Competitive Screening under Heterogeneous Information*

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Abstract

We study competition in price-quality schedules in a setting where consumers privately know their willingness-to-pay for quality (type), and are heterogeneously informed about the offers available in the market. While firms are ex-ante identical, the menus offered in equilibrium are dispersed. Equilibrium menus are ordered so that more generous menus leave more surplus uniformly over types. Firms that offer more generous menus engage in less quality distortions, have larger product lines with lower prices over all products, and derive a greater fraction of their profits from sales to consumers with low willingness-to-pay. In the limit as the number of firms grow large, or as consumers become perfectly informed about the firms' offers, equilibrium menus are efficient for all types.

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1 Introduction

Price discrimination is pervasive in economics. Firms often design sales strategies to obtain large profits from those consumers with higher willingness to pay for quality. Airline companies charge twice more for a little more space and a slightly better service. Car companies often demand a few extra thousand dollars for some luxuries such as leather trims. Amusement parks offer V.I.P. treatment for exorbitant prices. Two conditions are necessary for a firm to successfully engage in such practices. First, there must be different class of consumers with different willingness to pay for quality. Second, the firm should have some degree of market power in order to make larger profits from a certain type of consumer.

In the study of price competition models, economists often assume the existence of a monopolist who enjoys full market power (see the seminal contributions of Mussa and Rosen (1978), Maskin and Riley (1984) and Goldman, Leland and Sibley (1984)). This assumption allows the researcher to understand how different willingness to pay shape optimal price policies. Economist have obtained several important insights and are now able to understand two-part tariffs, optimal offer of menus and the design of quality distortions. The conclusions from these models are often used by competition police experts to set up regulatory norms in several industries.

Nonetheless, the monopolist structure is far from the rule in most industries in which we see price discrimination. Indeed, one can often choose among different airline companies, car producers and TV manufacturers. Moreover, not only oligopoly is rather the norm in several industries in which we see price discrimination. But also, the amount of oligopolistic power varies tremendously across these industries.

In this paper, we write a tractable model to study competition in nonlinear price schedules by firms that face asymmetric information about customers' marginal willingness to pay for some product characteristic. For that, we model competition through sales function. Sales functions determine the revenue that a firm obtains from a particular type of consumer as a function of the level of competition (mass of firms in the market) and attractiveness of the offer made to this type of consumer relative to the other firms. Sales functions are smooth objects that allow us to explicitly solve for optimal sorting strategies in a rather complex environment.

As the mass of firms increases, our equilibrium approaches the Bertrand outcome in which each consumer purchases the socially optimal quality and leaves no profits for the firm. As the firms decreases, our equilibrium approaches the well studies monopoly paradigm studied by Mussa and Rosen (1978). Therefore, our model shows that the Bertrand outcome is robust to a small level of market power and the monopoly outcome is robust to a small amount of competition. More importantly, as we vary the mass of firms in the markets, we continuously span all competition spectrum and are able to show general properties of sales policies as well as obtain new and testable predictions about optimal screening strategies in oligopolistic settings.

Our analysis reveals that equilibrium outcomes involve dispersion in nonlinear price schedules. The firms' choice of schedules balances retention, profit, and efficiency considerations. The analysis of this triadic trade-off illuminates the interplay between competition and incentive constraints (and their associated distortions relative to efficiency). For each customer type, firms trade-off rent extraction and volume of sales. Across consumer types, firms trade-off efficiency and rent extraction (as dictated by incentive constraints). We show that improving the offer of consumers with low valuation is strategically complementary to offering a better deal to consumers with high valuation. These property implies that equilibria involve *sorting in generosity* and *sorting in distortions*. Sorting in generosity implies that firms that offer better deals to consumers with low willingness to pay also leave more rents to consumer with high willingness to pay. Moreover, these more generous firms also sort themselves in distortions. That is, generous the firms offer more efficient qualities to all kind of consumers.

We proceed with the analysis of a more general model in which consumers have a continuum of types. This set up involves new technical challenges. In particular, the equilibrium characterization requires the solution of a partial differential equation subject to nontrivial boundary conditions. We specialize the model assuming quadratic costs of quality provision and a uniform distribution of types. We are then able to obtain an explicit characterization of differentiable equilibria. This exercise provides robustness of our findings in the first section of our paper. More importantly, the continuum of types analysis shows that firms sort themselves in the extensive margin too. That is, some firms are more generous, offer more efficient qualities and sell to consumers with lower valuations.

Our findings are consistent with stores selling the same kind of good engaging in different sales strategies. On one polar case, we have stores that specialize in selling to consumers with high willingness to pay for quality. These stores typically charge higher mark-ups, engage more actively in price discrimination, and obtain a higher fraction of their profits from consumers with high valuations. On the other extreme, we have stores that specialize in selling to more price-sensitive consumers. These stores engage in less quality distortions (have a more complete line of products), have lower mark-ups, and derive a greater share of their profits from consumers with lower valuations. These stores compensate lower prices with higher volume of sales.

1.1 Related Literature

This paper brings the theory of nonlinear pricing under asymmetric information (Mussa and Rosen (1978), Maskin and Riley (1984) and Goldman, Leland and Sibley (1984)) to a competitive setting where consumers are heterogeneously informed about the offers made by firms. Other related literature include:

Price Dispersion. The seminal papers of Butters (1977), Salop and Stigitz (1977), Varian

(1980) and Burdett and Judd (1987) study oligopolistic competition in settings where consumers are differently informed about the prices offered by firms. In these papers, there is complete information about consumer preferences, and firms compete only on prices. Relative to this literature, we introduce asymmetric information about consumers' tastes, and allow firms to compete on price *and* quality.

Competition in Nonlinear Pricing. The contributions of Armstrong and Vickers (2001), Rochet and Stole (1997, 2002) and Yang and Ye (2008) study a duopolistic version of Mussa and Rosen (1978) in a setting where consumers buy an indivisible object from either one of two firms. Firms enjoy some degree market power due to horizontal differentiation, as consumers have idiosyncratic preferences (e.g., due to transportations costs or brand loyalty) for the product offered by either firm. Our model offers an alternative to the Hotelling approach followed by this literature, and derives a distinctively new set of predictions. First, our equilibria always leads to menu dispersion, while the papers above study symmetric duopolistic equilibria. Second, we show that in equilibrium firms are endogenously segmented in a spectrum that ranges from "pricy and exclusive" to "bargain and inclusive" stores, in accord to the extensive empirical literature on price/quality dispersion.

Calzolari and Denicolo (2013) also study duopolistic competition in nonlinear price schedules with horizontal differentiation, but drop the exclusivity assumption from the aforementioned papers. The focus of their work is on the welfare effects of exclusive contracts and market-share discounts (i.e. discounts that depend on the seller's share of a customer's total purchases). Their analysis is relevant for markets where consumers make non-exclusive purchasing decisions, while our analysis is relevant for markets where exclusivity is the rule (e.g., markets for durable goods).

