Information delays and cycles

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Abstract

We study a dynamic team problem à la Bonatti and Hörner (2011) in which two players continuously contribute to a common project whose reward is uncertain. The probability that the project is successful increases with the level of effort made by the two players. We introduce delay in this model by assuming that players are informed of the success or failure of their partner after a time lag Δ. This delay is interpreted as a technological constraint on information transmission. Regardless of the delay, there is a unique symmetric equilibrium in which players alternate maximal and minimal effort. The symmetric equilibrium payoff thus follows a regular cyclical pattern and and oscillates around the symmetric equilibrium payoff without delay.

1 Introduction (preliminary)

Free-riding incentives exist in all pure public goods provision problems. In some of them, the social benefit expected from the contribution of one player is delayed. The free-

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riding phenomenon comes then from the strategic substitution between a player's current provision and her opponents' past provisions.

In order to study the dynamics of contributions in this kind of situation, we consider a stylized model of public goods provision problem with two players. Both players are engaged in a common project which is known to be good, but provides a benefit only when it is completed, the completion date being random. The probability of completing the project depends on the level of contribution of both players. When a player completes the project, the game stops for him, and the other player receives the benefit after some exogenous delay. The contribution efforts chosen by a player are unobserved by the other player.

This stylised model encompasses various public good provision situations.

- Research and development teamwork with distant partners is a special case of our model. All members of the team get a profit from the completion of the project, no matter who finished it, and the time at which the product is released is uncertain. It is not unusual that team members work in different places, so that they cannot monitor the (costly) effort of each other, and communicate about their success or failure with some technological delay.

- In some pollution abatement situations, players receive the collective benefit at different times. Imagine for instance two villages located at opposite banks of the same (large) river, engaged in a common pollution abatement project. If the river is large enough, it is reasonable to assume that 1) village A does not perfectly observe village B's cleaning efforts, 2) the two villages communicate with each other with some delay, and 3) one village receives the benefit in terms of fish quality from pollution abatement earlier than the other one.

- For security reasons, fight against terrorism within military alliances is also a public good provision problem with delays in communication, and therefore, in payoffs.

We show that there exists a unique symmetric equilibrium, that exhibits the following features:

- *Agents alternate between intense effort phases and rest phases.* Without delay, agents are indifferent at all dates between the maximal and the minimal contribution, and

thus always exert the same intermediate contribution level. With a positive delay however, players are never indifferent. The equilibrium contribution strategy is cyclical, with an infinite switch between maximal and minimal contribution levels. Furthermore, cycles are regular, in the sense that effort phases (in which the maximal contribution is exerted), as well as rest phases (in which the minimal contribution is exerted), all have the same length.

- *Payoffs follow a regular cyclical pattern.* In the case without delay the equilibrium payoff is constant over time and it is equal to the net present value of the project minus the cost of a unit of effort. When benefits come with a delay, agents' payoff is not constant over time, but oscillates around the equilibrium payoff without delay. The smaller the delay, the shorter the cycles and the smaller the amplitude of cycles. The payoff converges toward the equilibrium payoff without delay when the delay vanishes.

This result can be generalized easily to n players, $n > 2$.

Besides the equilibrium analysis, we check that this instability comes from information delay and not from one of the other assumptions. In the last section of the paper we use convex cost instead of linear cost and check that the effort choice is not constant over time. Instability also arises when the information revelation is random.

From a manager point of view, the optimum is to have a maximal delay, with the two players exerting the maximal effort. It is obviously very costly for the players. Then a social criterium could be a weighted average between the manager and the worker's payoff. There would trivially exists an optimal value for the delay, whose value would be ad-hoc and would depend on the chosen weights. Despite studying an adhoc criterium, we prefer to analyse the behavior of some interesting value that may help understand the welfare impact of the delay. More precisely, we analyze the expected time of arrival of the breakthrough and the frequency of effort. We show that if players are sufficiently patient these variation are increasing with the delay in the neighborhood of zero.

Related literature. Our paper is very close to Bonatti and Hörner (2011)'s paper about dynamic choice in teams. They consider a continuous time model in which a finite

number of agents participate to a common project. Contrary to the present paper, the quality of the project is there unknown, and can be good or bad. A bad project never gives a success, while a good one gives a success at a random time. Effort made by players are private information, and impact the probability distribution of the success arrival time. The second difference with this paper is that feedbacks are instantaneous. Campbell, Ederer and Spinnewijn (2014) study a two period team problem with endogenous feedback delays. At each period players first choose an effort level, and whether to send feedbacks or not at the end of the period. Sending a positive feedback at the end of a period may reinforce incentives to free ride in the second period, thus players may want to delay feedbacks. The two main difference with our paper are that delay is not part of a strategy and that we assume an infinite number of periods. Finally, Bimpikis and Drakopoulos (2014) also consider delayed feedback in teams but from a mechanism design perspective. In the models developed by the literature on dynamic choice in teams, free riding arises because partners' efforts produce free valuable information. In other words, these models feature positive information externalities. Bimpikis and Drakopoulos (2014) design a mechanism in which an external agent gather feedbacks and can send them to the different members of team at a given date. Choosing this date properly may prevent from free riding.

2 The model

Two agents $i = 1, 2$ are engaged in a project which is commonly known to be good. The project ends for an agent if he observes a breakthrough, whose probability of occurrence depends on the level of effort exerted by the two agents at each date. Precisely, agent i continuously chooses a level of effort $k_t^i \in [0, \lambda]$, with $\lambda \in]0, 1[$, over the infinite horizon $\mathbb{R}_+$. Assuming there was no breakthrough before, a breakthrough is generated by agent i's effort at date t with instantaneous probability k_i^t. A strategy for agent i is thus a piecewise left-continuous function $k^i : \mathbb{R}_+ \to [0, \lambda]$ with the interpretation that k_t^i is the instantaneous effort exerted by i at time t, conditional on no breakthrough having occurred.

A breakthrough is worth a net present value of 1 to an agent who observes it, independently of who is actually responsible for this breakthrough. On the other hand, the level of effort k_i^t entails an instantaneous cost to agent i of αk_i^t with $\alpha > 0$. Finally, both agents discount time at rate r. If agent i plays the strategy k^i, and is informed of the

breakthrough at time $t < \infty$, he gets 1 at time t and the game stops for him. He gets a discounted payoff e^{-rt} and incurs a discounted total cost of $\int_0^t e^{-rs}\alpha k_s^i ds$. His discounted payoff is thus

$$e^{-rt} - \int_0^t e^{-rs}\alpha k_s^i ds$$

A player's objective is to choose the level of effort so as to maximize his expected payoff.

We assume that agents do not observe their partner's effort choice. All they observe is whether a breakthrough occurred or not for their partner after some time lag Δ. At time $t < \Delta$, agent i has access only to his own experience. The instantaneous cost to player i is αk_t^i. If a breakthrough occurs, which happens with instantaneous probability k_t^i, agent i gets 1 and the game stops for him. If not, he gets 0 and the game goes on. It follows that the expected instantaneous reward to i at time $t < \Delta$ is $k_t^i(1-\alpha)$. Given that the probability that a breakthrough has not occurred for i by time $t < \Delta$ is $e^{-\int_0^t k_s^i ds}$, the expected payoff to agent i on $[0, \Delta]$ is

$$\int_0^\Delta e^{-rt} k_t^i \left(1-\alpha\right) e^{-\int_0^t k_s^i ds} dt$$

At time $t > \Delta$, agent i has also access to agent j's experience. The expected instantaneous reward to i at time $t > \Delta$ is thus $(k_t^i + k_{t-\Delta}^j) - \alpha k_t^i$. Given that the probability that a breakthrough has not occurred for i by time $t > \Delta$ is $e^{-\int_0^\Delta k_s^i ds} \times e^{-\int_\Delta^t (k_s^i + k_{s-\Delta}^j) ds}$, the expected payoff to agent i on $[\Delta, +\infty[$ is

$$\int_\Delta^\infty e^{-rt}\left(k_t^i(1-\alpha) + k_{t-\Delta}\right) e^{-\int_0^\Delta k_s^i ds} \times e^{-\int_\Delta^t (k_s^i + k_{s-\Delta}^j) ds} dt$$

Rearranging a bit, the average expected payoff to agent i given effort strategies k^i and k^j at time 0 is

$$U_0^i(k^i, k^j) := \int_0^\Delta e^{-\int_0^t (k_s^i + r) ds} k_t^i (1-\alpha) dt + e^{-\int_0^\Delta (k_s^i + r) ds} \int_\Delta^\infty e^{-\int_\Delta^t (k_s^i + k_{s-\Delta}^j + r) ds} (k_t^i (1-\alpha) + k_{t-\Delta}^j) dt \tag{2.1}$$

3 The strategic problem

We consider the problem faced by two players acting non-cooperatively. Player i seeks to maximize (2.1) conditional on player j's effort strategy. By the optimality principle,

his payoff at time t maximizes the sum of his current payoff and his discounted expected payoff. In the interval $[t, t+dt[$, he gets $\left(k_t^i(1-\alpha) + k_{t-\Delta}^j \mathbb{1}_{t\geq\Delta}\right) dt$. In $t+dt$ he gets 1 and the game stops his he observes a breakthrough, which happens with probability $(1-(k_t^i + k_{t-\Delta}^j \mathbb{1}_{t\geq\Delta})dt)$, and gets the continuation payoff conditional on no breakthrough having occurred U_{t+dt}^i otherwise. Approximating e^{-rdt} by $(1-rdt)$, i's payoff in t is approximated by the Hamilton-Jacobi-Bellman equation

$$U_t^i = \max_{k_t\in[0,\lambda]} \left\{ \left(k_t(1-\alpha) + k_{t-\Delta}^j \mathbb{1}_{t\geq\Delta}\right) dt + e^{-rdt}(1-(k_t + k_{t-\Delta}^j \mathbb{1}_{t\geq\Delta})dt)U_{t+dt}^i \right\}$$

Eliminating terms to the order $(dt)^2$ and rearranging gives

$$U_t^i = \max_{k_t^i\in[0,\lambda]} \left\{ k_t^i \left(1-\alpha-U_{t+dt}^i\right) dt + k_{t-\Delta}^j \mathbb{1}_{t\geq\Delta} dt + (1-rdt-k_{t-\Delta}^j dt \mathbb{1}_{t\geq\Delta}))U_{t+dt}^i \right\} \tag{3.1}$$

The linearity in k_t^i implies that the effort exerted by a player depends on whether, in the (t, U)-plane, his payoff is above or below the line $1-\alpha$.

Let us first characterize the condition under which it is dominant for players to always exert the maximal effort. Suppose that player j always exerts effort λ. Under what conditions is it a best-response for player i to do the same? From time Δ on, the problem faced by player i is stationary. Therefore, if he plays λ in time Δ, he will play λ in any time $t \geq \Delta$. The condition $U_\Delta < 1-\alpha$ is thus sufficient for $k_t^i = \lambda$ to be a best-response for $t \geq \Delta$. Furthermore, player i's payoff before time Δ is smaller than U_Δ. Indeed, for $t \leq \Delta$, i's payoff is C^1 and satisfies $rU_t = \max_{k_t\in[0,\lambda]} k_t(1-\alpha-U_t) + U_t'$. The fact that U_t is necessarily larger than the single player payoff $\frac{\lambda(1-\alpha)}{\lambda+r}$ implies that, for $t \leq \Delta$, $U_t' \geq 0$ and $U_t \leq U_\Delta$. The condition $U_\Delta \leq 1-\alpha$ is thus also sufficient for $k_t^i = \lambda$ to be a best-response for $t \leq \Delta$. If $U_\Delta > 1-\alpha$ however, then no matter how long the delay, i will make no effort after time Δ, thus the condition is also necessary. Since efforts are strategic substitutes, player i will have even more interest in playing λ if j plays another effort strategy. Therefore, playing $k_t = \lambda$ for all t is a dominant strategy if and only if $U_\Delta \leq 1-\alpha$.

Replacing the continuation payoff U_Δ by its value $\frac{\lambda(2-\alpha)}{2\lambda+r}$ gives the condition $r \geq \frac{\alpha\lambda}{1-\alpha}$: the cost of procrastination to a player increases with the player's impatience.[1]

[1] Note that this threshold is the same as in BH, since the value of $\frac{\lambda(2-\alpha)}{2\lambda+r}$ is the agent's payoff in time 0 when $\Delta = 0$.

Lemma 1. *The game has a unique equilibrium in which players always exert the maximal effort if and only if* $r \geq \frac{\alpha\lambda}{1-\alpha}$.

From now on, we assume that agents are sufficiently patient to consider postponing effort at some dates, that is

$$r < \frac{\alpha\lambda}{1-\alpha}$$

We shall use the notation $K := \alpha\lambda - r(1-\alpha)$ in the rest of the paper. Clearly, $K > 0$ for patient players.

3.1 Preliminary results

In this section, we establish two results that will simplify the equilibrium analysis. First, players do not play "interior" strategies in the sense that they exert either the maximal level of effort ($k_t = \lambda$), or no effort at all ($k_t = 0$).

