $\omega \in K_R(E)$ and the sender never sends those messages at any $\omega \in K_S K_R(E)$. Define a receiver strategy ρ'_R by

$$\rho'_R(\omega, m) = \begin{cases} a_t^R & \text{if } m = m_t \text{ and } \omega \in K_R(E) \\ \rho_R^*(\omega, m) & \text{otherwise} \end{cases}$$

Since ρ'_R differs from ρ_R^* only at language states and messages that are not received at those states the payoffs from the strategy profiles $e^* = ((\sigma_S^*, \rho_S^*, \rho_R^*))$ and $((\sigma_S^*, \rho_S^*, \rho'_R))$ are the

same. Define a signaling rule σ'_S for the sender by

$$\sigma'_S(\omega, t) = \begin{cases} m_t & \text{if } \omega \in \mathcal{O}^S(\hat{\omega}) \\ \sigma^*_S(\omega, t) & \text{otherwise} \end{cases}$$

and an action rule ρ'_S for the sender by

$$\rho'_S(\omega, t) = \begin{cases} a^S_t & \text{if } \omega \in \mathcal{O}^S(\hat{\omega}) \\ \rho^*_S(\omega, t) & \text{otherwise.} \end{cases}$$

Since we assumed that e^* is not *ex post* optimal at $\hat{\omega}$ and because $\mathcal{O}^S(\hat{\omega}) \subseteq K_S K_R(E)$, the strategy profile $e' := ((\sigma'_S, \rho'_S, \rho'_R))$ achieves a strictly higher expected payoff than $((\sigma^*_S, \rho^*_S, \rho'_R))$ for the sender at $\hat{\omega}$, the maximal *ex post* payoff at every state $\omega \in \mathcal{O}^S(\hat{\omega})$ and does not change the sender's expected payoff at any other language state. Therefore the expected payoff from the profile e' is strictly higher than from e^* and hence e^* cannot have been optimal. It follows that for e^* to be *ex ante* optimal it must be *ex post* optimal at $\hat{\omega}$.

□

In contrast to Proposition 2, version 1 of the *Opportunity* example demonstrates that if we want to ensure *ex post* optimality at language state $\hat{\omega}$ then, regardless of the size of the set of messages available to the sender at $\hat{\omega}$, we cannot ignore the sender's second-order knowledge. To make this explicit it helps to adopt a notion of strong violation of the rich language condition.

Definition 2 *The rich language condition is strongly violated at $\hat{\omega}$ if*

$$\hat{\omega} \in K_S K_R(E) \Rightarrow |\lambda_{\hat{\omega}}| < |E| \ \forall E \subseteq \Omega$$

Thus, the rich language condition is strongly violated at $\hat{\omega}$ if every event that is second-order knowledge of the sender is large in an appropriate sense in relation to the size of the set of messages available to the sender at $\hat{\omega}$. The violation is termed "strong" because unlike in the negation of the rich language condition, which would involve $|\lambda_{\hat{\omega}}| < (|E| + 1)|T|$, here we use $|\lambda_{\hat{\omega}}| < |E|$. What we found for version 1 of the opportunity example then implies the following observation:

Observation 2 *For $|T| = 2$ and all $N \in \mathbb{N}_{>0}$ the class of common interest communication games with a state $\hat{\omega}$ at which the rich language condition is strongly violated contains a game in which* ex post *optimality at $\hat{\omega}$ fails, $|\lambda_{\hat{\omega}}| > N$ and $\lambda_{\hat{\omega}}$ is mutual knowledge.*

A slight modification of the example (in which one increases the number of payoff types to an arbitrary finite $|T|$; for each payoff type lets each player have $|T|+1$ actions; payoffs from the receiver's $|T|+1$st action are 4 unless the sender also takes her $|T|+1$st action, in which case they are 5; there is a unique type specific action pair with a payoff of 6; and, all other payoffs are zero) shows that the observation holds for all $|T|$. Hence, merely requiring that the sender has a large set of messages available or that this is mutual knowledge is not enough.