Search and Matching. This paper belongs to the literature that studies competition under matching and search frictions (see Butters (1977), Burdett and Judd (1987), Moen (1997) and Burdett and Mortensen (1998) for seminal contributions). This paper contributes to this literature by studying the interplay between nonlinear pricing and matching/information frictions. Inderst (2001) shows in a dynamic search model with adverse selection that inefficiencies vanish when matching frictions are sufficiently small under alternate-offers bargaining. Inderst's work provide foundations for competitive outcomes, while our model focuses on non-vanishing frictions. Faig and Jerez (2005) studied the effect of private information into a directed-search model. They prove that private information has no bite under directed search: The same equilibrium outcome is obtained whether the worker's private information is observed or not. In turn, Guerrieri, Shimer and Wright (2010) show that private information lead to inefficiencies in a directed search environment with common values. Indeed, the best-separation allocation arises as the unique equilibrium outcome. Our model is closer to Faig and Jerez (2005), as we study private values. In contrast to Faig and Jerez (2005), our model leads to a menu dispersion and distortions. Our paper is also related to Moen and Rosén (2011), who introduce private information on match quality and effort choice in a labor market

with search frictions. Our paper is complimentary to theirs, since we focus on private information about willingness to pay (the same for all firms), while workers have private information about the match-specific shock in their model.

Finally, the works of Burdett, Shi and Wright (2001) and Eeckhout and Kircher (2010a, 2010b) consider models where principals post prices (rather than mechanisms) and agents engage in directed search. This literature focus on the coordination frictions that arise in un-mediated matching markets, and ignores the mechanism-design issues that lie at the core of this work.

Competing Auctioneers. McAfee (1993), Peters (1997), Peters and Severinov (1997) and Pai (2012) study competition *in mechanisms* (e.g., second-price auctions with reserve prices). A key ingredient of these papers is that sellers face capacity constraints (each seller has *one* indivisible good to sell), and offer homogenous goods which quality is exogenous. Our paper differs from this literature in three important respects. First, sellers in our model control both the price and the quality of the good to be sold. Second, we assume away capacity constraints. Third, buyers are heterogeneously informed about the offers made by sellers.

2 Model and Preliminaries

The economy is populated by a unit-mass continuum of consumers with single-unit demands for a vertically differentiated good. If a consumer with valuation per quality θ purchases a unit of the good with quality q at a price x, his utility is

$$u(q, x, \theta) \equiv \theta \cdot q - x.$$

Consumers are heterogeneous in their valuations per quality: the valuation of each consumer is an iid draw from a discrete distribution with support $\{\theta_l, \theta_h\}$, where $\Delta\theta \equiv \theta_h - \theta_l > 0$, and associated probabilities p_l and p_h. Consumers privately observe their valuations per quality. The utility from not buying the good is normalized to zero.

A continuum of firms with mass $v > 0$ compete by posting *menus* of contracts with different combinations of quality and price. Firms have no capacity constraints and share a technology that exhibits constant returns to scale. The per-unit profit of a firm who sells a good with quality q at a price x is

$$x - \varphi(q),$$

where $\varphi(q)$ is the per-unit cost to the firm of providing quality q. We assume that $\varphi(\cdot)$ is twice continuously differentiable, strictly increasing and strictly convex, with $\varphi(0) = \varphi'(0) = 0$. Furthermore, we assume that $\lim_{q\to\infty} \varphi'(q) = \infty$, which guarantees that surplus maximizer qualities are interior.

By the Revelation Principle we may assume that the firms post menus with two price-quality pairs: $\mathcal{M} \equiv ((q_l, x_l), (q_h, x_h)) \subset (\mathbb{R}_+ \times \mathbb{R})^2$, where (q_k, x_k) is the contract designed for the type

$k \in \{l, h\}$.[1] Furthermore, every menu has to satisfy the incentive compatibility constraint, for each type $k \in \{l, h\}$:

$$IC_k: \qquad u(q_k, x_k, \theta_k) \geq \max_{\hat{k} \in \{l,h\}} \theta_k \cdot q_{\hat{k}} - x_{\hat{k}},$$

as well as the individual rationality constraint (IR) $u(q_k, x_k, \theta_k) \geq 0$. A menu $\mathcal{M}$ that satisfies the IC and IR constraints is said to be *implementable*. The set of implementable menus is denoted by $\mathbb{I}$.

Let $\tilde{F}$ (with support $\mathbb{S} \subseteq \mathbb{I}$) be the (possibly degenerate) cross-section distribution over menus prevailing in the economy. The distribution over menus $\tilde{F}$ induces, for each type k, a marginal distribution over indirect utilities

$$F_k(\tilde{u}_k) \equiv \text{Prob}\left[\mathcal{M} : u(q_k, x_k, \theta_k) \leq \tilde{u}_k\right].$$

We denote by Υ_k the cross-section support of indirect utilities offered to type-k consumers.

The key feature of our model is that there is heterogeneity on the information possessed by consumers about the menus offered by firms. We model this heterogeneity in reduced-form by means of the *sales function*

$$\Phi\left(u_k | F_k, v, p_k\right),$$

which determines the mass of sales to type-k consumers obtained by a firm that (i) offers a contract with indirect utility u_k when (ii) the cross-section cdf of indirect utilities to k-types is F_k, (iii) there is a v-mass of firms in the market, and (iv) there is a p_k-mass of type-k consumers. In the next subsection, we will discuss at length the sales functions induced by different matching technologies between firms and consumers, and detail the economic and technical assumptions that define the class of sales functions considered in this paper.

A firm that faces a cross-section distribution of menus $\tilde{F}$ (with marginal cdf over type-k indirect utilities F_k) chooses a menu $((q_l, x_l); (q_h, x_h)) \in \mathbb{I}$ to maximize profits

$$\sum_{k=l,h} \Phi\left(u(q_k, x_k, \theta_k) | F_k, v, p_k\right) \cdot \left(x_k - \varphi(q_k)\right). \tag{1}$$

The next definition formalizes our notion of equilibrium in terms of the cross-section cdf over menus prevailing in the economy.

Definition 1 ***[Equilibrium]*** *An equilibrium is a distribution over menus $\tilde{F}$ (with marginal cdf over type-k indirect utilities F_k) such that $\mathcal{M} \in supp\ \tilde{F} \subset \mathbb{I}$ implies that $\mathcal{M}$ maximizes (1).*

Remark 1 *The equilibrium definition above renders itself to multiple interpretations. In one interpretation, firms follow symmetric mixed strategies by randomizing over menus according to the distribution $\tilde{F}$. Another interpretation is that each firm follows a pure strategy that consists in*

[1]The restriction to deterministic mechanisms comes with no loss of generality. As the analysis that follows will make clear, firms will always prefer to offer deterministic mechanisms in equilibrium.

posting the menu associated with a given quantile of the distribution $\tilde{F}$. Alternatively, firms might randomize over different subsets of the support $\mathbb{S}$ according to the conditional distributions induced by $\tilde{F}$.

The next subsection is devoted to the sales functions described above.

2.1 Sales Functions

A number of consumer search/matching models have been proposed to resolve both the Diamond and the Bertrand paradoxes. The key common feature of these approaches is that consumers are differently informed about the offers made by firms. In order to derive robust predictions, we proceed by identifying properties of sales functions that hold across a number of natural matching technologies. The next assumption describes the class of sales functions considered in this paper.

Assumption 1 *Let $\tilde{F}$ be a distribution over menus with supp $\tilde{F} \subset \mathbb{I}$, and marginal distribution over type-k indirect utilities F_k, with support Υ_k.*

At any continuity point $u_k \in \Upsilon_k$ of F_k, the sales function $\Phi\left(u_k|F_k, v, p_k\right)$ can then be written as

$$\Phi\left(u_k|F_k, v, p_k\right) \equiv p_k \cdot \Lambda\left(F_k(u_k)|v\right), \tag{2}$$

where the kernel $\Lambda\left(y|v\right) : [0,1] \times \mathbb{R}_{++} \to \mathbb{R}_{++}$:

1. *is continuously differentiable and strictly increasing in y,*
2. *satisfies the feasibility constraint*

$$v \cdot \int_0^1 \Lambda\left(y|v\right) dy \leq 1 \tag{3}$$

for all $v \in \mathbb{R}_{++}$.[2]

According to the functional form (2), the mass of sales among type-k consumers obtained by a firm offering indirect utility u_k (i) is proportional to the mass of type-k consumers in the market, and (ii) depends only on the ranking relative to the cross section distribution of indirect utilities, $F_k(u_k)$. The first property rules out "externalities" among consumers (such as rule-of-mouth effects). As illustrated by the examples below, the second "ranking property" is a common feature of large markets where consumers are aware of different groups of firms.