Lemma 2. *There is no equilibrium in which an agent plays* $k_t \in]0, \lambda[$ *over a non empty interval of time.*

Proof. We proceed by contradiction. Suppose that agent i plays $0 < k_t^i < \lambda$ over some interval $[t_1, t_2]$. From the linearity in k_t^i in equation (3.1) and the continuity of U_t^i, it must be the case that $U_t^i = 1 - \alpha$ for all $t \in [t_1, t_2[$. This implies that for all $t \in [t_1, t_2]$,

$$k_{t-\Delta}^j \mathbb{1}_{t \geq \Delta} = \frac{r(1-\alpha)}{\alpha}$$

If $t < \Delta$, this is a contradiction. If $t \geq \Delta$, this implies $k_{t-\Delta}^j = \frac{r(1-\alpha)}{\alpha}$. The assumption that agents $r < \frac{\alpha\lambda}{1-\alpha}$ implies that

$$0 < k_{t-\Delta}^j < \lambda$$

Applying the same argument as for player i, this implies that player j was indifferent among all level of efforts for all $t \in [t_1 - \Delta, t_2 - \Delta]$. Using equation (3.1) for player j, this implies that $k_{t-\Delta}^i \mathbb{1}_{t \geq \Delta} = \frac{r(1-\alpha)}{\alpha}$ for all $t \in [t_1 - 2\Delta, t_2 - 2\Delta]$. If $t < \Delta$, this is a contradiction. If not, we apply the same argument again. We go on recursively until some level $n \geq (t_2 - \Delta)/\Delta$ such that i mixes over the level of effort for all $t \in [t_1 - n\Delta, t_2 - n\Delta]$. For $t \leq t_2 - n\Delta$, the indifference condition $k_t^j \mathbb{1}_{t \geq \Delta} = \frac{r(1-\alpha)}{\alpha}$ cannot be satisfied. $\square$

The fact that players make either the maximal or the minimal level of effort implies that player i necessarily is in one of the following four phases at any time $t \geq \Delta$:

1. an **exploration phase** (E), in which he makes an effort while learning nothing from j: $k_t^i = \lambda, k_{t-\Delta}^j = 0$;
2. a **waiting phase** (W), in which he makes no effort and learns nothing from j: $k_t^i = 0, k_{t-\Delta}^j = 0$;
3. a **free-riding phase** (F), in which he makes no effort and learns information from j: $k_t^i = 0, k_{t-\Delta}^j = \lambda$;
4. an **joint effort phase** (J), in which he makes an effort and learns information from j: $k_t^i = \lambda, k_{t-\Delta}^j = \lambda$.

Now we show that a player's payoff increases with time when he does not learn information from his opponent (namely in experimentation and waiting phases), and decreases otherwise (namely in free-riding and joint-effort phases). Intuitively, phases E and W are lean periods in terms of information acquisition: as time goes by, the perspective of catching up good news gets closer. On the other hand, players benefit from free information in phases J and F, and their payoff decreases as lean phases approach.

Lemma 3. *A player's payoff is non-decreasing in E- and W-phases, and non-increasing in F- and J-phases.*

Consider the phases associated to player i. Let us first notice that, within each phase, j's past action is constant. It follows that i's payoff is differentiable in each phase, and that equation (3.1) rewrites

$$rU_t^i = \max_{k_t^i \in \{0,\lambda\}} \left\{ k_t^i(1-\alpha-U_t^i) + (1-U_t^i)k_{t-\Delta}^j + U_t^{i\prime} \right\}$$

In a E-phase, $k_t^i = \lambda$ and $k_{t-\Delta}^j = 0$. The E-phase payoff to player i thus satisfies $rV_t^E = (1-\alpha-V_t^E)\lambda + V_t^{E\prime}$, that is

$$V_t^E = \frac{\lambda(1-\alpha)}{\lambda+r} + e^{t(\lambda+r)}C^E, \tag{3.2}$$

where C^E is a constant of integration. In a W-phase, $k_t^i = k_{t-\Delta}^j = 0$. Thus the W-phase payoff satisfies $rV^W = V_t^{W\prime}$, that is

$$V_t^W = e^{tr} C^W, \tag{3.3}$$

where C^W is a constant of integration.

In both of these phases, i learns nothing from j but can stop effort whenever it is not optimal. Therefore, V_t^E and V_t^W must be greater than the payoff he would get, was he playing λ forever after, with player j making no effort, which is the single-player payoff $\int_0^\infty e^{-(r+\lambda)t}\lambda(1-\alpha)dt = \frac{\lambda(1-\alpha)}{\lambda+r}$. Inequalities $V_t^E \geq \frac{\lambda(1-\alpha)}{\lambda+r}$ and $V_t^W \geq \frac{\lambda(1-\alpha)}{\lambda+r}$ imply that constants of integration C^E and C^W are non-negative, and that i's payoff is non-decreasing in both phases.

In a F-phase, $k_t^i = 0$ and $k_{t-\Delta}^j = \lambda$. Thus the F-phase payoff satisfies $rV^F = (1 - V_t^F)\lambda + V_t^{F\prime}$, that is

$$V^F = \frac{\lambda}{\lambda + r} + e^{t(\lambda+r)} C^F, \tag{3.4}$$

with C^F a constant of integration. Since j may stop effort in the future, i would be better-off if j was constrained to play λ forever after. In this case, he would get the free-rider payoff $\int_0^\infty e^{-(r+\lambda)t}\lambda dt = \frac{\lambda}{\lambda+r}$. The inequality $V_t^F \leq \frac{\lambda}{\lambda+r}$ implies $C^F \leq 0$ and that i's payoff is non-increasing in a free-riding phase.

Finally, in a J-phase, both players play λ. Player i's payoff satisfies $rV^J = (2 - \alpha - 2V_t^J)\lambda + V_t^{J\prime}$, that is

$$V^J = \frac{(2-\alpha)\lambda}{2\lambda + r} + e^{t(2\lambda+r)} C^J, \tag{3.5}$$

with C^J a constant of integration. For maximal effort to be a best-response for player i, it must be the case that $V_t^J < 1 - \alpha$. Since $r < \frac{\lambda\alpha}{1-\alpha}$, this inequality implies $C^J < 0$ and that i' payoff decreases in a J-phase.

We now turn to the equilibrium analysis.

3.2 Symmetric equilibrium

The aim of this section is to show that, when $r < \frac{\alpha\lambda}{1-\alpha}$, there is a unique symmetric equilibrium, which exhibits a first E-phase followed by the infinite repetition of W-F-J-E.

Equilibrium cycles are *regular*, in the sense that, except for the first E-phase, the length of each phase is the same in any cycle.

The cyclical feature of equilibrium comes from the fact that no phase can have an infinite duration. Indeed, at a symmetric equilibrium, an infinite phase would be either a perpetual J or a perpetual W. Imagine a perpetual W-phase starting at t_0. The continuation value in t_0 would be 0. Clearly, 0 is smaller than $1 - \alpha$ and exerting no effort cannot be a best-response for player i to no more information from j. Imagine now a perpetual J-phase starting at t_0. The continuation value is $\frac{\lambda(2-\alpha)}{2\lambda+r}$. Yet the condition $r < \frac{\alpha\lambda}{1-\alpha}$ implies that $\frac{\lambda(2-\alpha)}{2\lambda+r} \geq 1-\alpha$, so that playing λ cannot be a best-response for player i to j playing λ forever.

It follows that the equilibrium effort strategy alternates between *effort intervals*, namely intervals of time in which the player payoff is below $1 - \alpha$, and *rest intervals*, where the player payoff is above $1 - \alpha$. An effort interval is necessarily made up of J-and E-phases. Since the payoff is increasing in the former and decreasing in the latter, a search interval necessarily ends with a E-phase. Analogously, we call *rest interval* an interval of time in which i's payoff is above $1 - \alpha$. For the same reasons, a rest interval is made up of W and F phases, and necessarily ends with a F-phase. It follows that a rest interval preceded by an effort interval necessarily starts with an W-phase, and an effort interval preceded by a rest interval necessarily starts with a J phase.

Now, remark that the equilibrium sequence can start only with a E-phase. Before time Δ, players receive no information from their opponent, hence the first phase can be only a E- or a W-phase. Suppose that the sequence starts with a W. Since no phase has an infinite duration, this first W must stops at some date t_0. Regardless of whether $t_0 < \Delta$ or $t_0 > \Delta$, this first W must be followed by an E phase at a symmetric equilibrium, since there is no delayed information to be learned by players. Yet we know that a W-phase cannot be followed by a E-phase. Thus the first effort interval only contains a E-phase. After the first effort interval, the first rest interval necessarily starts with a W-phase and ends with a F-phase. At a symmetric equilibrium, the number of F-and J-phases depends on the number of preceding effort intervals. Since the first rest interval is preceded by only one effort interval, it can have at most one F-phase. Therefore, the first rest interval is necessarily $W - F$. By the symmetry argument, the second effort interval is preceded by

only one rest interval and can thus contain at most one E-phase. Since it follows a rest interval, it has to start with a J-phase. Therefore, it is necessarily $J-E$. The second rest interval being preceded by only one effort interval, it contains only one F-phase. Since it follows an effort interval, it has to start with a W-phase. Therefore it is necessarily $W-F$. And so on and so forth.

It is now easy to see what the symmetric equilibrium strategy must be. Given that players start the game with an E-phase, then alternate between rest and effort intervals, the symmetric equilibrium strategy can be characterized by a pair of increasing sequences $(\underline{\tau}_n, \bar{\tau}_n)_n$, with $\underline{\tau}_0 = 0$ and $\underline{\tau}_n < \bar{\tau}_n < \underline{\tau}_{n+1} < \infty$ for all n such that

$$k_t = \begin{cases} \lambda \text{ if } t \in [\underline{\tau}_n, \bar{\tau}_n[; \\ 0 \text{ if } t \in [\bar{\tau}_n, \underline{\tau}_{n+1}[\end{cases}$$

We call such an effort strategy a *cyclical strategy.*

The last step is to show that the symmetric equilibrium strategy must be *regular*, in the sense that lengths of effort and rest intervals must be constant. We denote by $\varepsilon_n := \bar{\tau}_n - \underline{\tau}_n$ the length of the n-th effort interval, and by $\rho_{n+1} := \underline{\tau}_{n+1} - \bar{\tau}_n$ the length of the $(n+1)$-th rest interval.

We know that players switch from one level of effort to the other one every time their payoff crosses the $(1-\alpha)$-line, which implies that $U_{\bar{\tau}_{n-1}} = U_{\underline{\tau}_n} = 1-\alpha$ for all $n \geq 1$. This allows to characterize players payoff in any of the four phases of the equilibrium sequence.

Fix some integer $n \geq 1$. The n-th W-phase starts in $\bar{\tau}_{n-1}$. The relation $V^W_{\bar{\tau}_{n-1}} = 1-\alpha$ gives the value of the constant in equation (3.3) and thus the value of the payoff function in the n-th W-phase

$$V_t^W = (1-\alpha)e^{r(t-\bar{\tau}_{n-1})}$$

The n-th F-phase stops in $\underline{\tau}_n$. The relation $V^F_{\underline{\tau}_n} = 1-\alpha$ gives the value of the constant in equation (3.4) and the value of the payoff function in the n-th F-phase

$$V_t^F = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r} e^{(t-\underline{\tau}_n)(\lambda+r)}$$

Players start the n-th J phase in $\underline{\tau}_n$. The relation $V^J_{\underline{\tau}_n} = 1 - \alpha$ gives the value of the constant in equation (3.5) and the value of the payoff function in the n-th J-phase

$$V_t^J = \frac{\lambda(2-\alpha)}{2\lambda + r} - \frac{K}{2\lambda + r} e^{(t-\underline{\tau}_n)(2\lambda+r)}$$

Finally, The (n)-th E-phase stops in $\bar{\tau}_n$. The relation $V^E_{\bar{\tau}_n} = 1-\alpha$ gives the value of the constant in equation (3.2) and thus the value of the payoff function in the (n)-th E-phase

$$V_t^E = \frac{\lambda(1-\alpha)}{\lambda + r} + \frac{r(1-\alpha)}{\lambda + r} e^{(t-\bar{\tau}_n)(\lambda+r)}$$

Within the n-th rest phase, players switch from the n-th W-phase to the n-th F-phase when they starts receiving information from their opponent's previous effort interval, namely in $\underline{\tau}_{n-1} + \Delta$. Continuity of the payoff function at the transition from the W- to the F-phase implies $V^W_{\underline{\tau}_{n-1}+\Delta} = V^F_{\underline{\tau}_{n-1}+\Delta}$, which gives the relation

$$(1-\alpha)e^{r(\underline{\tau}_{n-1}+\Delta-\bar{\tau}_{n-1})} = \frac{\lambda}{\lambda + r} - \frac{K}{\lambda + r} e^{(\underline{\tau}_{n-1}+\Delta-\underline{\tau}_n)(\lambda+r)}$$

Similarly, the player stops receiving information from his opponent in the n-th effort interval in $\bar{\tau}_{n-1} + \Delta$. Continuity of the payoff function at the transition from the J- to the E-phase implies $V^J_{\bar{\tau}_{n-1}+\Delta} = V^E_{\bar{\tau}_{n-1}+\Delta}$ and

$$\frac{\lambda(2-\alpha)}{2\lambda + r} - \frac{K}{2\lambda + r} e^{(\bar{\tau}_{n-1}+\Delta-\underline{\tau}_n)(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda + r} + \frac{r(1-\alpha)}{\lambda + r} e^{(\bar{\tau}_{n-1}+\Delta-\bar{\tau}_n)(\lambda+r)}$$

Rewriting the last two equations in terms of effort and rest interval lengths, we obtain the following system

$$(1-\alpha)e^{r(\Delta-\varepsilon_{n-1})} = \frac{\lambda}{\lambda + r} - \frac{K}{\lambda + r} e^{(\Delta-\rho_n-\varepsilon_{n-1})(\lambda+r)} \tag{3.6}$$

$$\frac{\lambda(2-\alpha)}{2\lambda + r} - \frac{K}{2\lambda + r} e^{(\Delta-\rho_n)(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda + r} + \frac{r(1-\alpha)}{\lambda + r} e^{(\Delta-\rho_n-\varepsilon_n)(\lambda+r)} \tag{3.7}$$

Equation (3.6) implicitly defines ρ_n as a function $f(\varepsilon_{n-1})$ of the $(n-1)$-th effort interval. Analogously, equation (3.7) defines ε_n as a function $g(\rho_n)$ of the n-th rest interval. It follows that the sequence of effort intervals in a symmetric equilibrium is defined by the recurrence relation $\varepsilon_n = g \circ f(\varepsilon_{n-1})$ and some initial condition $\varepsilon_0 = \bar{\tau}_0$.

Basic analysis allows to establish that there exists a unique possible sequence $\{\varepsilon_n\}_n$ of elements in $[0, \Delta]$ satisfying $\varepsilon_n = g \circ f(\varepsilon_{n-1})$. This is the constant sequence $\varepsilon_n = \varepsilon^*$, where $\varepsilon^* \in]0, \Delta[$ is the unique fix point of $g \circ f$. Consequently, the duration of rest intervals is also constant, determined by $\rho_n = \rho^*$ for all n with $\rho^* = f(\varepsilon^*)$.

The next theorem, proved in the appendix using Pontryagin's principle, describes the strategy on the symmetric equilibrium path.