Next, I establish an analogue of Proposition 2 for beliefs. It replaces the rich language condition from above that was in terms of knowledge by a similar condition on beliefs. The value of this result is that it identifies a (sometimes strictly) larger class of states $\omega \in \Omega$ at which *ex post* optimality at ω can be achieved, at the expense of invoking the prior distribution over the language state space. For any event $F \subseteq \Omega$ and any $p \in [0,1]$ define $B_i^p(F) := \{\omega \in \Omega | \text{Prob}[F|\mathcal{O}^i(\omega)] \geq p\}$, the set of language states ω at which player i assigns at least probability p to the event F, or "p-believes" F. Again, a rich-language condition, expressed in terms of beliefs, is key to the result.

Suppose that at language state $\hat{\omega} \in E$ the receiver assigns high probability to E, the sender assigns high probability to the receiver having this belief, E is small relative to the set of messages available to the sender at E and the ratio of the *ex ante* probabilities of the sender's information set at $\hat{\omega}$ and the event that the receiver assigns at least probability p to E is bounded away from zero. Then, as the next result shows there is an equilibrium that is *ex post* optimal at $\hat{\omega}$.

A sketch of the argument is as follows. There is an *ex ante* optimal pure-strategy equilibrium. In any such equilibrium there is a large set of messages that are not sent at E. If the receiver believes with high probability that E obtains (i.e. at $B_R^p(E)$ for high p), then he expects to observe these messages with only small probability. Therefore, if he changes his responses to those messages to ones consistent with *ex post* optimality, the expected payoff loss at $B_R^p(E)$ is small. If one now modifies the sender's strategy at $\hat{\omega}$ so that she sends the messages that were formerly not sent at E and takes actions that complement the receiver actions, then at her information set $\mathcal{O}^S(\hat{\omega})$ the sender expects a positive payoff gain that is bounded away from zero if the original strategy profile is not *ex post* optimal at $\hat{\omega}$. If the ratio of the probability of the event where the sender expects a gain, $\mathcal{O}^S(\hat{\omega})$, relative to the probability of the event at which the receiver might expect a loss, $B_R^p(E)$, is bounded away from zero, then for large p the gain from adjusting the sender's strategy more than offsets the potential lowering of payoff that resulted from the alteration of the receiver strategy, which contradicts *ex ante* optimality.

The rich language condition expressed in terms of p-beliefs takes the following form:

Definition 3 *For $c > 0$, the p-rich language condition is satisfied at language state $\hat{\omega}$ if*

$$\exists E \subseteq \Omega \text{ } s.t. \text{ } \hat{\omega} \in E \cap B_R^p(E) \cap B_S^p B_R^p(E), \frac{Prob(\mathcal{O}^S(\hat{\omega}))}{Prob(B_R^p(E))} > c \text{ and } |\lambda_{\hat{\omega}}| \geq (|E|+1)|T|.$$

The following result establishes that for sufficiently large p satisfaction of the p-rich-language condition at language state $\hat{\omega}$ is sufficient for the existence of an *ex post* optimal equilibrium at that language state.

Proposition 3 *For any set of fundamentals $\mathcal{F}$ and any $c > 0$ there exists a $p \in (0,1)$ such that if the p-rich language condition is satisfied in the game $\mathcal{G} = (\mathcal{F}, \mathcal{I})$ at language state $\hat{\omega}$, the game $\mathcal{G}$ has an equilibrium that is* ex post *optimal at $\hat{\omega}$.*

Proof: In any communication game with common interests $\mathcal{G} = (\mathcal{F}, \mathcal{I})$ there exists an *ex ante* optimal strategy profile $e^* = (\sigma_S^*, \rho_S^*, \rho_R^*)$ that is in pure strategies. Evidently, e^* is an equilibrium. Suppose that e^* is not *ex post* optimal at $\hat{\omega}$. Then there exists at least one $t \in T$ such that $u(\rho_S^*(\hat{\omega}, t), \rho_R^*(\hat{\omega}, \sigma_S^*(\hat{\omega}, t))) < u(a_t^S, a_t^R)$. Since $|\lambda_{\hat{\omega}}| \geq (|E|+1)|T|$ and $\hat{\omega} \in E$ there are at least $|T|$ messages, $m_1, m_2, \ldots, m_{|T|}$ that are available to the sender at $\hat{\omega}$ that are not used at any $\omega \in E$, including $\hat{\omega}$. Define a receiver strategy ρ'_R by

$$\rho'_R(\omega, m) = \begin{cases} a_t^R & \text{if } m = m_t \text{ and } \omega \in B_R^p(E) \\ \rho_R^*(\omega, m) & \text{otherwise} \end{cases}$$