[2]In case F_k exhibits mass points, we adopt the uniform rationing rule. Formally, if $u_k \in \Upsilon_k$ is a mass point of F_k, we let

$$\Phi\left(u_k|F_k, v, p_k\right) \equiv p_k \cdot \left(F_k(u_k) - \lim_{\tilde{u}_k \uparrow u_k} F_k(\tilde{u}_k)\right)^{-1} \cdot \int_{\lim_{\tilde{u}_k \uparrow u_k} F_k(\tilde{u}_k)}^{F_k(u_k)} \Lambda\left(y|v\right) dy.$$

Finally, set $\Phi\left(u_k|F_k, v, p_k\right) = p_k \cdot \Lambda\left(1|v\right)$ if $u_k > \tilde{u}_k$ for all $\tilde{u}_k \in \Upsilon_k$, and $\Phi\left(u_k|F_k, v, p_k\right) = p_k \cdot \Lambda\left(0|v\right)$ if $0 \leq u_k < \tilde{u}_k$ for all $\tilde{u}_k \in \Upsilon_k$.

A crucial ingredient of Assumption 1 is that, for each consumer type, firms with the lowest indirect utility ranking make a positive number of sales, i.e., $\Lambda(0|v) > 0$. This property - which is only possible when consumers have heterogeneous information about the firms' offer - is what prevents the perfectly competitive outcome of Bertrand competition. Also important is the fact that the mass of sales is strictly increasing in the indirect utility ranking, therefore implying that $\Lambda(1|v) > \Lambda(0|v)$. This property rules out the Diamond Paradox, according to which all firms offering the monopolistic (Mussa-Rosen) menu constitutes an equilibrium.

Assumption 1 further requires that the kernel $\Lambda(y|v)$ is increasing and continuously differentiable in y. The latter requirement is technically convenient, but can be easily dispensed with. The monotonicity in y implies that offering better deals to consumers leads to higher sales. Finally, the feasibility constraint (3) requires that the total mass of sales in the market does not exceed the total demand from consumers.

The next examples discuss a number of random matching models that satisfy Assumption 1.

Example 1 ***[Generalized Burdett and Judd (1987)]*** *Let each consumer observe the menus of a sample of firms independently and uniformly drawn from the set of all firms. For each consumer, the size of the observed sample is $j \in \{0, 1, 2, \ldots\}$ with probability $\omega_j(v)$, where $\omega_1(v), \omega_2(v) > 0$ for all $v > 0$. The distribution over sample sizes $\Omega(v) \equiv \{\omega_j(v) : j = 0, 1, 2, \ldots\}$ is indexed v, so as to allow the mass of firms in the market to affect the amount of information observed by consumers.*

In this case, the sales function faced by firms has the functional form (2) with kernel

$$\Lambda(y|v) = \frac{1}{v} \cdot \sum_{j=1}^{\infty} j \cdot \omega_j(v) \cdot y^{j-1}. \tag{4}$$

Example 2 ***[Poisson-Burdett-Judd]*** *The Poisson-Burdett-Judd search model adds to the search model of Example 1 the feature that the size of the sample observed by each consumer is distributed according to a Poisson law with mean $\beta \cdot v$, where $\beta > 0$:*

$$\omega_j(v) = \frac{(\beta \cdot v)^j}{j!} \cdot \exp\{-\beta \cdot v\} \qquad \text{for} \qquad j = 0, 1, 2, \ldots. \tag{5}$$

Accordingly, as the mass of firms v increases, consumers observe larger samples of menus with higher probability (in the sense of likelihood ratio dominance). The parameter β measures how an increase in the mass of firms affects the distribution of sample sizes. The kernel of the Poisson-Burdett-Judd

model is:[3]

$$\Lambda(y|v) = \beta \cdot \exp\{-\beta \cdot v \cdot (1-y)\}. \tag{6}$$

Example 3 ***[Generalized Butters (1977)]*** *Let the menu offered by each firm be observed by exactly $n \geq 1$ consumers. The size-n subset of consumers reached by each firm is uniformly (and independently) drawn from the set of all n-size subsets of consumers. When the number of firms and consumers in the market is large (with ratio v), Butters (1977) shows that the sales function faced by firms has the functional form (2) with kernel*

$$\Lambda(y|v) = n \cdot \exp\{-v \cdot (1-y) \cdot n\}.$$

In the original Butters (1977) model, n is set to one.

It is interesting to note that the Generalized Butters and the Poisson-Burdett-Judd matching technologies lead to similar same sales functions. Another example of a matching technology satisfying Assumption 1 comes from the labor search literature.

Example 4 ***[Burdett and Mortensen (1998)]*** *The "on-the-job search" model of Burdett and Mortensen (1998) studies a dynamic economy in continuous time in which consumers receive ads (each ad describes the menu of a particular firm) according to independent Poisson processes with arrival rate λ. Consumers must make purchasing decisions as soon as an ad arrives, and there is no recall. The relationship between a consumer and a firm is dissolved exogenously according to independent Poisson processes with arrival rate γ. Finally, consumers may switch firms if they receive an ad describing a more attractive menu. There is no discounting.*

It follows from the analysis of Burdett and Mortensen (1998) that the steady-state outcome of this economy can be modeled as a static competition game which sales function has the functional form (2) with kernel

$$\Lambda(y|v) = \gamma \cdot \left[\frac{1}{\gamma + \lambda \cdot v \cdot (1-y)}\right]^2.$$

It is worthwhile noting that the "ranking property" of sales functions imposed by Assumption 1 distinguishes our model from spatial models of competition (such as Hotelling or Differentiated Bertrand). In such models, the mass of sales obtained by each firm is a function of the profile of *cardinal* indirect utilities offered to each consumer type. In contrast, in our model the mass of sales is

[3]Too see why, plug the Poisson densities (5) into the kernel (4) to obtain that

$$\Lambda(y|v) = \beta \cdot \exp\{-\beta \cdot v\} + \beta \cdot \exp\{-\beta \cdot v \cdot (1-y)\} \cdot \sum_{j=2}^{\infty} \frac{(\beta \cdot v \cdot y)^{j-1}}{(j-1)!} \cdot \exp\{-\beta \cdot v \cdot y\}$$

$$= \beta \cdot \exp\{-\beta \cdot v\} + \beta \cdot \exp\{-\beta \cdot v \cdot (1-y)\} \cdot (1 - \exp\{-\beta \cdot v \cdot y\}),$$

where the equality from the first to the second line follows from the fact that the Poisson pmf sums to one.

a function of the quantiles (relative to the cross-section) associated with the indirect utilities offered by a firm (i.e., it depends on *ordinal* properties of indirect utilities).

To simplify the exposition, in the baseline model described above, the information possessed by consumers is determined by an *exogenous* matching technology. Subsection 5.2 extends the baseline model to allow for active consumer search, where different types of consumer engage in fixed sample size search (in the spirit of Stigler (1961)).