Theorem 1. *There exists a unique symmetric equilibrium, in which, on the equilibrium path, the effort strategy of any player is characterized by the sequence of cutoffs* $\{\underline{\tau}_n, \bar{\tau}_n\}_{n\geq 0}$ *by*

$$k_t^* = \begin{cases} \lambda, \ \forall t \in [\underline{\tau}_n, \bar{\tau}_n) \\ 0, \ \forall t \in [\bar{\tau}_n, \underline{\tau}_{n+1}) \end{cases}$$

with $\underline{\tau}_n = n(\varepsilon+\rho)$ *and* $\bar{\tau}_n = \varepsilon + n(\varepsilon+\rho)$ *for all* $n \geq 0$, *and phase lengths* ε *and* ρ *determined by*

$$(1-\alpha)e^{r(\Delta-\varepsilon)} = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)} \tag{3.8}$$

$$\frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r}e^{(\Delta-\rho)(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)} \tag{3.9}$$

3.2.1 Comparative statics : à trier, sélectionner, interpréter

Let us now derive some comparative statics. A first relevant value is the sum $\varepsilon + \rho$, that we refer to as *cycle length.*

Cycle length

Lemma 4. *In the symmetric equilibrium, the cycle length :*

1. *increases in* Δ. *Precisely,* $0 < \frac{\partial\varepsilon}{\partial\Delta} < 1$, $0 < \frac{\partial\rho}{\partial\Delta} < 1$, *and* $1 < \frac{\partial\varepsilon+\partial\rho}{\partial\Delta}$.

2. *decreases in* α. *Precisely,* $\frac{\partial\varepsilon}{\partial\alpha} < 0$, $\frac{\partial\rho}{\partial\alpha} > 0$, *and* $\frac{\partial\varepsilon+\partial\rho}{\partial\alpha} < 0$.

3. *goes to infinity when* $\Delta \to \infty$, *and to 0 when* $\Delta \to 0$.

4. $\epsilon \to 0$ *and* $\rho \to \Delta$ *when* $r \to 0$.

Payoff variability

Also, it is interesting to know how Δ impacts the variability of players' payoff, measured by the *amplitude of cycles* $\max_t V_t - \min_t V_t$, and the time 0 payoff U_0.

Lemma 5. *In the symmetric equilibrium,*

1. *Players payoff U_0 decreases with Δ.*

2. *The amplitude of cycles increases with Δ.*

Expected time of a success

Another value of interest if the expectation of the random arrival time of a success *in the team*, that we denote by τ.

Let us compute the probability that a breakthrough occurs in time t. If t belongs to a rest phase, this is 0. If t belongs to the n^{th} effort interval, with $n \geq 1$, it is $2\lambda e^{-(n-1)2\lambda\varepsilon - 2\lambda(t-(n-1)(\varepsilon+\rho))}$. The density function of τ is thus, for all $n \geq 1$,

$$f(t) = \begin{cases} 0 & \text{if } t \in [\bar{\tau}_{n-1}, \underline{\tau}_n[\\ 2\lambda e^{-2\lambda(t-(n-1)\rho)} & \text{if } t \in [\underline{\tau}_{n-1}, \bar{\tau}_{n-1}[\end{cases}$$

The expectation of τ thus writes

$$E[\tau] = \int_0^\infty t f(t) dt = \sum_{n=0}^{\infty} \int_{\underline{\tau}_n}^{\bar{\tau}_n} 2\lambda t e^{-2\lambda(t-n\rho)} dt$$

A primitive for $2\lambda t e^{-2\lambda t}$ is $-(t + \frac{1}{2\lambda})e^{-2\lambda t}$. Therefore,

$$E[\tau] = \sum_{n=0}^{\infty} e^{2\lambda n\rho} \left[-(t + \frac{1}{2\lambda}) e^{-2\lambda t} \right]_{\underline{\tau}_n}^{\bar{\tau}_n}$$

Using the fact that $\sum_{k=1}^{\infty} kq^k = \frac{q}{(1-q)^2}$ when $|q| < 1$ and rearranging,

$$E[\tau] = \frac{1}{2\lambda} + \frac{\rho}{e^{2\lambda\varepsilon} - 1}$$

The effect of Δ on $E[\tau]$ in the neighborhood of zero depends on players' discount rate.

Lemma 6. *$E[\tau]$ decreases with Δ in the neighborhood of zero iff $r \leq \frac{\alpha\lambda(3-2\alpha)}{(1-\alpha)(3-\alpha)}$.*

Effort intensity in the cycle

Another interesting parameter is the effort intensity in the cycle, measured as $\frac{\varepsilon}{\varepsilon+\rho}$.

Lemma 7. *$\frac{\varepsilon}{\varepsilon+\rho}$ increases with Δ in the neighborhood of zero iff $r \leq \frac{\alpha\lambda}{(1-\alpha)(3-\alpha)}$.*

Manager's payoff.

The welfare effect of increasing players effort is not straightforward: in one hand, it increases the probability of an early breakthrough. In the other hand, it increases the total effort cost. Imagine for instance a firm with two employees in the same research team. The firm gets 1 once one of the employees completes the projects, and pays each of the employees α per unit of effort. Without delay, employes constantly make effort $\bar{k} = \frac{r(1-\alpha)}{\alpha}$, and the firms' discounted revenue is

$$\int_0^{\infty} 2\bar{k}(1-\alpha)e^{-(2\bar{k}+r)t}dt = \frac{2\bar{k}(1-\alpha)}{2\bar{k}+r}$$

In the symmetric equilibrium with delay, the probability that the project is completed during the $(n+1)$-th effort phase (namely at time $t \in [n(\varepsilon+\rho), n(\varepsilon+\rho)+\varepsilon]$) is $2\lambda e^{-2\lambda n\varepsilon}e^{-2\lambda(t-n(\varepsilon+\rho))}$. The probability that the project is completed during a rest interval is obviously 0. Therefore, the manager equilibrium payoff is

$$W^m = \sum_{n=0}^{\infty} \int_{n(\varepsilon+\rho)}^{n(\varepsilon+\rho)+\varepsilon} 2\lambda(1-\alpha)e^{-(2\lambda+r)t+2\lambda n\rho}dt$$

After easy simplifications, it comes

$$W^m = \frac{2\lambda}{2\lambda+r}\frac{1-e^{-(2\lambda+r)\varepsilon}}{1-e^{-(2\lambda+r)\varepsilon-r\rho}}$$

Lemma 8.

- $\lim_{\Delta\to 0} W^m = \frac{2\bar{k}(1-\alpha)}{2\bar{k}+r}$
- $\lim_{\Delta\to 0} \frac{\partial}{\partial\Delta} W^m > 0$

For the firm, the trade-off between a higher probability of completing the project and a higher total wage cost is positive in the neighborhood of zero. This implies that the firm benefits from imposing some small but positive delay in the information transmission among employees in the team.

3.3 Asymmetric equilibria

In the appendix we provide formal arguments proving that players cannot use cyclical strategies in an asymmetric equilibrium.

The only possible candidates are thus asymmetric equilibria in which one player (the pushover) always exerts the maximal effort from some time on, while the other one (the shirker) never makes any effort and waits for the arrival of the pushover's breakthrough from some other time on. We call this type of equilibrium a Pushover/Shirker equilibrium. Intuitively, if the delay is too large, waiting for a delayed payoff is very costly in terms of discounted payoff, which makes shirking not a good idea.

Let us show that if the delay is small enough, there exist an asymmetric equilibrium, in which the pushover always exert the maximal effort. Whether the shirker exerts effort λ before shirking depends on the magnitude of the delay.

Suppose that the pushover always plays λ. The problem faced by the potential shirker is thus stationary in Δ, since the signal received from the pushover is constant afterwards. Therefore, the best-response at time Δ is still a best-response at any time $t > \Delta$. If λ is a best-response in Δ, the continuation payoff in Δ is

$$\int_{\Delta}^{\infty} e^{-r(t-\Delta)}\lambda(2-\alpha)e^{-2\lambda(t-\Delta)}dt = \frac{\lambda(2-\alpha)}{2\lambda+r}$$

Yet for patient agents, $\frac{\lambda(2-\alpha)}{2\lambda+r} > 1-\alpha$, which implies that λ cannot be a best-response in Δ.

For the same reasons, if playing 0 is a best-response in Δ, it is still the case at any time after Δ, so that the shirker's continuation payoff in Δ from playing $k_\Delta = 0$ is

$$\int_{\Delta}^{\infty} e^{-r(t-\Delta)}\lambda e^{-\lambda(t-\Delta)}dt = \frac{\lambda}{\lambda+r}$$

For patient agents, $\frac{\lambda}{\lambda+r}$ is larger than $1-\alpha$, thus always shirking is a best-response from time Δ on.

Now, what is the shirker's best-response before Δ? Denote by $\bar{s}_0 \in [0, \Delta[$ the time at which the shirker starts making no effort. We know that the shirker's payoff in Δ is the free-rider payoff $\frac{\lambda}{\lambda+r}$. The shirker's payoff in time 0 is thus

$$\int_0^{\bar{s}_0} e^{-rt}e^{-\lambda t}\lambda(1-\alpha)dt + e^{-\lambda \bar{s}_0}e^{-r\Delta}\frac{\lambda}{\lambda+r}$$

Maximizing this expression with respect to $\bar{s}_0$ in $[0, \Delta[$ gives

$$\bar{s}_0 = \max\left\{0, \Delta - \frac{1}{r}\ln(\frac{\lambda}{(\lambda+r)(1-\alpha)}\right\}$$

It follows that if $\Delta \leq \underline{\Delta} := \frac{1}{r}\ln\left[\frac{\lambda}{(1-\alpha)(r+\lambda)}\right]$, the shirker never makes any effort. Always playing λ is indeed a best-response for the push over to this shirker strategy.

If $\Delta > \underline{\Delta}$ however, there exists an optimal value of $\bar{s}_0$ in $]0, \Delta[$. The pushover will thus receive information from his opponent between Δ and $\Delta + \bar{s}_0$, which may modify his incentives to play λ. Let us now study the conditions under which always playing λ is still a best-response for the pushover. The payoff yielded to the pushover by this strategy is as follows.

From time $\bar{s}_0 + \Delta$ on, the pushover never receives information from the shirker, and thus only gets his own signal. His continuation payoff is then

$$U^P_{\bar{s}_0+\Delta} = \frac{\lambda(1-\alpha)}{\lambda+r}$$

Between Δ and $\bar{s}_0 + \Delta$, the pushover receives a J-phase payoff of the form $\frac{(2-\alpha)\lambda}{2\lambda+r} + C^J e^{t(2\lambda+r)}$. Value-matching in $\bar{s}_0 + \Delta$ gives the value of the constant

$$C^J = \left(\frac{\lambda(1-\alpha)}{\lambda+r} - \frac{(2-\alpha)\lambda}{2\lambda+r}\right)e^{-(\bar{s}_0+\Delta)(2\lambda+r)}$$

The pushover payoff in Δ is thus

$$U^P_\Delta = \frac{(2-\alpha)\lambda}{2\lambda+r} - \lambda\frac{\alpha\lambda+r}{(\lambda+r)(2\lambda+r)}e^{-\bar{s}_0(2\lambda+r)}$$

Since U^P_Δ is the maximum value of the Pushover's payoff, $U^P_\Delta < 1-\alpha$ is a necessary and sufficient condition for always playing λ to be a best-response to the shirker's strategy. Plugging the value of U^P_Δ and $\bar{s}_0$ yields the following condition on Δ

$$\Delta < \overline{\Delta} := \underline{\Delta} + \frac{1}{2\lambda+r}\ln\left(\frac{(\alpha\lambda+r)\lambda}{K(\lambda+r)}\right)$$

4 Concluding remarks

4.1 Interpretations of the results

Management interpretation: scrum method We have seen that observation delay, even small, induces players to alternate between phase of hard work and resting phase. The method of scrum in managing product development suggests to organize team work around "time boxed effort", that is intense effort phases with limited duration. Our results suggest that such cyclical patterns arise naturally because information transmission is not instantaneous. Also, the sprint phases are longer as the information transmission is slow.

Theoretical interpretation: Purification Lemma 2 implies that delays rules out indifference. The seminal Aumann's notion of purification according to which mixed strategy are the limit of equilibrium payoffs of a game with incomplete information. Recall that intermediate effort in the symmetric equilibrium without delay relies on player's indifference. We face the same puzzling aspect than mixed strategy: since player are different with any effort level, why do they choose to make an effort level of $\frac{r(1-\alpha)}{\alpha}$ exactly? Our main result may provide a rational. Clearly, players' equilibrium payoffs are converging toward $1-\alpha$ as delay vanish. Then intermediate effort may be understood as the limit of equilibrium payoffs of a game with observation delay in which players alternate between making the maximal effort and making the minimal effort.

4.2 Where do cycles come from?

In this section we try to understand where the cyclical pattern of the symmetric equilibrium comes from.

4.2.1 Cost function

To emphasize that cycles are not driven by the linearity of the cost function, we show here that even with a convex cost function, a positive delay precludes the existence of stationary symmetric equilibria.

Specifically, let us amend the model so that the cost incurred by a player from effort k is $c(k)$, where $c(.)$ is a differentiable, strictly increasing and convex function from $[0, \lambda]$ into $\mathbb{R}_+$, with $c(0) = 0$ and $c'(0) < 1$.