The receiver strategy ρ'_R differs from ρ_R^* only at language states at which he assigns probability no larger than $1 - p$ to receiving one of the messages $m_1, m_2, \ldots, m_{|T|}$ and only following those messages. Therefore the receiver's *ex ante* expected payoff loss from switching from the strategy profile $e^* = ((\sigma_S^*, \rho_S^*, \rho_R^*))$ to the profile $((\sigma_S^*, \rho_S^*, \rho'_R))$ is bounded from above by

$$\text{Prob}(B_R^p(E))(1-p)\left(\max_{a^S, a^R, t} u(a^S, a^R, t) - \min_{a^S, a^R, t} u(a^S, a^R, t)\right).$$

Define a signaling rule σ'_S for the sender by

$$\sigma'_S(\omega, t) = \begin{cases} m_t & \text{if } \omega \in \mathcal{O}^S(\hat{\omega}) \\ \sigma_S^*(\omega, t) & \text{otherwise} \end{cases}$$

and an action rule ρ'_S for the sender by

$$\rho'_S(\omega, t) = \begin{cases} a_t^S & \text{if } \omega \in \mathcal{O}^S(\hat{\omega}) \\ \rho^*_S(\omega, t) & \text{otherwise.} \end{cases}$$

The sender strategy (σ'_S, ρ'_S) differs from (σ^*_S, ρ^*_S) only at language states at which he assigns probability at least p to the receiver (who uses ρ'_R) responding with a_t to any m_t in $\{m_1, \ldots, m_T\}$, since $\mathcal{O}^S(\hat{\omega}) \subseteq B^p_S B^p_R(E)$. Therefore the sender's *ex ante* expected payoff gain from replacing $(\sigma^*_S, \rho^*_S, \rho'_R)$ with $(\sigma'_S, \rho'_S, \rho'_R)$ is bounded from below by

$$\text{Prob}(\mathcal{O}^S(\hat{\omega})) \sum_{t \in T} \pi(t) \left\{ p \left(u(a_t^S, a_t^R, t) - u(\rho^*_S(\hat{\omega}, t), \rho^*_R(\hat{\omega}, \sigma^*_S(\hat{\omega}, t))) \right) \right.$$

$$\left. -(1-p) \left(\max_{a^S, a^R, \tau} u(a^S, a^R, \tau) - \min_{a^S, a^R, \tau} u(a^S, a^R, \tau) \right) \right\}.$$

Since $\text{Prob}(\mathcal{O}^S(\hat{\omega})) > c\text{Prob}(B^p_R(E))$, the prior distribution over payoff states π is fixed, $\pi(t) > 0$ for all $t \in T$ and the payoff differences $u(a_t^S, a_t^R, t) - u(\rho^*_S(\hat{\omega}, t), \rho^*_R(\hat{\omega}, \sigma^*_S(\hat{\omega}, t)))$ are all nonnegative with at least one being strictly positive, if p is sufficiently large the expected payoff gain from the sender changing her strategy outweighs the loss from the receiver changing his strategy. This contradicts the assumption that e^* was optimal. It follows that for sufficiently large p, in order for e^* to be *ex ante* optimal it must also be *ex post* optimal at $\hat{\omega}$. □

In our two-player environment an event E is ***p-evident*** if $E \subseteq B^p_i(E)$, $i = 1, 2$, and an event F is ***common p-belief*** at language state ω if there exists a p-evident event E with $\omega \in E$ and $E \subseteq B^p_i(F)$, $i = 1, 2$.