2.2 Incentive Compatibility and Indirect Utilities

A key step in our analysis is to formulate the firms' maximization problem in terms of the of indirect utilities offered to consumers. To this end, denote by

$$q_k^* \equiv \arg\max_q \ \theta_k \cdot q - \varphi(q),$$

the efficient quality for type-k consumers, and let $S_k^* \equiv \theta_k \cdot q_k^* - \varphi(q_k^*)$ be the social surplus associated with the efficient quality provision. The next lemma uses the incentive constraints and the optimality of equilibrium contracts to map indirect utilities into quality levels.

Lemma 1 *Consider a menu $\mathcal{M} = \{(q_l, x_j), (q_h, x_h)\}$ in the support of the equilibrium distribution over menus, $\tilde{F}$, and let $u_k \equiv u(q_k, x_k, \theta_k)$. Then, for all $k \in \{l, h\}$,*

$$q_k = \mathbf{1}_k(u_h - u_l) \cdot \frac{u_h - u_l}{\triangle\theta} + (1 - \mathbf{1}_k(u_h - u_l)) \cdot q_k^*, \tag{7}$$

where $\mathbf{1}_h(z)$ is an indicator function that equals one if and only if $z > q_h^* \cdot \triangle\theta$, and $\mathbf{1}_l(z)$ is an indicator function that equals one if and only if $z < q_l^* \cdot \triangle\theta$.

The result above is standard in adverse selection models. Consider some menu $\mathcal{M} \in \text{supp}(\tilde{F})$ offered in equilibrium. If the IC_k constraint does not bind under $\mathcal{M}$, then profit-maximization by firms implies that the quality provision to the other type of consumer (i.e., type $-k$) is efficient under $\mathcal{M}$. However, if the IC_k constraint does bind under $\mathcal{M}$, then the quality to consumers of type $-k$ is chosen to make type-k consumers indifferent between either contract. These facts are summarized in equation (7).

In light of Lemma 1, we can describe each menu in the support of $\tilde{F}$ in terms of the indirect utilities induced by $\mathcal{M}$. Accordingly, we shall write $\mathcal{M} = (u_l, u_h)$ to describe the menu $\mathcal{M} = ((q_l, x_l), (q_h, x_h))$, where the map between q's and u's follows from equation (7). In a similar fashion, for convenience, we will more often refer to the marginal distribution over indirect utilities, F_k, rather than to the distribution over menus $\tilde{F}$.

Two natural benchmarks play an important role in the analysis that follows. The first one is the static monopolistic (or Mussa-Rosen) solution. Under this benchmark, the quality provided to low

types, denote it q_l^m, is implicitly defined by:

$$\varphi'(q_l^m) = \max\left\{\theta_l - \frac{p_h}{p_l} \cdot \Delta\theta, 0\right\}. \tag{8}$$

We interpret $q_l^m = 0$ as meaning that low-type consumers are not served under the monopolistic solution. In turn, quality provision for high types is efficient: $q_h^m = q_h^*$. Finally, recall that, in the monopolistic solution, the indirect utility left to low types is zero, $u_l^m = 0$ (as the IR is binding), and the indirect utility left to high types is $u_h^m = q_l^m \cdot \triangle\theta$, as the IC_h is binding. Written in terms of indirect utilities, the menu $\mathcal{M}^m \equiv (0, q_l^m \cdot \triangle\theta)$ is the *monopolist* (or *Mussa-Rosen*) *menu*.

The second benchmark is the competitive (or Bertrand) solution. Under this benchmark, quality provision is efficient to both types, and firms derive zero profits from each contract in the menu. Written in terms of indirect utilities, the menu $\mathcal{M}^* \equiv (S_l^*, S_h^*)$ is the *competitive* (or *Bertrand*) *menu*. We can now proceed to characterizing the equilibrium of our model.

3 Screening and Competition

For each menu $\mathcal{M} = (u_l, u_h)$ offered in equilibrium, let

$$S_k(u_l, u_h) \equiv \theta_k \cdot q_k(u_l, u_h) - \varphi(q_k(u_l, u_h)) \tag{9}$$

be the social surplus induced by $\mathcal{M}$ for each consumer type, where the quality levels $q_k(u_l, u_h)$ are computed according to (7). We can then write the profit from type-k consumers produced by the menu $\mathcal{M} = (u_l, u_h)$ as $S_k(u_l, u_h) - u_k$.

Employing Lemma 1 and Assumption 1, we can rewrite the firm's profit-maximization problem (in response to the cross-section cdf's over indirect utilities $\{F_l, F_h\}$) as that of choosing menus (u_l, u_h) to maximize

$$\pi(u_l, u_h) \equiv \sum_{k=l,h} p_k \cdot \Lambda\left(F_k(u_k)|v\right) \cdot \left(S_k(u_l, u_h) - u_k\right), \tag{10}$$

subject to the constraint $u_h \geq u_l \geq 0$, which accounts for the IC and IR constraints that ensure implementability.

As a first step towards characterizing equilibria, the next lemma uses standard arguments to establish that, for each $k \in \{l, h\}$, the distribution over indirect utilities, F_k, is absolutely continuous, and has support on an interval that starts at the indirect utility associated with the monopolistic (Mussa-Rosen) menu.

Lemma 2 *[**Support**] In any equilibrium of this economy, the marginal cdf over indirect utilities, F_k, is absolutely continuous and has positive density over the support*

$$\Upsilon_k = [u_k^m, \bar{u}_k],$$

where $\bar{u}_k < S_k^*$, for $k \in \{l, h\}$.

In the proof of this lemma, we construct deviations to any candidate equilibrium where F_k is not continuous, or where Υ_k is not a closed interval starting at u_k^m. To obtain some intuition, assume that $\Upsilon_l \times \Upsilon_h$ contains an implementable menu whose indirect utilities (u_l, u_h) are such that $u_k < u_k^m$ for some k (in which case $\inf \Upsilon_k < u_k^m$). By deviating to the menu $(\max\{u_k, u_k^m\})_{k=l,h}$, the firm weakly raises its profits and sales for both consumer types, and makes strict gains for at least one type of consumer. Therefore, any menu in $\Upsilon_l \times \Upsilon_h$ has to satisfy $u_k \geq u_k^m$ for all k.

Alternatively, suppose there exists $\epsilon > 0$ such that $u_l > u_l^m + \epsilon$ for all $u_l \in \Upsilon_l$ (the argument for type $k = h$ is analogous). Pick the menu with the lowest indirect utility for low types, denoted $(u_l^\#, u_h^\#)$, and consider a deviation to $(u_l^m, u_h^\#)$. This deviation does not affect sales for either low or high types, but it strictly increases the profits obtained from the low types. Therefore, $\inf \Upsilon_k = u_k^m$ for all k. The proof contained in the appendix formalizes the heuristics above, and applies similar ideas to establish that each Υ_k is connected, and each F_k is absolutely continuous.