The single player problem is to solve

$$\max_{k\in\mathcal{K}} \int_0^{+\infty} e^{-rt}e^{-\int_0^t k_s ds}(k_t - c(k_t))dt$$

The solution payoff is stationary, and satisfies the Bellman equation:

$$r\bar{V} = \max_k\{k(1-\bar{V}) - c(k)\} \tag{4.1}$$

The first order condition expresses the fact that the marginal effort cost is equal to the marginal opportunity cost of procrastination :

$$c'(\bar{k}) = 1 - \bar{V}$$

Plugging the FOC into equation 4.1 gives the single player solution $\bar{k}$

$$(r+\bar{k})(1-c(\bar{k}) = \bar{k} - c(\bar{k})$$

The assumption $c'(0) < 1$ is necessary and sufficient for $\bar{k}$ to be positive[2]

Let us now turn to the strategic problem, and let us show that, because of the delay, player i's best response to a stationary strategy cannot be stationary. Suppose that j plays a stationary strategy $k_t^j = \kappa^j$. Player i optimal payoff is stationary and satisfies the Bellman equation

$$rV_t^i = \max_{k_t\in[0,\lambda]}(k_t(1-V_t^i) - c(k_t)) + \kappa^j(1-V_t^i)\mathbb{1}_{t\geq\Delta} + V_t^{i'}$$

The FOC of player i's best response problem is the same before or after Δ

$$c'(k_t) = 1 - V_t^i$$

If player i's best response was a stationary strategy κ^i, then i's payoff would also be stationary, and would satisfy two different equations before and after Δ:

$$rV^i = \kappa^i(1-V^i) - c(\kappa^i)$$
$$rV^i = \kappa^i(1-V^i) - c(\kappa^i) + \kappa^j(1-V^i)$$

Since $V^i < 1$, these equilibrium conditions only hold if $\kappa^j = 0$. In this case, player i's best-response is the single player solution $\bar{k} > 0$, and there is therefore no symmetric equilibrium in stationary strategies.

[2]Indeed, let $f(k) := (r+k)c'(k) - c(k) - r$. The convexity of c implies that f is increasing in k. Thus $f(k) = 0$ has a positive root only if $f(0) < 0$, that is if $c'(0) < 1$.

4.2.2 Pure informational delay

In the model, players get payoff 1 the first time they are informed of a breakthrough, either immediately if this breakthrough comes from their own effort, or with delay if it comes from their opponent's effort. Because players discount the future, there is delay both in information and in payoff. In this section, we show that a pure informational delay is enough for cycles to emerge in equilibrium. To do so, we slightly modify the model so that there is no delay in payoff. Precisely, we assume that when a player learns in $t+\Delta$ that his opponent received a breakthrough in t, he quits the game and receives, in $t+\Delta$, the payoff $e^{r\Delta}$, instead of receiving 1. With this specification, there is no difference, in terms of payoffs, between receiving the breakthrough and learning that one's opponent received it: delay is now purely informational.

Now, player i's best-response payoff satisfies

$$V_t = \max_{k\in[0,\lambda]} k(1-\alpha-V_{t+dt})dt + k^j_{t-\Delta}\mathbb{1}_{t\geq\Delta}(e^{r\Delta}-V_{t+dt})dt + (1-rdt)V_{t+dt}$$

The first-order condition is thus unchanged: the optimal strategy depends on whether the continuation payoff is above or below $1-\alpha$.

First, remark than the discount rate threshold above which exerting full effort is a dominant strategy is larger than in the case with delay in payoff. Indeed, always exerting full effort is a dominant strategy for player i if it is a best-response to full effort from player j. In this case, player i's payoff is $\frac{\lambda(1-\alpha+e^{r\Delta})}{r+2\lambda}$, which is smaller than $1-\alpha$ if and only if $K(\Delta) =: K + \lambda(e^{r\Delta}-1) < 0$. Since we assume K to be positive, $K(\Delta)$ is necessarily positive and exerting full effort is not a dominant strategy.

However, the arguments of section 4.2 can be used to show the existence of a unique symmetric equilibrium where players use regular and cyclical strategies. Equilibrium values of effort and rest intervals $\widetilde{\varepsilon}$ and $\widetilde{\rho}$ are then determined by the equations

$$(1-\alpha)e^{r(\Delta-\widetilde{\varepsilon})} = \frac{\lambda}{\lambda+r} - \frac{K(\Delta)}{\lambda+r}e^{(\Delta-\widetilde{\rho}-\widetilde{\varepsilon})(\lambda+r)}$$
$$\frac{\lambda(1-\alpha+e^{r\Delta})}{2\lambda+r} - \frac{K(\Delta)}{2\lambda+r}e^{(\Delta-\widetilde{\rho})(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{(\Delta-\widetilde{\rho}-\widetilde{\varepsilon})(\lambda+r)}$$

Proof. See Appendix. □

4.2.3 No first date

Imagine that there is no first date and 0 is the first date at which the modeler considers the game. The resolution of this problem, in particular the Pontryagin's Principle (see Appendix **??**), is the same as before except that there is no initial condition. Based on this observation, if there is a symmetric equilibrium in which players alternate effort and rest phases an infinite number of times then we must have cutoffs, $\{\underline{\tau}_n, \bar{\tau}_n\}_{n\in\mathbb{Z}}$ given by

$$k_t^* = \begin{cases} \lambda, \ \forall t \in [\underline{\tau}_n, \bar{\tau}_n) \\ 0, \ \forall t \in [\bar{\tau}_n, \underline{\tau}_{n+1}) \end{cases}$$

with $\underline{\tau}_n = n(\varepsilon + \rho)$ and $\bar{\tau}_n = \varepsilon + n(\varepsilon + \rho)$ for all $n \geq 0$, with ε and ρ determined by Equations 3.8 and 3.9. Since the fixed point analysis that allows to prove the existence of such equilibrium does not depend on any initial condition, we can conclude that this strategy profile is still an equilibrium.

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Appendix

A Proofs of Theorems 1 and ??

The proof of theorems 1 1 and **??** is built as follows.

Step 1 We use Pontryagin's principle to characterize the best-response of player i to his opponent's strategy k^j. We show that

Proposition 1 (Player i's best-response to k^j)**.** *Player i's best response to k^j is the strategy $(k_t^*)_t$ defined by*

$$k_t^* = \begin{cases} 0 & \text{if } V_t \geq 1-\alpha \\ \lambda & \text{if } V_t < 1-\alpha \end{cases}$$

where player i's best-response payoff V_t is entirely determined by

- *an initial condition V_Δ;*
- *the dynamics $V_t' - rV_t = -\left[k_t^*(1-\alpha-V_t) + k_{t-\Delta}^j(1-V_t)\mathbb{1}_{t\geq\Delta}\right]$, at any point at which V is differentiable;*

Step 2 We know from Proposition 1 that, in equilibrium, players exert either full effort or no effort at all. It follows that any equilibrium effort strategy k can be characterized by a pair of increasing cutoffs sequences $((\underline{\tau}_n)_n, (\bar{\tau}_n)_n)$ with $\underline{\tau}_0 = 0$, such that $k_t = \lambda$ for $t \in [\underline{\tau}_n, \bar{\tau}_n[$ and $k_t = 0$ for $t \in [\bar{\tau}_n, \underline{\tau}_{n+1}[$. We denote by $\varepsilon_n^i := \bar{\tau}_n^i - \underline{\tau}_n^i$ the length of player i's n-th effort interval, and by $\rho_n^i := \underline{\tau}_{n+1}^i - \bar{\tau}_n^i$ the length of player i's n-th rest interval.

A *cyclical strategy* is an effort strategy containing no "infinite" interval, and a cyclical equilibrium is an equilibrium where both players use cyclical strategies. A *regular strategy* is a particular cyclical strategy in which effort and rest intervals are regular, in the sense that $\varepsilon_n^i = \varepsilon^i$ and $\rho_n^i = \rho^i$ for all n. Accordingly, a *regular equilibrium* is an equilibrium in which players use regular strategies.

In Step 2, we show that the three following assertions are equivalent.

(i) The equilibrium is cyclical;

(ii) The equilibrium is regular;

(iii) The equilibrium is symmetric;

This, in particular, implies that a symmetric equilibrium is necessarily regular, and an asymmetric equilibrium is necessarily non regular.

Step 3 We prove Theorem 1 by showing that there exists a unique symmetric equilibrium, in which players alternate effort and rest intervals of respective lengths ε^* and ρ^*, determined by

$$(1-\alpha)e^{r(\Delta-\varepsilon)} = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)}$$
$$\frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r}e^{(\Delta-\rho)(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)}$$

Let us now turn to the detailed steps of the proof.

A.1 Step 1

The best-response problem of player i is to maximize $u_0^i(k,k^j)$, that is to solve the problem $\mathcal{P}_0(k^j)$

$$\max_{k\in\mathcal{K}} \int_0^\Delta e^{-rt-\int_0^t k_s ds} k_t(1-\alpha)dt + e^{-\int_0^\Delta k_s ds} \int_\Delta^\infty e^{-rt-\int_\Delta^t (k_s+k^j_{s-\Delta})ds}(k_t(1-\alpha)+k^j_{t-\Delta})dt$$

where $\mathcal{K}$ denotes the set of piece-wise continuous functions from $\mathbb{R}_+$ into $[0,\lambda]$.

We denote by $U_0^i(k^j)$ the value of problem $\mathcal{P}_0(k^j)$. Consider the subgame starting at Δ in which player i has not observed a success yet. At time Δ, player i must solve the problem $\mathcal{P}_\Delta(k^j)$

$$\max_{k\in\mathcal{K}} \int_\Delta^\infty e^{-rt}\left(k_t(1-\alpha)+k^j_{t-\Delta}\right)e^{-\int_\Delta^t (k_s+k^j_{s-\Delta})ds}dt$$

We denote by $U_\Delta^i(k^j)$ the value of the problem $\mathcal{P}_\Delta(k^j)$. Since i does not observe the actions of player j, his continuation payoff in time Δ does not depend on his effort choice before Δ. Formally, the optimal effort choice from time Δ on, conditional on having observed no breakthrough, is independent on the optimal effort choice before Δ. The

problem $\mathcal{P}_0(k^j)$ can thus be rewritten as two nested problems, the second depending on the first one only through the continuity of the objective function.

$$U_0^i(k^j) = \max_{\{k_s\}_{s\in[0,\Delta]}} \int_0^\Delta e^{-rt} k_t(1-\alpha)e^{-\int_0^t k_s ds} dt + e^{-\int_0^\Delta k_s ds} U_\Delta^i(k^j)$$

We first solve for $\mathcal{P}_\Delta(k^j)$.

A.1.1 Resolution of $\mathcal{P}_\Delta(k^j)$

With the notation $x_t := e^{-\int_\Delta^t (k_s + k_{s-\Delta}^j)}$, $F(t, x_t, k_t) := (k_t(1-\alpha) + k_{t-\Delta}^j)x_t$, and $f(t, x_t, k_t) := -(k_t + k_{t-\Delta}^j)x_t$, the problem $\mathcal{P}_\Delta(k^j)$ rewrites

$$\begin{aligned} &\max_{k\in\mathcal{K}} \int_\Delta^\infty e^{-rt} F(t, x_t, k_t) dt \\ &w.r.t.\ x_t' = f(t, x_t, k_t) \text{ and } x_\Delta = 1 \end{aligned}$$

Let us first remark that the integral $\int_\Delta^\infty e^{-rt} F(t, x_t, k_t) dt$ is well-defined, and that a solution exists.[3] We solve the problem with the Pontryagin principle. Because player j plays either λ or 0, k^j is not continuous and the implicit price in the Hamiltonian will not be differentiable everywhere. By standard results in this case,[4] the optimal control k^* and the associated trajectory x^* must satisfy the following necessary conditions:

Lemma 9 (Necessary conditions). *If (x^*, k^*) is a solution of $\mathcal{P}_\Delta(k^j)$, there exists a continuous, piecewise continuously differentiable function $V : [\Delta, +\infty[\to \mathbb{R}$ defined by V_Δ and*

(i) At any point at which V is differentiable,[5] $V_t' - rV_t = -H_x(t, x_t^*, k_t^*, V_t)$.

(ii) For all admissible control k, $H(t, x_t^*, k_t, V_t) \leq H(t, x_t^*, k_t^*, V_t)$.

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[3] The integral is well-defined because $F(t, x_t, k_t)$ is bounded by $(2-\alpha)\lambda$. A solution exists because a) f is continuous and Lipschitz in x_t: $|f(t, x_t, k_t) - f(t, x_t', k_t)| \leq 2\lambda|x_t' - x_t|$, b) f is of linear growth at ∞: $f(t, x_t, k_t) \leq C(1 + |x_t|)$, c) the set $\mathcal{K}$ is compact, d) F is continuous in x_t and k_t, and e) the set $N(t, x) := \{f(t, x_t, k_t), k \in \mathcal{K}\}$ is convex for all (t, x) (indeed, $\forall\, k, k' \in \mathcal{K}$, $\forall\, \alpha \in [0, 1]$, $\alpha f(t, x_t, k_t) + (1 - \alpha) f(t, x_t, k_t') = f(t, x_t, \alpha k_t + (1-\alpha)k_t')$ and $\alpha k_t + (1-\alpha)k_t' \in \mathcal{K}$).

[4] See for instance in Halkin (1974) or Michel (1982).

[5] In the sequel, the fact that V_t' is defined only in points ar which V_t is differentiable will be implicit.

where $H(t, x_t, k_t, V_t) := F(t, x_t, k_t) + V_t f(t, x_t, k_t)$ *is the (discounted) Hamiltonian of the problem.*[6]

Remark that the implicit price V_t measures the marginal cost of an increase in x_t. Since x_t is the probability that i has observed no breakthrough by time t, V_t is the equivalent of the continuation payoff of Bellman's Principle at time t. Considering that $H(t, x_t, k_t, V_t) = x_t \left[k_t(1-\alpha-V_t) + k^j_{t-\Delta}(1-V_t)\right]$, the first necessary condition rewrites

$$(i)\ V_t' - rV_t = -\left[k^*_t(1-\alpha-V_t) + k^j_{t-\Delta}(1-V_t)\right]$$

The second condition rewrites (ii) $(k^*_t - k_t)(1-\alpha-V_t) \geq 0$ for all admissible control k. According to it, if $V_t < 1-\alpha$, then k^*_t must be larger than any admissible control, which is true only if $k^*_t = \lambda$. If, on the contrary, $V_t > 1-\alpha$, k^*_t must be smaller than any admissible control, which is true only if $k^*_t = 0$. Therefore, condition (ii) rewrites

$$(ii)\ \begin{cases} k^*_t = \lambda \Leftrightarrow V_t \leq 1-\alpha \\ k^*_t = 0 \Leftrightarrow V_t \geq 1-\alpha \end{cases}$$

We now show that, quite conveniently, conditions (i) and (ii) are also sufficient.