Definition 4 *For $c > 0$, the common p-rich language condition is satisfied at language state $\hat{\omega}$ if there exists $E \subseteq \Omega$ that is common p belief at $\hat{\omega}$ with $Prob(\mathcal{O}^S(\hat{\omega})) > cProb(B^p_R(E))$ and $|\lambda_{\hat{\omega}}| \geq (|E| + 1)|T|$.*

Evidently the common p-rich language condition implies the p-rich language condition. Therefore we have,

Corollary 1 *For any set of fundamentals $\mathcal{F}$ and any $c > 0$ there exists a $p \in (0, 1)$ such that if the common p-rich language condition is satisfied in the game $\mathcal{G} = (\mathcal{F}, \mathcal{I})$ at language state $\hat{\omega}$, the game $\mathcal{G}$ has an equilibrium that is* ex post *optimal at $\hat{\omega}$.*

4.1 Opportunity revisited

To compare the rich language condition and the p-rich language condition, consider the class of common-interest communication games that have the same payoff function and payoff type distribution as the opportunity game considered before, a message space M with $|M| \geq 6$, a language state space $\Omega = \{\omega_1, \ldots, \omega_{2|M|+3}\}$ and the following information structure (for easy reference language types as a function of language states and the prior probabilities of the language states have been indicated beneath the corresponding states):

$$\begin{array}{l}
\mathcal{O}^S = \{\{\omega_1\}, \ldots, \{\omega_{|M|}\}, \{\omega_{|M|+1}, \omega_{|M|+2}\}, \{\omega_{|M|+3}\}, \{\omega_{|M|+4}\} \ldots, \{\omega_{2|M|+3}\}\} \\
\mathcal{O}^R = \{\{\omega_1, \ldots\ldots \omega_{|M|},\ \omega_{|M|+1}\}, \{\omega_{|M|+2},\quad \omega_{|M|+3},\quad \omega_{|M|+4} \ldots\ldots, \omega_{2|M|+3}\}\} \\
\lambda(\omega) = \{m_1\} \ldots \{m_{|M|}\} \quad M_6 \qquad M_6 \qquad M_6 \qquad \{m_1\} \quad \ldots \quad \{m_{|M|}\} \\
q(\omega) = \quad y \quad \ldots\ldots \quad y \qquad \epsilon \qquad \epsilon^2 \qquad \epsilon^2 \qquad \epsilon^3 \quad \ldots\ldots \quad \epsilon^3
\end{array}$$

Thus $\lambda_{\omega_n} = \{m_n\}$ for $n = 1, \ldots |M|$, $\lambda_{\omega_{|M|+1}} = \lambda_{\omega_{|M|+2}} = \lambda_{\omega_{|M|+3}} = \{m_1, \ldots, m_6\} =: M_6$ and $\lambda_{\omega_{|M|+3+j}} = \{m_j\}$ for $j = 1, \ldots, |M|$. The common prior, q, is given by $q(\omega_n) = y$ for $n = 1, \ldots, |M|$, $q(\omega_{|M|+1}) = \epsilon$, $q(\omega_{|M|+2}) = q(\omega_{|M|+3}) = \epsilon^2$ and $q(\omega_n) = \epsilon^3$ for $n > |M| + 3$, where $\epsilon \in (0, 1)$, $\epsilon + 2\epsilon^2 + |M|\epsilon^3 < 1$ and $y = \frac{1-\epsilon-2\epsilon^2-|M|\epsilon^3}{|M|}$.

Consider the language state $\omega_{|M|+3}$. The smallest set $E \subseteq \Omega$ that satisfies the property $\omega_{|M|+3} \in K_S K_R(E)$ is $\{\omega_{|M|+2}, \ldots, \omega_{2|M|+3}\}$. The cardinality of that set is $|M|+1$ and hence $\left|\lambda_{\omega_{|M|+3}}\right| = 6 < (|\{\omega_{|M|+2}, \ldots, \omega_{2|M|+3}\}| + 1)|T| = 2(|M| + 2)$ and therefore the rich language condition fails to be satisfied at $\omega_{|M|+3}$. Thus *from the knowledge structure alone we cannot infer whether there is an equilibrium that is* ex post-*efficient at* $\omega_{|M|+3}$.[4]

Now introduce the prior and consider the set $E' = \{\omega_{|M|+2}, \omega_{|M|+3}\}$. Note that for every $p \in (0, 1)$ we can find $\underline{\epsilon} \in (0, 1)$ such that for all $\epsilon \in (0, \underline{\epsilon})$ we have

$$\omega_{|M|+3} \in E' \cap B_R^p(E') \cap B_S^p B_R^p(E').$$

Since at the same time

$$\left|\lambda_{\omega_{|M|+3}}\right| = 6 = (|E'| + 1)|T|$$

and

$$\frac{\mathrm{Prob}(\mathcal{O}^S(\omega_{|M|+3}))}{\mathrm{Prob}(B_R^p(E'))} = \frac{\epsilon^2}{2\epsilon^2 + |M|\epsilon^3} \to \frac{1}{2} \text{ for } \epsilon \to 0,$$