As usual in mechanism design, we will proceed by assuming that IC_l is slack in equilibrium, in which case IC_h is the only potentially binding constraint. As will become clear, this is indeed true in any equilibrium of this economy. In light of Lemma 2, we can take first-order conditions with respect to u_h to obtain that

$$\underbrace{p_h \cdot \Lambda_1 \left(F_h(u_h)|v\right) \cdot f_h(u_h) \cdot (S_h^* - u_h)}_{\text{sales gains}} - \underbrace{p_h \cdot \Lambda \left(F_h(u_h)|v\right)}_{\text{profit losses}} + \underbrace{p_l \cdot \Lambda \left(F_l(u_l)|v\right) \cdot \frac{\partial S_l}{\partial u_h}(u_l, u_h)}_{\text{efficiency gains}} = 0, \quad (11)$$

and with respect to u_l to obtain that

$$\underbrace{p_l \cdot \Lambda_1 \left(F_l(u_l)|v\right) \cdot f_l(u_l) \cdot (S_l(u_l, u_h) - u_l)}_{\text{sales gains}} - \underbrace{p_l \cdot \Lambda \left(F_l(u_l)|v\right)}_{\text{profit losses}} + \underbrace{p_l \cdot \Lambda \left(F_l(u_l)|v\right) \cdot \frac{\partial S_l}{\partial u_l}(u_l, u_h)}_{\text{efficiency losses}} = 0. \quad (12)$$

Intuitively, the firms' choice of menus balances sales, profit, and efficiency considerations. Let us start with the first-order condition for high types, given by equation (11). The first two terms in (11) are standard. By increasing the indirect utility u_h, the firm increases sales (the first term), but decreases profits (the second term). The third term captures the effect of an increase in u_h into the quality offered to low-type consumers. When IC_h is slack (i.e., $u_h > u_l + \Delta\theta \cdot q_l^*$), high types have no incentive to imitate low types, and this term is zero. Let us then focus on the complementary case where IC_h is binding. As implied by profit-maximization, the low-type quality is set to satisfy the constraint $u_h \geq u_l + \Delta\theta \cdot q_l$ with equality. As a consequence, an increase in u_h relaxes this constraint, and allows the firm to marginally increase the quality to low-type consumers by

$$\frac{\partial q_l(u_l, u_h)}{\partial u_h} = \left(\frac{1}{\Delta\theta}\right).$$

Therefore, the efficiency gains from increasing the quality of high types are generated by the decrease in distortions of the contract to low types, and equal

$$p_l \cdot \Lambda \left(F_l(u_l)|v\right) \left(\frac{\theta_l - \varphi'(q_l)}{\Delta\theta}\right) > 0. \quad (13)$$

Let us now consider the first-order condition for low types, given by equation (12). The first two terms capture the sales gains and the profit losses from increasing u_l. In contrast to (11), however, increasing u_l has the effect of tightening the incentive constraint IC_h, which implies that the quality distortion present in the low types' contract has to increase. This efficiency loss is the third term in equation (12). By the same reasoning described above, this term has the same magnitude of (13), but opposite sign.

It is interesting to note that, *ceteris paribus*, increasing the indirect utility of high and low types have opposing effects on how much firms optimally distort the quality in low-type contracts. Our equilibrium analysis of the next subsections will clarify how such countervailing forces unfold in equilibrium.

3.1 Ordered Equilibrium

We construct an equilibrium in which firms that cede high indirect utilities to high types also cede high indirect utilities low types. We say that equilibria that satisfy this property are ordered.

Definition 2 *[**Ordered Equilibrium**] An equilibrium is said to be ordered if, for any two menus $\mathcal{M} = (u_l, u_h)$ and $\mathcal{M}' = (u_l', u_h')$ offered in equilibrium, $u_l < u_l'$ if and only if $u_h < u_h'$. In this case, the menu (u_l', u_h') is said to be more generous than the menu (u_l, u_h).*

As the next proposition establishes, there always exists a unique ordered equilibrium, which, under natural conditions to be clarified below, constitute the unique equilibrium of this economy. Ordered equilibria have the following important property.

Remark 2 *[**Support Function**] In every ordered equilibrium, the support of indirect utilities offered by firms can be described by a strictly increasing and bijective support function $\hat{u}_l : \Upsilon_h \to \Upsilon_l$ such that, for every menu $\mathcal{M} = (u_l, u_h)$ in $\Upsilon_l \times \Upsilon_h$, $u_l = \hat{u}_l(u_h)$.*

Remark 2 tells us that there is a strictly increasing function $\hat{u}_l$ that determines the utility offered to the low type as a function of the utility of the high type. Proposition 1 characterizes the unique ordered equilibrium of the economy. For notational convenience, we will denote the identity function according to $\hat{u}_h(u_h) = u_h$.

Proposition 1 *[**Equilibrium Characterization**] There exists a unique ordered equilibrium. In this equilibrium, the support of indirect utilities offered by firms is described by the support function $\hat{u}_l : [u_h^m, \bar{u}_h] \to [0, \bar{u}_l]$ that is the unique solution to the differential equation*

$$\hat{u}_l'(u_h) = \frac{S_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)}{S_h^* - u_h} \cdot \frac{1 - \frac{p_l}{p_h} \cdot \frac{\partial S_l}{\partial u_h}(\hat{u}_l(u_h), u_h)}{1 - \frac{\partial S_l}{\partial u_l}(\hat{u}_l(u_h), u_h)} \tag{14}$$

with boundary condition $\hat{u}_l(u_h^m) = 0$.

The equilibrium distribution over menus solves

$$\frac{\Lambda\left(F_h(u_h)|v\right)}{\Lambda\left(0|v\right)} = \frac{\sum_{k=l,h} p_k \cdot \left(S_k(0, u_h^m) - u_k^m\right)}{\sum_{k=l,h} p_k \cdot \left(S_k(\hat{u}_l(u_h), u_h) - \hat{u}_k(u_h)\right)}, \tag{15}$$

and the supremum point $\bar{u}_h$ *is determined by* $F_k(\bar{u}_h) = 1$.

The existence (and uniqueness) of ordered equilibrium is intimately related to complementarities in the profit function (10). Consider a menu $\mathcal{M} = (u_l, u_h)$ in which IC_h is the only binding constraint. An increase in u_h, *ceteris paribus*, increases the quality that can be supplied to low-type customers in an incentive-compatible menu. Therefore, the profit that firms can earn from sales to low types is increased. This in turn increases the firms' benefit of making marginal sales to low-type consumers (or, equivalently, of ceding a higher indirect utility u_l). This implies that the indirect utilities ceded to low and high type consumers are strategic complements:

$$\frac{\partial^2 \pi(u_l, u_h)}{\partial u_h \partial u_l} = \frac{p_l \cdot \Lambda\left(F_l(u_l)|v\right) \cdot \varphi''(q_l)}{\left(\Delta\theta\right)^2} > 0, \tag{16}$$

as can be verified from the profit function (10).

In contrast, when IC_h is slack, the firms' decisions of which indirect utility to leave to each type of consumer are (locally) separable. Formally, the firms' profits have locally strict increasing (constant) differences if and only if the incentive constraint IC_h is binding (slack).

Remark 3 *[**Increasing Differences**] Consider a menu that provides indirect utilities* (u_l, u_h).

1. *If* $u_h - u_l < \triangle\theta \cdot q_l^*$, *in which case* IC_h *is binding, the profit function* $\pi(u_l, u_h)$ *satisfies strict increasing differences in an open neighborhood of* (u_l, u_h).

2. *If* $u_h - u_l > \triangle\theta \cdot q_l^*$, *in which case* IC_h *is slack, the profit function* $\pi(u_l, u_h)$ *is modular in an open neighborhood of* (u_l, u_h), *that is,* $\pi(u_l, u_h) = \sum_{k\in\{l,h\}} \pi_k(u_k)$, *for some choice of* $\pi_k(\cdot)$.

This increasing difference property is fundamental for the construction of ordered equilibria. As we will see in more detail below, it implies that the local optimality of a menu $(\hat{u}_l(u_h), u_h)$ guarantees its global optimality. With these observations in mind, let us now describe the main ideas behind the proof of Proposition 1.