Lemma 10 (**Necessary conditions are also sufficient**). *Let* (x^*, k^*) *satisfying* (i) *and* (ii). *For any admissible pair* (x, k), $\int_0^\infty e^{-rt}F(t, x_t, k_t) \leq \int_0^\infty e^{-rt}F(t, x^*_t, k^*_t)$.

Proof. Consider (x^*, k^*) satisfying (i) and (ii), and let us show that, for all admissible pair (x, k),

$$e^{-rt}F(t, x_t, k_t) - e^{-rt}F(t, x^*_t, k^*_t) \leq \frac{\partial}{\partial t}\left[(x^*_t - x_t)e^{-rt}V_t\right]$$

Indeed, $F(t, x_t, k_t) - F(t, x^*_t, k^*_t) = x_t\left[k_t(1-\alpha) + k^j_{t-\Delta}\right] - x^*_t\left[k^*_t(1-\alpha) + k^j_{t-\Delta}\right]$. Using (i), it is easy to check that

$$x^*_t\left[k^*_t(1-\alpha) + k^j_{t-\Delta}\right] = -x^*_t(V_t' - rV_t) - V_t {x^*_t}'$$

and that

[6] The Hamiltonian of the problem is $\widehat{H}(t, x_t, k_t, p_t) := e^{-rt}F(t, x_t, k_t) + p_t f(t, x_t, k_t)$. For expositional reasons, we set $V_t := e^{rt}p_t$ and use the discounted Hamiltonian $H(t, x_t, k_t, V_t) = e^{rt}\widehat{H}(t, x_t, k_t, p_t)$. The first necessary condition is adapted from the standard one $\dot{p}_t = -\widehat{H}_x(t, x_t, k_t, p_t)$.

$$x_t\left[k_t(1-\alpha)+k^j_{t-\Delta}\right] = -x_t(V_t'-rV_t)-V_tx_t^{'}+x_t(1-\alpha-V_t)(k_t-k_t^*)$$

Using these expressions,

$$F(t,x_t,k_t)-F(t,x_t^*,k_t^*) = (x_t^*-x_t)(V_t'-rV_t)+V_t(x_t^{*'}-x_t')+x_t(1-\alpha-V_t)(k_t-k_t^*)$$

By condition (ii), $(1-\alpha-V_t)(k_t-k_t^*)\leq 0$ for all admissible control k. It follows that

$$F(t,x_t,k_t)-F(t,x_t^*,k_t^*)\leq (x_t^*-x_t)(V_t'-rV_t)+V_t(x_t^{*'}-x_t')$$

Multiplying both hands by e^{-rt}, one finds that

$$e^{-rt}F(t,x_t,k_t)-e^{-rt}F(t,x_t^*,k_t^*)\leq \frac{\partial}{\partial t}\left[(x_t^*-x_t)e^{-rt}V_t\right] \tag{A.1}$$

To prove the result, we now have to integrate A.1 on $[0,+\infty]$. Since V is not C^1 everywhere, some precautions have to be taken. Let $\{T_n\}_{n\geq 1}$ be the set of points at which V is not differentiable. For any $n\geq 2$, it is possible to integrate (A.1) between T_{n-1} and T_n. This straightforwardly gives

$$\int_{T_{n-1}}^{T_n}\left[e^{-rt}F(t,x_t,k_t)-e^{-rt}F(t,x_t^*,k_t^*)\right]dt \leq e^{-rT_n}V_{T_n}(x^*_{T_n}-x_{T_n})-e^{-rT_{n-1}}V_{T_{n-1}}(x^*_{T_{n-1}}-x_{T_{n-1}})$$

Summing over n up to some integer N gives

$$\int_{\Delta}^{T_N}\left[e^{-rt}F(t,x_t,k_t)-e^{-rt}F(t,x_t^*,k_t^*)\right]dt \leq e^{-rT_N}V_{T_N}(x^*_{T_N}-x_{T_N})-e^{-r\Delta}V_{\Delta}(x^*_{\Delta}-x_{\Delta})$$

By definition, $x^*_\Delta = x_\Delta = 1$. Furthermore, because any possible trajectory x_t belongs to $[0,1]$, and because V_t is bounded above by 1 and below by the single player payoff $\frac{\lambda(1-\alpha)}{\lambda+r}$, there exists $M>0$ such that $e^{-rt}|V_t(x_t^*-x_t)|<Me^{-rt}$. Taking $T_N\rightarrow+\infty$ gives the result

$$\int_{\Delta}^{\infty}\left[e^{-rt}F(t,x_t,k_t)-e^{-rt}F(t,x_t^*,k_t^*)\right]dt\leq 0$$

□

Therefore, the solution of $\mathcal{P}_\Delta(k^j)$ is $(k_t^*)_t$ defined by

$$\begin{cases} k_t^* = \lambda \Leftrightarrow V_t < 1-\alpha \\ k_t^* = 0 \Leftrightarrow V_t > 1-\alpha \end{cases}$$

where V_t is a continuous, piecewise differentiable function defined by the initial condition V_Δ and, at any point in which it is differentiable,

$$V_t' - rV_t = -\left[k_t^*(1-\alpha-V_t) + k_{t-\Delta}^j(1-V_t)\right]$$

We now turn to the resolution of the main problem.

A.1.2 Resolution of $\mathcal{P}_0(k^j)$

The value of $\mathcal{P}_0(k^j)$ is

$$U_0^i(k^j) = \max_{\{k_s\}_{s\in[0,\Delta]}} \int_0^\Delta e^{-rt}k_t(1-\alpha)e^{-\int_0^t k_s ds}dt + e^{-\int_0^\Delta k_s ds}e^{-r\Delta}U_\Delta^i(k^j)$$

where $U_\Delta^i(k^j)$ is the value of $\mathcal{P}_\Delta(k^j)$ and does not depend on $(k_s)_{s\le\Delta}$. Before time Δ, player i receives no information from his opponent. Using the argument of the preceding section, his optimal payoff before Δ thus satisfies

$$V_t' - rV_t = -k_t^*(1-\alpha-V_t)$$

Therefore, it is optimal for player i to play either λ or 0, according to whether his payoff is smaller or larger than $1-\alpha$. Furthermore, the fact that V_t is larger than the single player payoff $\frac{\lambda(1-\alpha)}{\lambda+r}$ implies that V_t is increasing on $[0,\Delta]$, and that there exists some cutoff $\bar{\tau}_0^i \in [0,\Delta]$ such that $k_t^* = \lambda$ if $t \le \bar{\tau}_0^i$ and $k_t^* = 0$ otherwise. This cutoff is given by $V_{\bar{\tau}_0^i} = 1-\alpha$. Between $\bar{\tau}_0^i$ and Δ, player i receives no information from his opponent, and makes no effort. His payoff thus satisfies $V_t' = rV_t$. Integrating between $\bar{\tau}_0^i$ and Δ yields

$$V_{\bar{\tau}_0^i} = e^{-r(\Delta-\bar{\tau}_0^i)}V_\Delta$$

Therefore, the initial condition V_Δ, together with the condition from the Bellman equation $V_{\bar{\tau}_0^i} = 1-\alpha$, entirely characterize player i's best-response before Δ.

A.2 Step 2

We first remark that, in any equilibrium, if a player's strategy is cyclical, then his opponent's strategy has to be cyclical too, with effort and rest intervals of both players overlapping as described in the following Lemma.

Lemma 11 (**Overlapping cyclical strategies**). *In a cyclical equilibrium, any effort interval of player i starts in a delayed effort interval of player j, and any rest interval of player i starts in a delayed rest interval of player j. Formally, for all $n \geq 0$, there exist m, m' such that $0 \leq m \leq m'$ and*

$$\underline{\tau}^j_m + \Delta \leq \underline{\tau}^i_n \leq \bar{\tau}^j_m + \Delta$$

$$\bar{\tau}^j_{m'} + \Delta \leq \bar{\tau}^i_n \leq \underline{\tau}^j_{m'+1} + \Delta$$

Proof. • Suppose that player j plays a cyclical strategy. Let us first remark that player i's payoff is non-increasing on delayed effort interval of player j, and non-decreasing on delayed rest intervals of player j.

In a delayed effort interval, $k^j_{t-\Delta} = \lambda$. From Proposition 1, player i's payoff satisfies

$$V'_t = rV_t - k^*_t(1 - \alpha - V_t) - \lambda(1 - V_t)$$

If $V_t < 1 - \alpha$, then $k^*_t = \lambda$ and $V'_t = (r + 2\lambda)V_t - \lambda(2 - \alpha)$. Because $V_t < 1 - \alpha$ and $\alpha\lambda - r(1 - \alpha) \geq 0$, $V'_t \leq 0$.

If $V_t > 1 - \alpha$, then $k^*_t = 0$ and $V'_t = (\lambda + r)V_t - \lambda$. If player j's strategy is cyclical, j's past effort interval is finite, thus i's payoff is smaller than the free-rider payoff $\frac{\lambda}{\lambda+r}$. It follows that $V'_t \leq 0$.

In a delayed rest interval, player i's payoff satisfies

$$V'_t = rV_t - k^*_t(1 - \alpha - V_t)$$

The payoff to player i is always larger than the single player payoff $\frac{\lambda(1-\alpha)}{\lambda+r}$. It follows that $V'_t \geq \frac{(1-\alpha)}{\lambda+r} r(\lambda - k^*_t)$ and is thus positive.

• Now, if player j is playing a cyclical strategy in equilibrium, it has to be the case that his payoff oscillates around the $(1 - \alpha)$- line (Proposition 1), that is infinitely alternates

between increasing and decreasing intervals. Yet it has been proved in the previous lines that if player i's strategy were stationary, player j's payoff would be monotonic. Therefore, player i's strategy is cyclical too.

- Finally, we know that if player i starts an effort interval in $\underline{\tau}^i_n$, then it must be the case that his payoff crosses the $(1-\alpha)$-line in $\underline{\tau}^i_n$ and stays smaller than $1-\alpha$ for all $t \in [\underline{\tau}^i_n, \bar{\tau}^i_n]$. This implies that player i's payoff is locally decreasing around $\underline{\tau}^i_n$, and thus that $\underline{\tau}^i_n$ belongs to an interval in which player i is receiving information from player j. Therefore, there exists m such that $\underline{\tau}^j_m + \Delta \leq \underline{\tau}^i_n \leq \bar{\tau}^j_m + \Delta$. After $\bar{\tau}^j_m + \Delta$, player i's payoff alternates between increasing and decreasing phases according to player j's rest and effort past intervals. As long as his payoff stays in the area below the $(1-\alpha)$-line, player i plays λ. $\bar{\tau}^i_n$ is the first time his payoff crosses the $(1-\alpha)$-line after $\underline{\tau}^i_n$. Since his payoff was smaller than $1-\alpha$ for $t \leq \bar{\tau}^i_n$, his payoff must be locally increasing around $\bar{\tau}^i_n$. Thus $\bar{\tau}^i_n$ belongs to a delayed rest interval of player j. Therefore, there exists m' such that $\bar{\tau}^j_{m'} + \Delta \leq \bar{\tau}^i_n \leq \underline{\tau}^j_{m'+1} + \Delta$. Since this delayed rest interval is posterior to the delayed effort interval to which $\underline{\tau}^i_n$ belongs, $m' \geq m$.

$\square$

Now, let us consider a strategy profile $k = (k^i, k^j)$, and let us show that (i) k is a cyclical equilibrium $\Rightarrow$ (ii) k is a regular equilibrium $\Rightarrow$ (iii) k is a symmetric equilibrium $\Rightarrow$ (i) k is a cyclical equilibrium.

A.2.1 $(i) \Rightarrow (ii)$

Consider $\underline{\tau}^i_n$, $\bar{\tau}^i_n$, and $\underline{\tau}^i_{n+1}$. By Lemma 11, there exist m, m', m'', with $m \leq m' < m''$, such that

$$\underline{\tau}^j_m + \Delta \leq \underline{\tau}^i_n \leq \bar{\tau}^j_m + \Delta$$
$$\bar{\tau}^j_{m'} + \Delta \leq \bar{\tau}^i_n \leq \bar{\tau}^j_{m'+1} + \Delta$$
$$\underline{\tau}^j_{m''} + \Delta \leq \underline{\tau}^i_{n+1} \leq \bar{\tau}^j_{m''} + \Delta$$

By Lemma 11, the behavior of player i's payoff on $[\underline{\tau}^j_m + \Delta, \bar{\tau}^j_{m''} + \Delta]$ can be described as follows : it decreases on $[\underline{\tau}^j_m + \Delta, \bar{\tau}^j_m + \Delta]$ and $[\underline{\tau}^j_{m''} + \Delta, \bar{\tau}^j_{m''} + \Delta]$, and increases on $[\bar{\tau}^j_{m'} + \Delta, \underline{\tau}^j_{m'+1} + \Delta]$. Furthermore,

If $m < m'$, it is non monotonic on $[\bar{\tau}^j_m + \Delta, \bar{\tau}^j_{m'} + \Delta]$, and smaller than $1 - \alpha$.

If $m' + 1 < m''$, it is non monotonic on $[\underline{\tau}^j_{m'+1} + \Delta, \underline{\tau}^j_{m''} + \Delta]$, and larger than $1 - \alpha$.

We denote by f_n the length of the F-phase immediately preceding $\underline{\tau}^i_n$, and by j_n the length of the J-phase immediately following $\underline{\tau}^i_n$. One has $f_n = \underline{\tau}^i_n - (\underline{\tau}^j_m + \Delta)$ and $j_n = \bar{\tau}^j_m + \Delta - \underline{\tau}^i_n$. We also denote by e_n the length of the E-phase immediately preceding $\bar{\tau}^i_n$, and by w_n the length of the W-phase immediately following $\bar{\tau}^i_n$. One has $e_n = \bar{\tau}^i_n - (\bar{\tau}^j_{m'} + \Delta)$ and $w_n = \underline{\tau}^j_{m'+1} + \Delta - \bar{\tau}^i_n$. Finally, we denote by f_{n+1} the length of the F-phase immediately preceding $\underline{\tau}^i_{n+1}$: $f_{n+1} = \underline{\tau}^i_{n+1} - (\underline{\tau}^j_{m''} + \Delta)$.

proof-eps-converted-to.pdf

We first use Proposition 1 to evaluate player i's payoff in $\underline{\tau}^j_m + \Delta$, $\bar{\tau}^j_m + \Delta$, $\bar{\tau}^j_{m'} + \Delta$, $\underline{\tau}^j_{m'+1} + \Delta$, and $\underline{\tau}^j_{m''} + \Delta$.