[4]Indeed if we modify the prior so that $q(\omega_{|M|+3}) = \epsilon^4$ and $y = \frac{1-\epsilon-\epsilon^2-\epsilon^4-|M|\epsilon^3}{|M|}$, then for small ϵ there is no equilibrium that is *ex post*-efficient at $\omega_{|M|+3}$. Any strategic use of messages at $\omega_{|M|+3}$ is swamped by the nonstrategic use of messages at $\omega_{|M|+3}, \ldots, \omega_{2|M|+3}$.

Proposition 3 implies that for all sufficiently small ϵ there exists an $\omega_{|M|+3}$-efficient equilibrium. Thus, *there are prior distributions of language types in this example for which the condition on the second-order beliefs of the sender (Proposition 3) ensures the existence of an equilibrium that is* ex post*-efficient at $\omega_{|M|+3}$, when the earlier condition on the sender's second-order knowledge (Proposition 2) fails to do so.* The rich language condition is easier to check than the p-rich language condition since it only refers to the knowledge structure, but less effective in identifying language states ω for which there is an equilibrium that is *ex post*-efficient at $\omega_{|M|+3}$.

Now consider the stronger common p-rich language condition. Fix p close to 1 and let ϵ be small enough so that $\omega_{|M|+3} \in E' \cap B^p_R(E') \cap B^p_S B^p_R(E')$ for $E' = \{\omega_{|M|+2}, \omega_{|M|+3}\}$. Then the smallest p-evident event that contains $\omega_{|M|+3}$ must include $\{\omega_1, \dots, \omega_{|M|+3}\}$ and hence E' is not common p belief at $\omega_{|M|+3}$. Therefore, *there are prior distributions for which the p-rich language condition guarantees the existence of an equilibrium that is* ex post*-efficient at $\omega_{|M|+3}$ when the common p-rich language condition fails to hold.*

5 Approximate *ex ante* optimality

The previous section gave sufficient conditions for higher-order uncertainty about message availability not to interfere with efficient communication given players' information about message availability. This involved an interim perspective, focussing on a given language state and the players' knowledge (belief) at that state. Here I consider instead an *ex ante* perspective and ask what epistemic condition ensures that communication succeeds "on average," taking expectations over the possible language states players may find themselves in. The condition in this section is mild in that it does not involve higher-order knowledge (or belief) or a rich-language requirement: as long as there is high probability that the receiver has strong beliefs about the sender's language type, the expected efficiency loss from higher-order uncertainty about message availability is negligible.

Let $\Omega_\lambda := \{\omega \in \Omega | \lambda_\omega = \lambda\}$ be the set of language states at which the sender's language type is λ. For each information structure $\mathcal{I} = (\Omega, \mathcal{L}, \mathcal{O}^S, \mathcal{O}^R, q)$ and each $p \in (0, 1)$, define

$$\Omega^p(\mathcal{I}) := \{\omega \in \Omega | \exists \lambda \in \Lambda \text{ with } \omega \in B^p_R(\Omega_\lambda)\},$$

as the set of language states at which the receiver assigns at least probability p to some language type. For large p this is the set of language states at which the receiver is confident about being able to predict the sender's language type.

Let $U((\mathcal{F},\mathcal{I}))$ denote the (*ex ante*) payoff from an optimal equilibrium $e((\mathcal{F},\mathcal{I}))$ in the game $\mathcal{G}=(\mathcal{F},\mathcal{I})$. Let $U^*((\mathcal{F},\mathcal{I}^*))$ denote the payoff from an *ex ante* optimal equilibrium $e^*((\mathcal{F},\mathcal{I}^*))$ of the game $\mathcal{G}^*=(\mathcal{F},\mathcal{I}^*)$ with an information structure $\mathcal{I}^*$ that is obtained from $\mathcal{I}$ by replacing the knowledge partitions of both players by the finest partition, so that the language type is common knowledge at every language state. The next result considers sequences of information structures $\mathcal{I}_n=(\Omega_n,\mathcal{L}_n,\mathcal{O}_n^S,\mathcal{O}_n^R,q_n)$ that converge to the receiver assigning high probability to some language type with high probability.