Proof Sketch of Proposition 1. We proceed in three steps. First, we construct the support function $\hat{u}_l(\cdot)$. In the second step, we derive the equilibrium distribution over menus. In the last step, we show that firms cannot benefit from deviating to an out-of-equilibrium menu.

Step 1 *Constructing the support function*

Because of the ranking property of kernels, it follows that in any ordered equilibrium with support function $\hat{u}_l(\cdot)$,

$$\Lambda\left(F_h(u_h)|v\right) = \Lambda\left(F_l(\hat{u}_l(u_h))|v\right). \tag{17}$$

The equation above implies that the ratio of sales to high and low types is equal to the mass ratio across types, $\frac{p_h}{p_l}$. Accordingly, the support function $\hat{u}_l(\cdot)$ describes the locus of indirect utility pairs $(\hat{u}_l(u_h), u_h)$ that leaves the ratio of sales across types constant.

Differentiating the expression above, we obtain

$$\hat{u}_l'(u_h) = \frac{\Lambda_1\left(F_h(u_h)|v\right) \cdot f_h(u_h)}{\Lambda\left(F_h(u_h)|v\right)} \cdot \left[\frac{\Lambda_1\left(F_l(\hat{u}_l(u_h))|v\right) \cdot f_l(\hat{u}_l(u_h))}{\Lambda\left(F_l(\hat{u}_l(u_h))|v\right)}\right]^{-1}. \tag{18}$$

Intuitively, the slope of the support function, $\hat{u}_l'(u_h)$, equals the ratio between the semi-elasticities of sales with respect to indirect utilities for each type of consumer.

The first-order conditions (11) and (12) provide an alternative expression for these semi-elasticities. Evaluated at the locus $(\hat{u}_l(u_h), u_h)$, with the help of (17), equations (11) and (12) can be rewritten as

$$p_k \cdot \frac{\Lambda_1\left(F_k(\hat{u}_k(u_h))|v\right) \cdot f_k(\hat{u}_k(u_h))}{\Lambda\left(F_k(\hat{u}_k(u_h))|v\right)} \cdot \left(S_k(\hat{u}_l(u_h), u_h) - u_k\right) = p_k - p_l \cdot \frac{\partial S_l}{\partial u_k}(\hat{u}_l(u_h), u_h), \tag{19}$$

for $k = h$ and $k = l$, respectively. In equilibrium, the optimality of firms' menus requires that the support function $\hat{u}_l(\cdot)$ simultaneously satisfies the first-order conditions (19) and equation (18). Combining these two equations leads to the differential equation (14) which tells how the utility of the low type relates to the utility of the high type in the equilibrium menus.

From Lemma 2, we know that the less generous menu in equilibrium is the Mussa and Rosen menu $(0, u_h^m)$. Therefore, we impose the boundary conditions $\hat{u}_l(u_h^m) = 0$ to the differential equation (14). Finally, the increasing differences property of the profit function, established in Remark 3, implies that the solution to the differential equation (14) satisfies $\hat{u}_l'(u_h) > 0$, in which case the menus $(\hat{u}_l(u_h), u_h)$ are indeed ordered.

We also need to verify that IC_l is never binding in any menu of the $(\hat{u}_l(u_h), u_h)$. Indeed, for all $u_h \in [u_h^m, \bar{u}_h]$,

$$u_h - \hat{u}_l(u_h) \leq \bar{u}_h - \hat{u}_l(\bar{u}_h) < S_h^* - S_l^* < \triangle\theta \cdot q_h^*,$$

where the first and second inequalities follows from the fact that $\hat{u}_l'(u_h) < 1$ for all $u_h \in [u_h^m, \bar{u}_h]$.

Step 2 *Constructing the distribution over menus*

In view of the support function $\hat{u}_l(\cdot)$, we can describe the equilibrium distribution over menus in terms of the distribution of indirect utilities to high type consumers, $F_h(\cdot)$. The key idea in the construction is to choose, for each u_h, the quantile $F_h(u_h)$ in a way that all menus offered in

equilibrium lead to the same expected profits as the Mussa-Rosen menu $\mathcal{M}^m$. This is reflected in the indifference condition (15).

An important ingredient in the construction is to show that

$$\sum_{k=l,h} p_k \cdot (S_k(\hat{u}_l(u_h), u_h) - \hat{u}_k(u_h))$$

is a strictly decreasing function of u_h, Intuitively, the *expected* profit *per unit sold* decreases as menus become more generous along the locus $\{(\hat{u}_l(u_h), u_h) : u_h \in [u_h^m, \bar{u}_h]\}$. As a consequence, the right-hand side of the indifference condition (15) is strictly increasing in u_h. Because kernels $\Lambda(y|v)$ are strictly increasing in the quantile y, this implies that $F_h(\cdot)$ is strictly increasing in u_h.

In order to complete the construction of the distribution $F_h(\cdot)$, we need to determine the support of high type indirect utilities, Υ_h. By Lemma 2, Υ_h is a closed interval of the form $[u_h^m, \bar{u}_h]$, so we are only left to compute the upper limit of Υ_h, $\bar{u}_h$. In the appendix, we show that the solution to the differential equation (14) satisfies $\hat{u}_l(S_h^*) = S_l^*$, that is: When high types receive their Bertrand utility S_h^*, so do low types. This property implies that the right-hand side of the indifference condition (15) approaches infinity as $u_h \to S_h^*$, what guarantees that there exists a unique $\bar{u}_h < S_h^*$ for which $F_h(\bar{u}_h) = 1$.

Step 3 *Verifying the optimality of equilibrium menus*

Finally, we verify that no seller has a profitable deviation. Observe first that no deviation to a menu that leads to indirect utilities outside of the range $\Upsilon_l \times \Upsilon_h = [u_l^m, \hat{u}_l(\bar{u}_h)] \times [u_h^m, \bar{u}_h]$ can be optimal. Consider therefore a menu (u_l', u_h') such that $u_k' \in \Upsilon_k$ for all $k = l, h$. Let us analyze first the case where $u_l' < \hat{u}_l(u_h')$. Using straight-forward algebra, one can show that the gains from this deviation relative the the equilibrium menu $(\hat{u}_l(u_h'), u_h')$ equal

$$\pi(u_l', u_h') - \pi(\hat{u}_l(u_h'), u_h') = -\int_{\hat{u}_l^{-1}(u_l')}^{u_h'} \frac{\partial^2 \pi(\tilde{u}_l, u_h')}{\partial \tilde{u}_h \partial \tilde{u}_l} d\tilde{u}_h,$$

which is non-positive by virtue of the increasing-differences property established in Remark 3. By a similar argument one can handle the case where $u_l' > \hat{u}_l(u_h')$, which completes the proof of Proposition 1. Q.E.D.

In what follows, we focus attention on the ordered equilibrium described above. In subsection 3.4, we present a complete characterization of the equilibrium set, and show that little (if anything) is lost by restricting attention to ordered equilibrium.

3.2 Equilibrium Properties

Recall from the best-response analysis of subsection 4.1 that, ceteris paribus, increasing the indirect utility of high and low types have opposing effects on how much firms optimally distort the quality in low-type contracts. Which of these countervailing effects prevails in equilibrium?