Player i's best-response on $[\underline{\tau}^j_m + \Delta, \bar{\tau}^i_m + \Delta]$ is

$$\begin{aligned} k^*_t &= 0 \quad \text{if } t \in [\underline{\tau}^j_m + \Delta, \underline{\tau}^i_n[\\ k^*_t &= \lambda \quad \text{if } t \in]\underline{\tau}^i_n, \bar{\tau}^i_m + \Delta[\end{aligned}$$

Because $k^j_{t-\Delta} = \lambda$ on $[\underline{\tau}^j_m + \Delta, \bar{\tau}^j_m + \Delta]$, player i's payoff satisfies

$$\begin{aligned} &(a) \quad rV_t = \lambda(1 - \alpha - V_t) + \lambda(1 - V_t) + V'_t \quad &&\text{if } t \in]\underline{\tau}^i_n, \bar{\tau}^j_m + \Delta[\\ &(b) \quad V_t = 1 - \alpha &&\text{if } t = \underline{\tau}^i_n \end{aligned}$$

Integrating (a) between $\underline{\tau}^i_n$ and $\bar{\tau}^j_m + \Delta$, and using (b) yields

$$V_{\bar{\tau}^j_m+\Delta} = \frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r}e^{(2\lambda+r)j_n}$$

Furthermore, player i's best-response on $[\bar{\tau}^j_{m'}+\Delta, \underline{\tau}^i_{m'+1}+\Delta]$ is

$$\begin{aligned} k^*_t &= \lambda \quad \text{if } t \in [\bar{\tau}^j_{m'}+\Delta, \bar{\tau}^i_n[\\ k^*_t &= 0 \quad \text{if } t \in]\bar{\tau}^i_n, \underline{\tau}^i_{m'+1}+\Delta[\end{aligned}$$

Because $k^j_{t-\Delta} = 0$ on $[\bar{\tau}^j_{m'}+\Delta, \underline{\tau}^j_{m'+1}+\Delta]$, player i's payoff satisfies

$$\begin{aligned} &(c) \quad rV_t = \lambda(1-\alpha-V_t)+V'_t && \text{if } t \in [\bar{\tau}^j_{m'}+\Delta, \bar{\tau}^i_n[\\ &(d) \quad rV_t = V'_t && \text{if } t \in]\bar{\tau}^i_n, \underline{\tau}^i_{m'+1}+\Delta[\\ &(e) \quad V_t = 1-\alpha && \text{if } t = \bar{\tau}^i_n \end{aligned}$$

Integrating (c) between $\bar{\tau}^j_{m'}+\Delta$ and $\bar{\tau}^i_n$ and using (e) yields

$$V_{\bar{\tau}^j_{m'}+\Delta} = \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{-(\lambda+r)e_n}$$

Integrating d between $\bar{\tau}^i_n$ and $\underline{\tau}^i_{m'+1}+\Delta$ and using (e) yields

$$V_{\underline{\tau}_{m'+1}+\Delta} = (1-\alpha)e^{rw_n}$$

Finally, player i's best-response on $[\underline{\tau}^j_{m''}+\Delta, \underline{\tau}^i_{n+1}]$ is to play 0 while $k^j_{t-\Delta} = \lambda$. His payoff on this interval thus satisfies

$$\begin{aligned} &(f) \quad rV_t = \lambda(1-V_t)+V'_t && \text{if } t \in]\underline{\tau}^j_{m''}+\Delta, \underline{\tau}^i_{n+1}[\\ &(g) \quad V_t = 1-\alpha && \text{if } t = \underline{\tau}^i_{n+1} \end{aligned}$$

Integrating (f) on between $\underline{\tau}^j_{m''}+\Delta$ and $\underline{\tau}^i_{n+1}$ and using (g) yields

$$V_{\underline{\tau}^j_{m''}+\Delta} = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{-(\lambda+r)f_{n+1}}$$

Now we show that player i's payoff in $\bar{\tau}^j_{m'}+\Delta$ is a non-decreasing function of his payoff in $\bar{\tau}^j_m+\Delta$. Also, his payoff in $\underline{\tau}^j_{m''}+\Delta$ is a non-decreasing function of his payoff in $\underline{\tau}^j_{m'+1}+\Delta$:

Claim 1. *There exist $h, g : \mathbb{R} \to \mathbb{R}$ non-decreasing functions such that* $V_{\bar{\tau}^j_{m'}+\Delta} = h\left(V_{\bar{\tau}^j_m+\Delta}\right)$ *and* $V_{\underline{\tau}^j_{m''}+\Delta} = g\left(V_{\underline{\tau}^j_{m'+1}+\Delta}\right)$.

Proof. If $m = m'$, then it is trivially true. If $m < m'$, then player i's payoff exhibits finally many E- and J-phases between $\bar{\tau}^j_m + \Delta$ and $\bar{\tau}^j_{m'} + \Delta$. Let Q be the number of E-phases, and let $\underline{v}_q$ denote player i's payoff at the beginning of the q^{th} E-phase, and $\bar{v}_q$ player i's payoff at the beginning of q^{th} J-phase. With this notation, $\underline{v}_1 = V_{\bar{\tau}_m+\Delta}$ and $\underline{v}_Q = V_{\bar{\tau}_{m'}+\Delta}$. The lengths of the q^{th} E- and J-phases are exactly the lengths of the corresponding rest and effort intervals of player j. With a slight abuse of notation, we denote these lengths by ρ^j_q and ε^j_q.

The relation between player i's payoff at the beginning and at the end of the q^{th} E-phase is obtained by integrating (c) along player j's corresponding rest interval. The relation between player i's payoff at the beginning and at the end of the q^{th} J-phase is obtained by integrating (a) along player j's corresponding effort interval. It is straightforward to show that for all $q \in \{1, \ldots, Q-1\}$, one has

$$\underline{v}_q = \frac{\lambda(1-\alpha)}{\lambda + r}\left[1 - e^{-(\lambda+r)\rho^j_q}\right] + e^{-(\lambda+r)\rho^j_q}\bar{v}_q$$
$$\bar{v}_q = \frac{\lambda(2-\alpha)}{2\lambda + r}\left[1 - e^{-(2\lambda+r)\varepsilon^j_q}\right] + e^{-(2\lambda+r)\varepsilon^j_q}\underline{v}_{q+1}$$

Clearly, $\underline{v}_q$ is an increasing function of $\underline{v}_{q+1}$ for all $q \in \{1, \ldots, Q-1\}$. By induction, $\underline{v}_1$ is an increasing function of $\underline{v}_Q$.

The same reasoning applies between $\underline{\tau}^j_{m'+1} + \Delta$ and $\underline{\tau}^j_{m''} + \Delta$. If $m' + 1 = m''$, then it is trivially true. If $m' + 1 < m''$, then player i's payoff exhibits finally many J- and W-phases between $\underline{\tau}^j_{m'+1} + \Delta$ and $\underline{\tau}^j_{m''} + \Delta$. Suppose that there are Q F- and W-phases between $\underline{\tau}^j_{m'+1} + \Delta$ and $\underline{\tau}^j_{m''} + \Delta$, and let $\bar{v}_q$ denote player i's payoff at the beginning of the q^{th} F-phase, and $\underline{v}_q$ player i's payoff at the beginning of the q^{th} W-phase. With this notation, $\bar{v}_1 = V_{\underline{\tau}_{m'+1}+\Delta}$ and $\bar{v}_Q = V_{\underline{\tau}_{m''}+\Delta}$. The lengths of the q^{th} F- and W-phases are exactly the lengths of the corresponding effort and rest intervals of player j. Again, with a slight abuse of notation, we denote them by ε^j_q and ρ^j_q.

The relation between player i's payoff at the beginning and at the end of the q^{th} F-phase is obtained by integrating (f) along player j's corresponding effort interval. The relation between player i's payoff at the beginning and at the end of the q^{th} W-phase is obtained by

integrating (d) along player j's corresponding rest interval. It is straightforward to show that for all $q \in \{1, \ldots, Q-1\}$, one has

$$\overline{v}_q = \frac{\lambda}{\lambda + r}\left[1 - e^{-(\lambda+r)\varepsilon_q^j}\right] + e^{-(\lambda+r)\varepsilon_q^j}\underline{v}_q$$

$$\underline{v}_q = e^{-r\rho_q^j}\overline{v}_{q+1}$$

Here again, it is straightforward to conclude that $\bar{v}_1$ is an increasing function of $\bar{v}_Q$.

□

We are now able to establish non ambiguous relations between f_n, j_n, e_n, w_n, and f_{n+1}: First, by definition,

$$\begin{cases} f_n + j_n & = \bar{\tau}_m^j - \underline{\tau}_m^j \\ e_n + w_n & = \underline{\tau}_{m'+1}^j - \bar{\tau}_{m'}^j \end{cases}$$

such that f_n decreases with j_n and e_n decreases with w_n.

Second, by Claim 1,

$$\begin{cases} \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{-(\lambda+r)e_n} & = g\left(\frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r}e^{(2\lambda+r)j_n}\right) \\ \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{-(\lambda+r)f_{n+1}} & = h\left((1-\alpha)e^{rw_n}\right) \end{cases}$$

Because h and g are non-decreasing, it is straightforward to establish the following result:

Claim 2.

- f_{n+1} *is an increasing function of* f_n*;*
- j_{n+1} *is an increasing function of* j_n*;*
- e_{n+1} *is an increasing function of* e_n*;*
- w_{n+1} *is an increasing function of* w_n*;*

Proof. Standard. □

This result has the following implications. If $f_n \neq f_{n+1}$ for some n, then the sequence $(f_n)_n$ either diverges or goes to 0, and player i's strategy is not cyclical. The same argument applies for the sequences $(j_n)_n$, $(e_n)_n$, and $(w_n)_n$. Therefore, a cyclical equilibrium strategy must be regular, with $\varepsilon_n^i = \varepsilon^i$ and $\rho_n^i = \rho^i$ for all $n \geq 0$.

A.2.2 $(ii) \Rightarrow (iii)$

By Lemma 11, cyclical equilibrium strategies overlap as follows: for all n, there exist m, m', and m'' with $m \leq m' < m''$, such that

$\underline{\tau}^j_m + \Delta < \underline{\tau}^i_n < \bar{\tau}^j_m + \Delta$

$\bar{\tau}^j_{m'} + \Delta < \bar{\tau}^i_n < \underline{\tau}_{m'+1} + \Delta$

$\underline{\tau}^j_{m''} + \Delta < \underline{\tau}^i_{n+1} < \bar{\tau}^j_{m''} + \Delta$

- We first show that regularity implies $m = m' = n - 1$ and $m'' = n$.

Suppose that $m < m'$. Then player i's payoff exhibits finally many J- and E- phases between $\underline{\tau}^i_n$ and $\bar{\tau}^i_n$. From the proof of Claim 1, we know that for all q, player i's payoff at the beginning of the q^{th} E-phase is an increasing function of his payoff at the end of the q^{th} E-phase.

Let us now show that regularity implies that $\underline{v}_2 < \underline{v}_1$. Payoff values at the end of the first J- and E-phases are defined by

$$\underline{v}_1 = (1-\alpha)e^{(2\lambda+r)(\bar{\tau}^j_m+\Delta-\underline{\tau}^i_n)} + \frac{\lambda(2-\alpha)}{2\lambda+r}\left(1-e^{(2\lambda+r)(\bar{\tau}^j_m+\Delta-\underline{\tau}^i_n)}\right)$$

$$\underline{v}_2 = \bar{v}_1 e^{(2\lambda+r)\varepsilon^j} + \frac{\lambda(2-\alpha)}{2\lambda+r}\left(1-e^{(2\lambda+r)\varepsilon^j}\right)$$

Therefore, one has

$$\underline{v}_2 - \underline{v}_1 = \left(\frac{\lambda(2-\alpha)}{2\lambda+r} - (1-\alpha)\right)e^{(2\lambda+r)(\bar{\tau}^j_m+\Delta-\underline{\tau}^i_n)} - \left(\frac{\lambda(2-\alpha)}{2\lambda+r} - \bar{v}_1\right)e^{(2\lambda+r)\varepsilon^j}$$

If $m < m'$, then $\bar{v}_1 < (1-\alpha)$. Furthermore, regularity implies that $\bar{\tau}^j_m + \Delta - \underline{\tau}^i_n < \varepsilon^j$. It follows that $\underline{v}_2 < \underline{v}_1$. This, together with the fact that $\underline{v}_{q+1}$ is an increasing function of $\underline{v}_q$, implies that $(v_q)_q$ is a decreasing sequence, which contradicts the fact that there are finitely many E- and J-phases between $\underline{\tau}^i_n$ and $\bar{\tau}^i_n$. Therefore, $m = m'$.

Applying exactly the same arguments to the alternating W- and F- phases on $[\bar{\tau}^i_n, \underline{\tau}^i_{n+1}]$ yields to $m' + 1 = m''$.

Finally, because of the first date, m is necessarily equal to $n - 1$.