Proposition 4 *For any sequence of common-interest communication games* $\{\mathcal{G}_n\}_{n=1}^{\infty}=\{(\mathcal{F},\mathcal{I}_n)\}_{n=1}^{\infty}$ *and sequence of probabilities* $\{p_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty}p_n=1$ *and* $\mathrm{Prob}(\Omega^{p_n}(\mathcal{I}_n))\geq p_n$*,*

$$\lim_{n\to\infty}|U((\mathcal{F},\mathcal{I}_n))-U^*((\mathcal{F},\mathcal{I}_n^*))|=0.$$

Thus, when there is high probability that the receiver assigns high probability to some language type (which may vary with the language state), the expected payoff gain from making the sender's language type common knowledge would be insignificant.

Proof: Let $\Psi_S^\lambda=\{\psi_S:T\to\lambda\}$, $\Phi_S=\{\phi_S:T\to A_S\}$ and $\Phi_R^\lambda=\{\phi_R:\lambda\to A_R\}$ be the sets of pure-strategy signaling and action rules in the communication game where it is common knowledge that the sender's message space is λ. Let

$$(\psi_S^\lambda,\phi_S^\lambda,\phi_R^\lambda)\in\arg\max_{\substack{(\psi_S,\phi_S,\phi_R)\in\\ \Psi_S^\lambda\times\Phi_S\times\Phi_R^\lambda}}\mathbb{E}_tU(\phi_S(t),\phi_R(\psi_S(t)),t)$$

be a payoff maximizing strategy profile in the auxiliary game in which it is common knowledge that the set of messages available to the sender is λ. For the game $\mathcal{G}_n=(\mathcal{F},\mathcal{I}_n)$ define a signaling rule σ_S^n for the sender by

$$\sigma_S^n(\omega,t)=\psi_S^{\mathcal{L}_n(\omega)}(t)$$

and an action rule ρ_S^n for the sender by

$$\rho_S^n(\omega,t)=\phi_S^{\mathcal{L}_n(\omega)}(t).$$

The sender strategy (σ_S^n,ρ_S^n) would be part of an optimal sender strategy if we could simply ignore the receiver's informational constraints or, equivalently, if at at every language state the language type were common knowledge between the sender and receiver.

Let $\hat{\lambda}^n(\omega)\in\arg\max_{\lambda\in\Lambda}\mathrm{Prob}(\lambda|\mathcal{O}_n^R(\omega))$ be a language type of the sender to which the

the receiver assigns maximal probability given his information $\mathcal{O}_n^R(\omega)$ at state ω in game $\mathcal{G}_n$. Define a receiver strategy ρ_R^n by

$$\rho_R^n(\omega, m) = \phi_R^{\hat{\lambda}^n(\omega)}(m).$$

Given the strategy profile $(\sigma_S^n, \rho_S^n, \rho_R^n)$, at any state $\omega \in \Omega^{p_n}(\mathcal{I}_n)$ the receiver assigns at least probability p_n to a set of language states at which the expected payoff (with the expectation taken with respect to t) from $(\sigma_S^n, \rho_S^n, \rho_R^n)$ is the same as from $e^*((\mathcal{F}, \mathcal{I}_n^*))$. Since the event $\Omega^{p_n}(\mathcal{I}_n)$ itself has at least probability p_n, the receiver's *ex ante* expected payoff from profile $(\sigma_S^n, \rho_S^n, \rho_R^n)$ in game $\mathcal{G}_n$ is bounded from below by

$$U^*((\mathcal{F}, \mathcal{I}_n^*)) - (1 - p_n^2)\left(\max_{a^S, a^R, t} u(a^S, a^R, t) - \min_{a^S, a^R, \tau} u(a^S, a^R, t)\right).$$

Hence there is a sequence of strategy profiles $\{(\sigma_S^n, \rho_S^n, \rho_R^n)\}_{N=1}^{\infty}$ corresponding to the sequence of games $\{\mathcal{G}_n\}_{=1}^{\infty}$ for which

$$\lim_{n\to\infty} |U((\mathcal{F}, \mathcal{I}_n)) - U^*((\mathcal{F}, \mathcal{I}_n^*))| = 0.$$