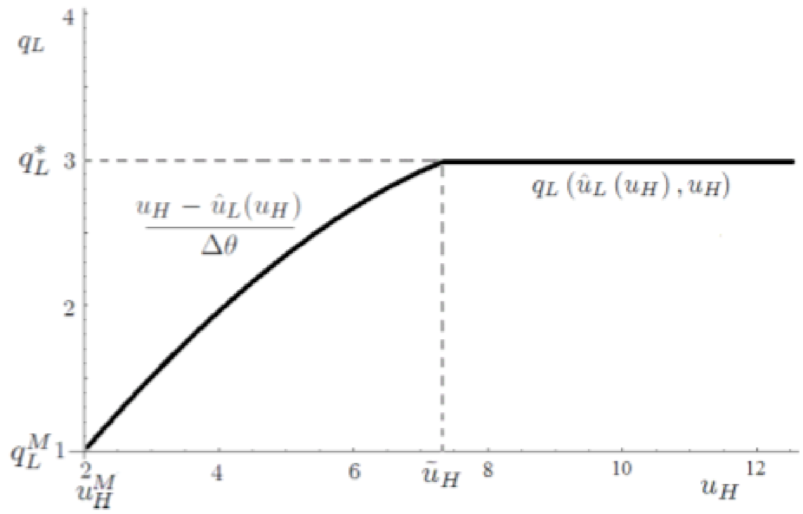

Figure 1: The quality schedule to low-type consumers as a function of u_h.

The characterization from Proposition 1 helps answering this question. In this regard, it is convenient to consider the function $\delta(u_h) \equiv u_h - \hat{u}_l(u_h)$, which measures the difference between the indirect utility of high and low types along the support Υ_h.

A key property of the ordered equilibrium is that the utility difference $\delta(u_h)$ is strictly increasing in u_h, reaching its maximum at the upper limit of Υ_h, $\bar{u}_h$. Intuitively, this property reflects the fact that competition for high types is fiercer than competition for low types in equilibrium, as high-type consumers "have more surplus to share" with firms. An immediate consequence is that, whenever IC_h binds, the quality provided to low types,

$$q_l\left(\hat{u}_l\left(u_h\right), u_h\right) = \frac{\delta\left(u_h\right)}{\Delta\theta}, \tag{20}$$

strictly increases in u_h (see Figure 1 above). Therefore, as equilibrium menus become more generous, their social surplus go up (as distortions go down). This, and other properties of equilibrium, are the subject of the next proposition.

Proposition 2 ***[Equilibrium Properties]*** *The following properties hold in the ordered equilibrium.*

1. ***Efficiency:*** *Menus for which customers earn higher payoffs are more efficient. In particular, the social surplus produced by low-type contracts, $S_l(\hat{u}_l(u_h), u_h)$, is strictly increasing in u_h whenever $u_h < \tilde{u}_h \in (u_h^m, S_h^*)$, and equal to the efficient level S_l^* whenever $u_h \geq \tilde{u}_h$.*[4]

2. ***Profits:*** *Firms which offer more generous menus obtain a large fraction of their profits from low-type consumers.*

[4]The threshold $\tilde{u}_h$ belongs to support $\Upsilon_h = [u_h^m, \bar{u}_h]$ when its upper limit $\bar{u}_h$ is such that $\bar{u}_h \geq \tilde{u}_h$. Proposition 5 below, which discusses equilibrium uniqueness, provides necessary and sufficient conditions (in terms of primitives) for $\bar{u}_h \geq \tilde{u}_h$.

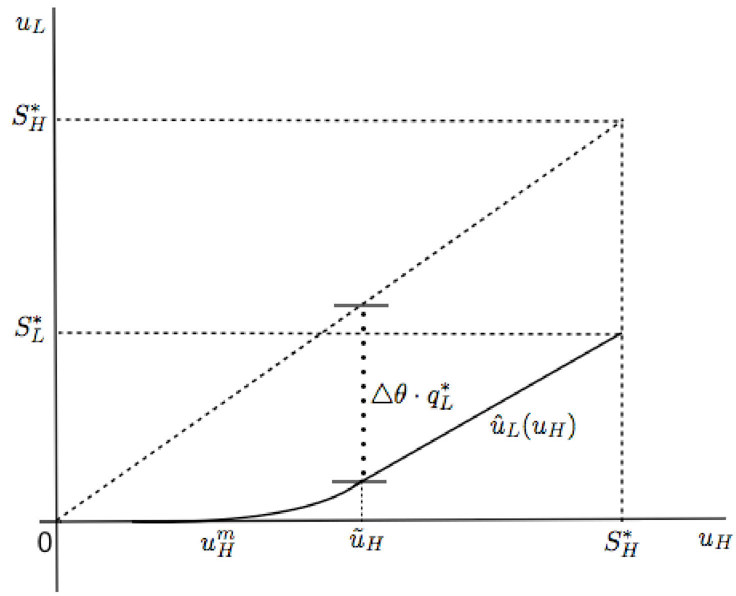

Figure 2: The equilibrium support function $\hat{u}_l(\cdot)$.

The first statement in Proposition 2 identifies the *IC-threshold* $\tilde{u}_h$ on the h-type indirect utility above which equilibrium menus $(\hat{u}_l(u_h), u_h)$ provide the efficient quality to low types (i.e., achieve the maximal social surplus). The IC-threshold $\tilde{u}_h$ corresponds to the lowest value of u_h such that the utility difference $\delta(\cdot)$ satisfies

$$\delta(u_h) \geq \triangle\theta \cdot q_l^*,$$

in which case the incentive constraint IC$_h$ is slack. Intuitively, because the indirect utility left to high types increases faster than that of low types, for generous enough menus, incentive constraints are slack and efficiency prevails. The next figure illustrates the support function and the utility difference between low and high-type consumers.

The second statement in Proposition 2 shows that firms sort themselves in equilibrium according to the composition of their profits. It establishes that firms that offer more generous (or equivalently, more efficient) menus derive a higher share of profits from low-type consumers. As menus become more generous, the ratio of profits derived from low and high types approaches the upper bound $\bar{s}$, as in constant at this level for all menus that provide quality efficiently for both types (i.e., those menus for which $u_h \geq \tilde{u}_h$).

The properties identified in Proposition 2 find support in the empirical literature. The work of Baylis and Perloff (2002) collect a data set consisting of prices and qualities offered by a large number of internet retailers of digital cameras and flabbed scanners. In line with the predictions of our model, they document dispersion on price-quality schedules. More importantly, they find evidence supporting the segmentation patterns of Proposition 2. Retailers offer price-quality schedules that range from high-price-low-quality menus (bad firms) to low-price-high-price menus (good firms). Bad

firms make sales to consumers with high willingness to pay, and little information about the offers available in the market. In contrast, good firms serve a larger set of consumers and charge lower prices for their products.

The next subsection provides additional testable predictions of our model by studying how the distributions over menus (and its support) vary with the degree of competition in the market.

3.3 Comparative Statics

We will now investigate how a higher degree of competition affects the equilibrium distribution over menus. Before stating results, we have to introduce a mild regularity condition on the kernel $\Lambda(y|v)$. This condition controls for how sales functions change with the mass of firms v.

Condition 1 ***[VM] V-Monotonocity:*** *The kernel ratio*

$$R(y|v) \equiv \frac{\Lambda(y|v)}{\Lambda(0|v)}$$

is strictly increasing in v for all $y \in (0,1]$.

The monotonicity requirement of Condition VM is satisfied by the Generalized Burdett-Judd matching model provided that, for any $\hat{v} > v$, the sample size distribution $\Omega(\hat{v})$ dominates the distribution $\Omega(v)$ in the likelihood-ratio order. In particular, this assumption is satisfied by the Poisson-Burdett-Judd matching model (and, therefore, by the Butters model, which shares a similar sales function). It is also satisfied by the Burdett-Mortensen matching model.