• Plugging these findings into the results of section A.2.1, one has that player i's best-response is entirely inductively determined by $\bar{\tau}_0^i$ and, for all $n \geq 0$, by the relations

$$\frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r} e^{(\lambda+r)(\bar{\tau}_n^j+\Delta-\bar{\tau}_{n+1}^i)} = \frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r} e^{(2\lambda+r)(\bar{\tau}_n^j+\Delta-\underline{\tau}_{n+1}^i)} \tag{A.2}$$

$$\frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r} e^{(\lambda+r)(\underline{\tau}_n^j+\Delta-\underline{\tau}_{n+1}^i)} = (1-\alpha) e^{r(\underline{\tau}_n^j+\Delta-\bar{\tau}_n^i)} \tag{A.3}$$

Regularity implies that $\underline{\tau}_n^l = n(\varepsilon^l + \rho^l)$ and $\bar{\tau}_n^i = n(\varepsilon^l + \rho^l) + \varepsilon^l$ for $l = i, j$. Plugging this into, for instance, equation A.3 yields

$$\frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r} e^{(\lambda+r)(n(\varepsilon^j+\rho^j-\varepsilon^i-\rho^i)+\Delta-\varepsilon^i-\rho^i)} = (1-\alpha) e^{r(n(\varepsilon^j+\rho^j-\varepsilon^i-\rho^i)+\Delta-\varepsilon^i)}$$

Since it must be true for all n, it must be the case that

$$\varepsilon^j + \rho^j = \varepsilon^i + \rho^i$$

Equation A.3 rewrites

$$\frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r} e^{(\lambda+r)(\Delta-\varepsilon^i-\rho^i)} = (1-\alpha) e^{r(\Delta-\varepsilon^i)}$$

Therefore, the cycle length $\varepsilon^i + \rho^i$ is implicitly defined as a strictly decreasing function of ε^i, that we shall denote $f(\varepsilon^i)$. Since this is true if the role of the two players are reversed, one has

$$\varepsilon^i + \rho^i = f(\varepsilon^i)$$

$$\varepsilon^j + \rho^j = f(\varepsilon^j)$$

which implies that $\varepsilon^i = \varepsilon^j$ and $\rho^i = \rho^j$.

A.2.3 $(iii) \Rightarrow (i)$

Consider some non-cyclical strategy $\widetilde{k}$ and suppose that both players play $\widetilde{k}$ in equilibrium. By definition of cyclicality, $\widetilde{k}$ contains an "infinite" interval, that is there exists some time $\widetilde{\tau}$ from on which players make a constant action.

There are two possible cases. If this constant action is λ, then from time $\widetilde{\tau}+\Delta$ on, both players exert full effort and receive information from their opponent: they enter in a infinite J-phase, and their continuation payoff in $\widetilde{\tau}+\Delta$ is $\dfrac{\lambda(2-\alpha)}{2\lambda+r}$. This payoff being larger than $1-\alpha$, players have profitable deviations from this strategy.

If the constant action is 0, then from time $\widetilde{\tau}+\Delta$ on, both players exert no effort and receive no information from their opponent: they enter in a infinite W-phase, thus their continuation payoff in $\widetilde{\tau}+\Delta$ is 0. This payoff being smaller than $1-\alpha$, players have profitable deviations from this strategy.

Therefore, a symmetric equilibrium strategy is necessarily cyclical.

A.3 Step 3

We know from Step 2 that a symmetric equilibrium is necessarily regular, so that it is characterized by ε and ρ such that $\underline{\tau}_n^i = n(\varepsilon+\rho)$ and $\bar{\tau}_n^i = n(\varepsilon+\rho)+\varepsilon$ for both players. Plugging this into equations (A.2) and (A.3), one has that equilibrium values must satisfy

$$\frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{(\lambda+r)(\Delta-\rho-\varepsilon)} = (1-\alpha)e^{r(\Delta-\varepsilon)} \tag{A.4}$$

$$\frac{r(1-\alpha)}{\lambda+r}e^{(\lambda+r)(\Delta-\rho-\varepsilon)} = \frac{\lambda(r+\alpha\lambda)}{(2\lambda+r)(\lambda+r)} - \frac{K}{2\lambda+r}e^{(2\lambda+r)(\Delta-\rho)} \tag{A.5}$$

Equation A.4 implicitly defines $\Delta-\rho$ as a function of ε by $\Delta-\rho = f(\varepsilon)$, and equation A.5 implicitly defines ε as a function of $\Delta-\rho$ by $\varepsilon = g(\Delta-\rho)$. Equilibrium values ε and ρ are thus solutions of

$$\begin{cases} g\circ f(\varepsilon) = \varepsilon \\ \rho = \Delta - f(\varepsilon) \end{cases}$$

We use the following Lemma.

Lemma 12. *The function $g\circ f$ has a unique fix point ε^*, which belongs to* $]0,\Delta[$.

Proof. The proof is basic analysis, and uses the fact tat the function $h(x) := g\circ f(x) - x$ is continuous and increasing on $[0,\Delta]$, with $h(0)<0$ and $h(\Delta)>0$.
To prove it, let us first show that 1) $f'(x)>1$, 2) $f(\Delta)=\Delta$, and 3) $f(0)<0$. f is defined by

$$\frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{(\lambda+r)(f(x)-x)} = (1-\alpha)e^{r(\Delta-x)}$$

1. Differentiating with respect to x gives $Ke^{(\lambda+r)(f(x)-x)}(f'(x)-1) = r(1-\alpha)e^{r(\Delta-x)}$, which implies that $f'(x) > 1$ for all x.

2. Taking $x = \Delta$ and rearranging gives $\frac{K}{\lambda+r}e^{(\lambda+r)(f(\Delta)-\Delta)} = \frac{K}{\lambda+r}$, which implies that $f(\Delta) = \Delta$.

3. Taking $x = 0$ gives $\frac{\lambda}{\lambda+r} - (1-\alpha)e^{r\Delta} = \frac{K}{\lambda+r}e^{(r+\lambda)f(0)}$. Since $\frac{\lambda}{\lambda+r} - (1-\alpha)e^{r\Delta} < \frac{K}{\lambda+r}$, this implies $f(0) < 0$.

Let us now show that 1) $g'(x) > 1$, 2) $g(0) = 0$, and 3) $g(\Delta) > \Delta$. g is defined by

$$\frac{r(1-\alpha)}{\lambda+r}e^{(\lambda+r)(y-g(y))} = \frac{\lambda(r+\alpha\lambda)}{(2\lambda+r)(\lambda+r)} - \frac{K}{2\lambda+r}e^{(2\lambda+r)y}$$

1. Differentiating with respect to y gives $r(1-\alpha)e^{(\lambda+r)(y-g(y))}(1-g'(y)) = -Ke^{(2\lambda+r)y}$, which implies that $g'(y) > 1$ for all y.

2. Taking $y = 0$ and rearranging gives $\frac{r(1-\alpha)}{\lambda+r}e^{-(\lambda+r)g(0)} = \frac{r(1-\alpha)}{\lambda+r}$, which implies that $g(0) = 0$.

3. Taking $y = \Delta$ gives $\frac{r(1-\alpha)}{\lambda+r}e^{(\lambda+r)(\Delta-g(\Delta))} = \frac{\lambda(r+\alpha\lambda)}{(2\lambda+r)(\lambda+r)} - \frac{K}{2\lambda+r}e^{(2\lambda+r)\Delta}$.
   Since $\frac{\lambda(r+\alpha\lambda)}{(2\lambda+r)(\lambda+r)} - \frac{K}{2\lambda+r}e^{(2\lambda+r)\Delta} < \frac{r(1-\alpha)}{\lambda+r}$, this implies $g(\Delta) > \Delta$.

Finally, let us show that $h(x)$ is increasing. Differentiation gives $h'(x) = g'(f(x))f'(x) - 1$. The fact that $g' > 1$ and $f' > 1$ implies that $h' > 0$. In addition, $h(0) = g(f(0))$. Since $f(0) < 0$ and g is increasing, $h(0) < g(0)$. Since $g(0) = 0$, $h(0) < 0$. Furthermore, $h(\Delta) = g(f(\Delta)) - \Delta$. Since $f(\Delta) = \Delta$ and $g(\Delta) > \Delta$, $h(\Delta) > \Delta$. By the theorem of intermediate values, there exists a unique value $\varepsilon^* \in]0, \Delta[$ such that $h(\varepsilon^*) = 0$. □

Therefore, there exists a unique pair (ε^*, ρ^*) satisfying the symmetric equilibrium conditions.

A.4 Asymmetric (in progress)

First consider that

Lemma 13. *There is no asymmetric equilibrium in cyclical strategy.*

Proof. In the paper. □

It follows that if (k^1, k^2) is an asymmetric equilibrium, one of the strategy is non cyclical, namely is stationary from some time on. Necessarily, the other strategy must be stationary from some time on too. Now consider that when strategies become stationary, it must be the case that one of the player is playing λ forever, and the other one is playing 0 forever (other cases are straightforwardly impossible). We call such an equilibrium a Shirker/Pushover equilibrium, where the Shirker is the player that makes no effort from some time on, and the Pushover the other player.

We denote by $SP(M, N)$ a Shirker/Pushover equilibrium in which the Shirker has M effort phases, and the Pushover N rest phases. Recall that a player's strategy can be identified with the sequences of cutoffs $(\underline{\tau}^i_n, \bar{\tau}^i_n)_n$, $\underline{\tau}^i_n \leq \bar{\tau}^i_{n\geq 0}$, $\underline{\tau}^i_0 = 0$, where $\underline{\tau}^i_n$ is the beginning of player i's $n+1$ effort interval, and $\overline{\tau}^i_n$ is the beginning of player i's n rest interval.

We denote by τ^P and τ^S the dates from which the pushover and the shirker respectively starts playing λ and 0. If the shirker has $N \geq 1$ effort intervals, then $\tau^S = \bar{\tau}^S_{N-1}$ If $N = 0$, then $\tau^S = 0$. If the pushover has N rest intervals, then $\tau^P = \underline{\tau}^P_N$.

Lemma 14 (Structure of Shirker/Pushover equilibria)**.** *In a $SP(M, N)$, either $M = N$ or $M = N + 1$. Furthermore,*

- *at $SP(N, N)$, $\tau^S < \underline{\tau}^P_{N-1} + \Delta$ and $\underline{\tau}^S_{N-1} < \tau^P < \tau^S + \Delta$*
- *at $SP(N + 1, N)$, $\tau^P < \overline{\tau}^S_{N-1} + \Delta$ and $\overline{\tau}^P_{N-1} + \Delta < \tau^S < \tau^P + \Delta$*

Proof. Consider a $SP(M, N)$. If the shirker has M effort intervals, then his payoff must cross the $(1-\alpha)$-line while being locally increasing in $\bar{\tau}^S_0, \bar{\tau}^S_1, \ldots, \tau^S$, that is M times. The first time occurs before Δ, and thus does not correspond to a rest interval of the Pushover. The other $(M - 1)$ times occur during a past rest interval of the Pushover. Therefore, the Pushover must have experienced at least $M - 1$ rest intervals, and thus $N \geq M - 1$.

Now, if the Pushover has N rest intervals, then his payoff must cross the $(1 - \alpha)$-line while being locally decreasing in $\underline{\tau}^P_1, \underline{\tau}^P_2, \ldots, \tau^P$, namely N times. All this dates occur during a past effort interval of the shirker. Therefore, the Shirker must have experienced at least N effort intervals, and thus $M \geq N$.

Putting this together, $M \geq N \geq M-1$.

Now consider a $SP(N,N)$. We now that τ^S occurs in a past rest interval of the pushover, thus there exists m such that $\bar{\tau}^P_m + \Delta < \tau^S < \underline{\tau}^P_{m+1} + \Delta$. We also know that τ^P occurs in a past effort interval of the shirker, thus there exists n such that $\underline{\tau}^S_n + \Delta < \tau^P < \bar{\tau}^S_n + \Delta$.

First, remark that it cannot be that $\bar{\tau}^S_n = \tau^S$ and $\underline{\tau}^P_{m+1} = \tau^P$. If it were the case, that τ^S and τ^P would be determined by

$$\tau^S = \tau^P - \Delta + \widetilde{\Delta}$$

$$\tau^P = \tau^D - \Delta + \underline{\Delta}$$

which cannot be generically true.

Now, in a $SP(N,N)$, rest and effort intervals of the two players must strictly overlap according to the pattern $\underline{\tau}^i_n < \bar{\tau}^i_{n-1} + \Delta < \bar{\tau}^i_n < \underline{\tau}^j_n + \Delta$... except once (otherwise it would imply $N+1$ effort intervals for the shirker, or $N+1$ rest intervals for the pushover. Because of the previous remark, the must be around the stopping time of one of the player.

Suppose that $\bar{\tau}^P_{N-1} + \Delta < \tau^S < \tau^P + \Delta$. Necessarily, the pushover has $N-1$ rest intervals. If, however, $\underline{\tau}^S_{N-1} < \tau^P < \tau^S + \Delta$, then the shirker has N rest intervals. □

Now work on $SP(N,N)$ and $SP(N+1,N)$ equilibria.

Lemma 15 (Yet a conjecture for $n \geq 1$)**.** *There exist a strictly increasing sequence of real cutoffs* $(\Delta_n)_{n\geq 0}$ *such that* $\Delta_0 = 0$ *and*

- $\Delta \in [\Delta_{2n}, \Delta_{2n+1}] \Leftrightarrow$ *there exists a unique asymmetric equilibrium, which is a* $SP(n,n)$.
- $\Delta \in [\Delta_{2n+1}, \Delta_{2(n+1)}] \Leftrightarrow$ *there exists a unique asymmetric equilibrium, which is a* $SP(n+1,n)$.

Proof. **Sketch**

- At a $SP(n,n)$,
  - Show that, at a $SP(n,n)$, conditional on τ^S and τ^P, there exist unique truncations of the shirker and the pushover's strategies $(\bar{\tau}^S_0, \underline{\tau}^S_1, \ldots, \underline{\tau}^S_{n-1})$ and $(\bar{\tau}^P_0, \underline{\tau}^S_1, \ldots, \underline{\tau}^P_{n-1}, \bar{\tau}^P_{n-1})$ that are mutual best-responses.

– At equilibrium, τ^S and τ^P are determined by

$\tau^P = \tau^S + \Delta - \widetilde{\Delta}$

$\tau^S = f(\tau^P, \bar{\tau}^P_{n-1}, \underline{\tau}^P_{n-1})$

Show that there exists a unique solution.

- Same kinf of reasonning for a $SP(n+1,n)$, except that then, the shirker has more cutoffs than the pushover.

□

B Comparative statics

Recall that equilibrium values for effort and rest intervals are determined by the system

$$(1-\alpha)e^{r(\Delta-\varepsilon)} = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)}(:= V_\Delta) \tag{B.1}$$

$$\frac{\lambda(2-\alpha)}{2\lambda+r} - \frac{K}{2\lambda+r}e^{(\Delta-\rho)(2\lambda+r)} = \frac{\lambda(1-\alpha)}{\lambda+r} + \frac{r(1-\alpha)}{\lambda+r}e^{(\Delta-\rho-\varepsilon)(\lambda+r)}(:= V_{\varepsilon+\Delta}) \tag{B.2}$$

We shall use the notation $x := \varepsilon + \rho$ and $Z := e^{(\Delta-x)(\lambda+r)}$.