For each $\mathcal{G}_n$ the payoff from an optimal strategy profile $(\sigma_S^{*n}, \rho_S^{*n}, \rho_R^{*n})$ is no less than the payoff from the profile $(\sigma_S^n, \rho_S^n, \rho_R^n)$. Since we are considering common-interest games, each optimal profile $(\sigma_S^{*n}, \rho_S^{*n}, \rho_R^{*n})$ is an equilibrium profile for $\mathcal{G}_n$. The claim follows. □

6 Conclusion and Discussion

Communication may be impaired by an insufficient repertoire of messages relative to the state space. Such a message deficit may be the result of a limited organizational code, lack of a shared language across academic disciplines or occupations, time constraints that prevent the formulation of elaborate messages, or simply lack of command of the local language. Jäger, Metzger, Riedel [13] study the optimal use of such a limited repertoire. Crémer, Garicano and Prat [8] and Sobel [29] investigate the implications for organizational structure. Uncertainty about the message repertoire further complicates matters. Blume and Board [3] show that optimal use of an uncertain repertoire results in indeterminacy of meaning, severing the link between messages and definite subsets of the payoff state space. In this paper I ask whether and how these concerns extend to higher-order uncertainty.

Version 1 of *Opportunity* suggests that the mere fear of being perceived as being restricted

to a small set of messages may render communication ineffective: in the example the sender commands a large repertoire of messages, the receiver knows this and yet there is no benefit at all from communication, when there would be with a commonly known language. Version 2 of *Opportunity* demonstrates that if in addition the size of the message space does not exceed that of the state space these concerns extend to situations where players have more than mutual knowledge of the message space being no smaller than the state space. Version 3 of *Opportunity* suggests, and the remainder of the paper confirms, that a combination of access to a large message space and mutual knowledge of the receiver knowing a small even in the language state space is enough to remove concerns about failures of higher order knowledge about message availability. A rich language is an antidote to the indeterminacy of meaning that may arise from higher-order uncertainty about the sender's repertoire of messages.

I have focussed on limited message availability both because of the precedents in the literature on communication games and because it expresses natural constraints on sender strategies that might be implied by unfamiliarity with an organizational code or a foreign language. Other constraints worth investigating include commitments to specific maps from payoff states into messages and actions resulting, for example, from conflicting externally given language uses. Such additional constraints make it easier to construct examples in the spirit of *Opportunity.* Still, I expect the positive results of this paper to go through when allowing for a larger class of constraints: at any language state where the rich language condition is satisfied and the sender knows that the receiver knows that there are no constraints in addition to those on message availability, there should be be an *ex post* optimal equilibrium.

This paper deals exclusively with common-interest games. The reason is that outside of this domain there can be a benefit from communicating with faulty devices (see, for example Blume, Board and Kawamura [2] and the references therein), of which an inadequate language is an just one example. If the interest is in the adverse effects of uncertainty about meaning, common-interest games are an obvious starting point.

There are at least two reasons for looking at the negative impact of higher-order uncertainty about language on communication through the lens of equilibria, in addition to the discipline that it imposes and the appeal of relying on a universal language for thinking about strategic interaction generally and strategic communication more specifically. First, prior work on common-interest communication games has considered equilibria; in fact, the bulk of work on information-transmission games has considered equilibria. Second, and more importantly, in common-interest games, which I argued are a natural point of departure for

investigating communication failures, one can think of optimal strategies, and those optimal strategies are equilibria. Optimal strategies provide a definite upper bound on what can be achieved with communicating with a faulty language. One would like to know whether the optimal strategies that are constrained by higher-order uncertainty about language come close to unconstrained optimal strategies, in which all available information is transmitted.

References to imprecision and unreliability of language abound in philosophy, from Locke [18] who speaks of the "imperfection of words ... the doubtfulness of their signification", through Wittgenstein's [31] advocacy of thinking of "family resemblances" (Familienähnlichkeit) in the different deployments of a word, to Quine's [25] insistence on the indeterminacy of translation. As a practical matter, concerns about language affect how we think about organizations (Arrow [1]), contract interpretation (Posner [24]), statutory interpretation (Posner [23], Eskridge, Frickey, and Garrett [9]) and lead to the creation of "trading zones" that facilitate communication across academic subcultures (Galison [10]). Lipman [16] connects vagueness of language to unforeseen contingencies.

This paper is part of a continuing effort to formalize some of these concerns. Specifically, it subjects optimal communication to a "stress test", where the source of the stress is higher-order uncertainty about message availability. The hope is that this may inspire other formal inquiries into the forces that help or hinder our ability to share meaning.

A Appendix: Silence

In *Opportunity* the potential for messages to be meaningful is undermined by the possibility that each message is the only one available and therefore does not carry information. We found that this may make it impossible to use available messages efficiently even if their availability is mutual knowledge. The following example shows how to arrive at the same conclusion when there is a message that, like silence, is always available.

Consider a variant of *Opportunity* with three equally likely payoff states t_1, t_2 and t_3, with the common payoffs given in the three tables below. The receiver actions are a_R, b_R, c_R and d_R and the sender actions a_S, b_S and c_S.

	a_S	b_S	c_S
a_R	6	0	0
b_R	0	0	0
c_R	4	4	5
d_R	-2	-2	-1

t_1

	a_S	b_S	c_S
a_R	0	0	0
b_R	0	6	0
c_R	4	4	5
d_R	-2	-2	-1

t_2

	a_S	b_S	c_S
a_R	-100	-100	-100
b_R	-100	-100	-100
c_R	-100	-100	-100
d_R	0	0	0

t_3

Figure 2: Payoff States; Prob(t_1) = Prob(t_2) = Prob(t_3) = $\frac{1}{3}$

The message space is $M = \{m_0, m_1, m_2, m_3\}$. Players' knowledge and beliefs about the language constraints the sender is facing are encoded in the language state space $\Omega = \{\omega_1, \ldots, \omega_6\}$. For convenience we arrange the information partitions, language types as a function of language states and the prior probabilities of language states side by side.

$$
\begin{array}{lcccccc}
\lambda(\omega) = & \{m_0\} & \{m_0, m_1\} & \{m_0, m_2\} & \{m_0, m_3\} & \{m_0, m_1, m_2, m_3\} & \\
\mathcal{O}^S = & \{\{\omega_1\}, & \{\omega_2\}, & \{\omega_3\}, & \{\omega_4\} & \{\omega_5, & \omega_6\}\} \\
\mathcal{O}^R = & \{\{\omega_1, & \omega_2, & \omega_3, & \omega_4, & \omega_5\}, & \{\omega_6\}\} \\
q(\omega) = & x & \epsilon & \epsilon & \epsilon & \epsilon^2 & \epsilon^3
\end{array}
$$

where $x = 1 - 3\epsilon - \epsilon^2 - \epsilon^3$.

Key features of this information structure are that the sender cannot distinguish language states ω_5 and ω_6, the receiver cannot distinguish language states $\omega_1, ..., \omega_5$, the sender is

language constrained at $\omega_1, \ldots, \omega_4$, the event $\{\omega_5, \omega_6\}$ (in which all messages are available) is mutual knowledge at ω_6, and message m_0 is always available.

For small ϵ, at $\mathcal{O}^R(\omega_1)$ message m_0 conveys little information regardless of the sender's strategic behavior at those states where the sender has more than one message available. Therefore at $\mathcal{O}^R(\omega_1)$ the receiver's unique best reply to message m_0 is to respond with action d_R.

If at $\mathcal{O}^R(\omega_1)$ the receiver responds to message m_j, $j = 1, 2, 3$, by taking any action other than d_r with positive probability, then at $\mathcal{O}^S(\omega_{j+1})$ both payoff types t_1 and t_2 of the sender send message m_j with probability one. It follows that in any equilibrium at $\mathcal{O}^R(\omega_1)$ the receiver responds to message m_0 with d_r and responds to message m_j, $j = 1, 2, 3$ with either c_R or d_R. If message m_j, $j = 1, 2, 3$, is not used in equilibrium, the receiver responds to m_j with action d_R at $\mathcal{O}^R(\omega_1)$. As a result, *in every equilibrium after every message that the sender sends with positive probability at* ω_6 *she takes action* c_S *when her payoff type is either* t_1 *or* t_2.

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