Proof of Lemma 4. 1. Differentiating Equations B.1 and B.2 with respect to Δ gives the system

$$r(1-\alpha)e^{r(\Delta-\varepsilon)}(1-\varepsilon') = -Ke^{(\Delta-\rho-\varepsilon)(\lambda+r)}(1-\varepsilon'-\rho') \tag{B.3}$$

$$-Ke^{(\Delta-\rho)(2\lambda+r)}(1-\rho') = r(1-\alpha)e^{(\Delta-\rho-\varepsilon)(\lambda+r)}(1-\varepsilon'-\rho') \tag{B.4}$$

With the notation

$A := \frac{K}{r(1-\alpha)}e^{(\lambda+r)(\Delta-\varepsilon-\rho)-r(\Delta-\varepsilon)}$

$B := \frac{r(1-\alpha)}{K}e^{(\lambda+r)(\Delta-\varepsilon-\rho)-(2\lambda+r)(\Delta-\rho)}$

one finds that

$$\varepsilon' = \frac{1+B}{1+A+B}$$

$$\rho' = \frac{1+A}{1+A+B}$$

Since A and B are positive parameters, one straightforwardly finds that

$0 < \varepsilon' < 1$, $0 < \rho' < 1$, $1 < \varepsilon' + \rho'$.

2. Differentiating equations B.1 and B.2 with respect to α gives the following equations

$$\varepsilon' \lambda(1 - V_\Delta) = \frac{\lambda}{K}(1 - \frac{V_\Delta}{1-\alpha}) - KZ\rho' \tag{B.5}$$

$$\rho' \lambda(1 - V_{\varepsilon+\Delta}) = \frac{\lambda}{K}(1 - \frac{V_{\varepsilon+\Delta}}{1-\alpha}) - r(1-\alpha)Z\varepsilon' \tag{B.6}$$

The system of equations (B.5) and (B.6) rewrite

$$\begin{cases} \varepsilon' = a + b\rho' \\ \rho' = c + d\varepsilon' \end{cases}$$

with $a := \frac{\lambda}{K}(1 - \frac{V_\Delta}{1-\alpha} / (\lambda(1 - V_\Delta))$, $b := -KZ/(\lambda(1 - V_\Delta))$, $c := \frac{\lambda}{K}(1 - \frac{V_{\varepsilon+\Delta}}{1-\alpha} / (\lambda(1 - V_{\varepsilon+\Delta}))$, and $d := -r(1-\alpha)Z/(\lambda(1 - V_{\varepsilon+\Delta}))$.

The solution to it is

$$\varepsilon'(1 - bd) = a + bc$$
$$\rho'(1 - bd) = c + ad$$

Because $1 - \alpha < V_\Delta < 1$ and $V_{\varepsilon+\Delta} < 1 - \alpha$, a, b, and d are negative and c is positive. Therefore, $a + bc$ is negative and $c + ad$ is positive. Let us determine the sign of $1 - bd$. By definition, $-KZ = (\lambda + r)V_\Delta - \lambda$, thus $b = -1 + \frac{rV_\Delta}{\lambda(1-V_\Delta)}$ and $b > -1$. Furthermore, by definition, $r(1-\alpha)Z = (\lambda + r)V_{\varepsilon+\Delta} - \lambda(1-\alpha)$ thus $d = 1 - \frac{\alpha\lambda + rV_{\varepsilon+\Delta}}{\lambda(1-V_{\varepsilon+\Delta})}$. The fact that $V_{\varepsilon+\Delta} < \frac{\lambda(2-\alpha)}{2\lambda+r}$ implies $-d < 1$. Therefore, $1 - bd > 0$. Putting these findings together, one finds that an increase in α has a negative effect on effort phases, and a positive one on rest phases.

Let us now turn to the overall effect on cycle length. One has

$$\varepsilon' + \rho' = \frac{a(1+d) + b(1+c)}{1-bd}$$

Since $1+d > 0$, and $a(1+d)+b(1+c) < 0$, and the length of cycles decreases with α.

3. We know that both the effort and the rest intervals increase with the delay. When Δ goes to $+\infty$, four cases are possible: there exist $\bar{\varepsilon}$ and $\bar{\rho}$ such that $\varepsilon \to \bar{\varepsilon}$ and $\rho \to \bar{\rho}$, or $\varepsilon \to +\infty$ and $\rho \to \bar{\rho}$, or $\rho \to +\infty$ and $\varepsilon \to \bar{\varepsilon}$, and both durations diverge. Let us show that the first three cases are impossible. Multiplying both hands of equation (B.3) by $e^{-(\lambda+r)\Delta}$ gives:

$$(1-\alpha)e^{-\lambda\Delta - r\varepsilon} = \frac{\lambda}{\lambda+r}e^{-(\lambda+r)\Delta} - \frac{K}{\lambda+r}e^{-(\lambda+r)(\rho+\varepsilon)}$$

If both durations converge, limits would satisfy $0 = e^{-(\lambda+r)(\bar{\rho}+\bar{\varepsilon})}$, which is impossible.

Now, combining (B.3) and (B.4) gives

$$(1-\alpha)\left[1 - e^{r(\Delta-\varepsilon)}\right] = \frac{K^2}{r(1-\alpha)(2\lambda+r)}\left[1 - e^{(\Delta-\rho)(2\lambda+r)}\right]$$

which rewrites

$$e^{(\Delta-\rho)(2\lambda+r)} = a + be^{r(\Delta-\varepsilon)}$$

with $a, b > 0$.

First, multiplying both hands by $e^{-(2\lambda+r)\Delta}$ allows to show that $\rho \to +\infty$. Then, incorporating this expression of $e^{(\Delta-\rho)(2\lambda+r)}$ into (B.3) gives

$$(1-\alpha)e^{r(\Delta-\varepsilon)} = \frac{\lambda}{\lambda+r} - \frac{K}{\lambda+r}e^{-\varepsilon(\lambda+r)}\left[a + be^{r(\Delta-\varepsilon)}\right]^{\frac{\lambda+r}{2\lambda+r}}$$

Again, multiplying both hands by $e^{-r\Delta}$ allows to state that $\varepsilon \to +\infty$.

$\square$

Proof of Lemma 5.

1. By definition, $U_0 = \int_0^{\varepsilon} \lambda e^{-(\lambda+r)t}(1-\alpha)dt + e^{-(\lambda+r)\varepsilon}(1-\alpha)$.

   Because ε increases with Δ, U_0 straightforwardly decreases with Δ.

2. Because equilibrium cycles are regular, the amplitude of cycles is defined by the distance $V_\Delta - V_{\varepsilon+\Delta}$. From equations (B.3) and (B.4), one has

$$V_\Delta - V_{\varepsilon+\Delta} = \frac{\alpha\lambda}{\lambda+r}(1 - e^{(\lambda+r)(\Delta-\varepsilon-\rho)})$$

Differentiating this expression with respect to Δ gives

$$\frac{\partial(V_\Delta - V_{\varepsilon+\Delta})}{\partial\Delta} = -\alpha\lambda(1-\varepsilon'-\rho')e^{(\lambda+r)(\Delta-\varepsilon-\rho)}$$

Because $1-\varepsilon'-\rho' < 0$ is negative, the amplitude of cycles increases with the delay.

□

B.1 Proofs of Lemmas 6, 7, 8

Preamble

Recall the notation

$$A := \frac{K}{r(1-\alpha)} e^{(\lambda+r)(\Delta-\rho-\varepsilon)-r(\Delta-\varepsilon)}$$

$$B := \frac{r(1-\alpha)}{K} e^{(\lambda+r)(\Delta-\rho-\varepsilon)-(2\lambda+r)(\Delta-\rho)}$$

One has

$$\varepsilon' = \frac{1+B}{1+A+B}$$

and

$$\rho' = \frac{1+A}{1+A+B}$$

$$\varepsilon'' = \frac{A}{(1+A+B)^3}\left[\lambda + r + rA(1+B) - B^2(2\lambda+r)\right]$$

$$\rho'' = \frac{B}{(1+A+B)^3}\left[\lambda + r + (2\lambda+r)B(1+A) - rA^2\right]$$

For $X \in \{\varepsilon, \rho, A, B, \varepsilon', \rho', \varepsilon'', \rho''\}$, we denote by X_0 the limit value of X when Δ goes to 0. One has $\varepsilon_0 = \rho_0 = 0$, and $A_0 = \frac{1}{B_0} = \frac{K}{r(1-\alpha)}$.

Proof of Lemma 6. Differentiating $E[\tau]$ with respect of Δ yields

$$\frac{\rho'(e^{2\lambda\varepsilon}-1)-\rho 2\lambda\varepsilon' e^{2\lambda\varepsilon}}{(e^{2\lambda\varepsilon}-1)^2} \tag{B.7}$$

Let n and d respectively denote the numerator and the denominator in B.7. Because they both go to 0 as Δ goes to 0, we have to apply L'Hôpital's rule twice to remove the indeterminacy. Differentiating twice n and d,

$$\lim_{\Delta\to 0} n'' = 2\lambda(\rho_0''\varepsilon_0' - \rho_0'\varepsilon_0'' - 2\lambda\varepsilon_0'^2\rho_0')$$

and

$$\lim_{\Delta\to 0} d'' = 2(2\lambda\varepsilon_0')^2$$

The second expression is positive. The sign of $\frac{\partial E[\tau])}{\partial\Delta}$ in the neighborhood of zero is thus the sign of n'' in 0. Simple calculation boil down to

$$\rho_0''\varepsilon_0' - \rho_0'\varepsilon_0'' - 2\lambda\varepsilon_0'^2\rho_0' = \frac{(1+B_0)^2}{B_0^2(1+A_0+B_0)^3}\left(rB_0^2 - 3\lambda B_0 - r\right)$$

Because $B_0 = \frac{r(1-\alpha)}{K}$, $rB_0^2 - 3\lambda B_0 - r < 0 \Leftrightarrow r < \dfrac{\alpha\lambda(3-2\alpha)}{(1-\alpha)(3-\alpha)}$.

□

Proof of Lemma 7. Differentiating $\frac{\varepsilon}{\varepsilon+\rho}$ with respect to Δ yields

$$\frac{\varepsilon'\rho - \varepsilon\rho'}{(\varepsilon+\rho)^2} \tag{B.8}$$

The numerator and the denominator both go to 0 as Δ goes to 0. We apply L'Hôpital's rule twice to remove the indeterminacy. Let n and d respectively denote the numerator and the denominator in B.8. One has

$n'' = \varepsilon'''\rho + \varepsilon''\rho' - \varepsilon'\rho'' - \varepsilon\rho'''$

$d''' = 2(\varepsilon' + \rho')^2 + 2(\varepsilon+\rho)(\varepsilon'' + \rho'')$

$$\varepsilon_0''\rho_0' - \varepsilon_0'\rho_0'' = \frac{(1+B)^2}{B^2(1+A+B)^3}\left[-B^2(2\lambda + r) + \lambda B + r\right]$$

□

Proof of Lemma 8. Let W^m rewrite $W^m = \dfrac{2\lambda}{2\lambda + r}\dfrac{u}{v}$, with $u = 1 - e^{-(2\lambda+r)\varepsilon}$ and $v = 1 - e^{-(2\lambda+r)\varepsilon - r\rho}$.

Clearly,

$u' = (2\lambda + r)\varepsilon'(1 - u)$

$v' = ((2\lambda + r)\varepsilon' + r\rho')(1 - v)$

$u'' = ((2\lambda + r)\varepsilon'' - (2\lambda + r)^2\varepsilon'^2)(1 - u)$

$v'' = ((2\lambda + r)\varepsilon'' + r\rho'' - [(2\lambda + r)\varepsilon' + r\rho']^2)(1 - v)$

Differentiating $\frac{u}{v}$ with respect to Δ yields $\dfrac{\partial W^m}{\partial \Delta} = \dfrac{u'v - uv'}{v^2}$. The numerator and the denominator both go to 0 when Δ goes to 0. We apply L'Hôpital's rule twice. With $n := u'v - uv'$ and $d := v^2$, one has

$n' = u''v - uv''$

$d' = 2vv'$

$n'' = u'''v + u''v' - u'v'' - uv'''$

$d'' = 2v'^2 + 2vv''$

Because $d'' \to 2v_0'^2$ when $\Delta \to 0$, the sign of $\lim\limits_{\Delta \to 0} \dfrac{\partial W^m}{\partial \Delta}$ is the sign of $u_0''v_0' - u_0'v_0''$, that is the sign of

$$\varepsilon_0''\rho_0' - \varepsilon_0'\rho_0'' + \varepsilon_0'\rho_0' \left[(2\lambda + r)\varepsilon_0' + r\rho_0'\right]$$

Because $\varepsilon_0'\rho_0'^2 r > 0$,

$$\varepsilon_0''\rho_0' - \varepsilon_0'\rho_0'' + \varepsilon_0'\rho_0' \left[(2\lambda + r)\varepsilon_0' + r\rho_0'\right] \geq \varepsilon_0''\rho_0' - \varepsilon_0'\rho_0'' + (2\lambda + r)\varepsilon_0'^2\rho_0'$$

Moreover,

$\varepsilon_0''\rho_0' \leq -(2\lambda + r)\frac{A(1+A)}{(1+A+B)^4}B^2$

$-\rho_0''\varepsilon_0' \geq -(2\lambda + r)\frac{B(1+B)}{(1+A+B)^4}(2 + B)$

$(2\lambda + r)\varepsilon_0'^2\rho_0' = (2\lambda + r)\frac{(1+B)^2(1+A)}{(1+A+B)^3}$

Thus

$$\begin{aligned} \varepsilon_0''\rho_0' - \rho_0''\varepsilon_0' + (2\lambda + r)\varepsilon_0'^2\rho_0' &\geq \tfrac{(2\lambda+r)}{(1+A+B)^4}\left[-A(1 + A)B^2 - B(1 + B)(2 + B) + (1 + B)^2(1 + A)(1 + A + B\right. \\ &\geq 0 \end{aligned}$$

$\square$