Intuitively, this condition states that, relative to the least generous menu in the cross-section, the proportional gains on sales from offering a contract which indirect utility lie in some quantile $y > 0$ increases with the mass of competing firms v.

The next proposition establishes that, when competition increases, firms more often offer menus that lead to high indirect utilities for both consumer types. As implied by Proposition 2, the mass of firms that offer inefficient qualities in equilibrium decreases as competition gets fiercer.

Proposition 3 ***[Competition and Distortions: Comparative Statics]*** *Assume that condition VM holds, and denote by F_k and $\hat{F}_k$ (with support Υ_k and $\hat{\Upsilon}_k$) the equilibrium distributions over indirect utilities when the mass of firms is v and $\hat{v}$, respectively. If $v > \hat{v}$, then*

1. *F_k first-order stochastically dominates $\hat{F}_k$, with $\hat{\Upsilon}_k \subseteq \Upsilon_k$, for $k \in \{l, h\}$*

2. *the fraction of firms offering inefficient qualities weakly decreases: $F_h(\tilde{u}_h) \leq \hat{F}_h(\tilde{u}_h)$.*[5]

[5] If the IC-threshold $\tilde{u}_h$ belongs to support $\Upsilon_h = [u_h^m, \bar{u}_h]$, an increase in v can be shown to *strictly* decrease the mass of firms offering inefficient qualities. See Proposition 5 below for necessary and sufficient conditions (in terms of primitives) under which $\bar{u}_h \geq \tilde{u}_h$.

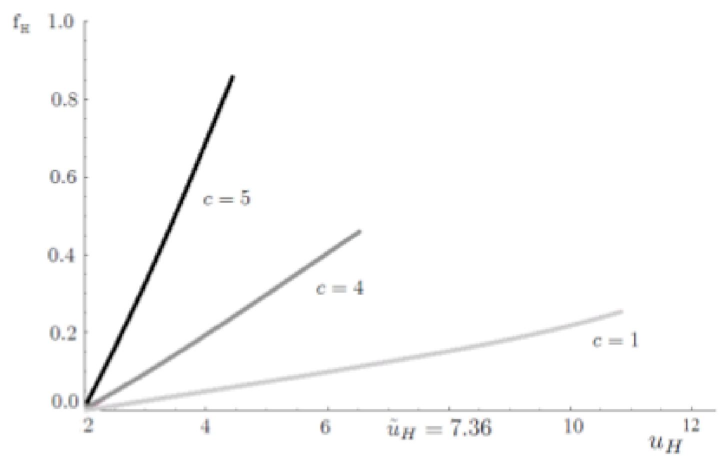

Figure 3: The equilibrium densities over the indirect utilities offered to high types for $v = 1, 4$ and 5, respectively.

Figure 3 illustrates the result from Proposition 3 by plotting the densities over the indirect utilities offered to high-type consumers. The distribution of high-type payoffs are ordered in terms of first-order stochastic dominance, and its supports are nested, as stated in the proposition above. The fraction of firms offering inefficient qualities is the area below each curve in the interval $[0, \tilde{u}_h]$.

The proposition above captures changes in the degree of market competition by varying the mass of firms, v. An alternative and intimately related notion of competition keeps v fixed, but varies the level of *frictions* of the random matching technology. This is explored in the next remark.

Remark 4 *[**Frictions and Distortions**] We say that the matching technology associated with the kernel $\Lambda(y|v)$ is less frictional than the matching technology associated with the kernel $\tilde{\Lambda}(y|v)$ if for all $y \in [0,1]$,*

$$\frac{\Lambda(y|v)}{\Lambda(0|v)} \geq \frac{\hat{\Lambda}(y|v)}{\hat{\Lambda}(0|v)}.$$

In the Generalized Burdett-Judd model, the matching technology becomes less frictional as the distribution of sample sizes increases in the sense of likelihood ratio dominance. In the Poisson-Burdett-Judd, the level of frictions is captured by the parameter β, which measures how the mass of firms v impacts the average sample size observed by consumers. In the Butters model, the level of frictions is captured by the parameter n, which is the number of consumers aware of the menu of each firm.

Proposition 3 can be recast in terms of the degree of frictions of the matching technology: As the matching technology becomes less frictional, e.g. when β or n increase, the distributions of indirect utilities increase in the sense of first-order stochastic dominance, and the fraction of firms offering efficient qualities increases.

The next proposition studies limiting properties of equilibria as the mass of firms in the market converges to zero or infinity.

Proposition 4 ***[Competition and Distortions: Limiting Cases]*** *Assume that conditions VM holds.*

1. *If* $\lim_{v\to 0} R(1|v) = 1$, *then, as the mass of firms converges to zero,* $v \to 0$, *the equilibrium distribution over menus converges to a degenerate distribution centered at the monopolistic (Mussa-Rosen) menu* $\mathcal{M}^m$. *In particular, the fraction of firms offering inefficient menus is one for small enough* v.

2. *If* $\lim_{v\to\infty} R(y|v) = \infty$ *for all* $y \in (0,1]$, *then, as the mass of firms grows large,* $v \to \infty$, *the distribution over menus converges to a degenerate distribution centered at the competitive (Bertrand) menu* $\mathcal{M}^*$. *In particular, the fraction of firms offering efficient menus converges to one.*

The first part of Proposition 4 investigates the limit properties of equilibrium when $v \to 0$. Besides V-Monotonicity, it requires that the proportional gains on sales from offering the most generous contract in the cross-section, relative to offering the least generous contract, converges to zero when the mass of competing firms approaches zero. This is a rather weak technical condition satisfied by the matching technologies of Examples 2, 3 and 4. It also holds for the Generalized Burdett-Judd model of Example 1 provided that the collection of sample size distributions $\{\Omega(v) : v > 0\}$ satisfies weak regularity conditions.[6]

To understand the result, note that, as the mass of firms v approaches zero, the support of h-type indirect utilities converges to u_h^m, the Mussa-Rosen indirect utility. As a consequence, the distribution over menus approach a degenerate distribution centered at the monopolistic menu. When the parameters of the price-discrimination problem dictate that $q_l^m = 0$ (see equation (8)), low types are excluded in the limit as $v \to 0$.

The second part of Proposition 4 investigates the limit properties of equilibria when $v \to \infty$. Besides V-Monotonicity, it requires that the proportional gains on sales, relative to the least generous contract, from offering a contract at *any* quantile $y > 0$, grows large as $v \to \infty$. This condition is satisfied by the Generalized Burdett-Judd matching technology provided that weak regularity conditions are satisfied (see footnote 6). It is also satisfied by the Poisson-Burdett-Judd and the Butters matching technologies. However, this condition is not satisfied by the Burdett-Mortensen matching technology. Under this technology, the distributions of indirect utilities converge to non-degenerate distributions. It can be shown however that the distribution of indirect utilities from

[6]Namely, assuming that $\lim_{v\to 0} \sum_{k\geq 1} \frac{\omega_k(v)}{\omega_1(v)} < \infty$, $\lim_{v\to 0} \frac{\omega_k(v)}{\omega_1(v)} = 0$ for all $k \geq 2$, and $\lim_{v\to\infty} \frac{\omega_k(v)}{\omega_1(v)} = \infty$ for some $k \geq 2$ implies condition L. These conditions are satisfied by the Poison model, for example.

"accepted offers" indeed converges to a degenerate distribution centered at the Bertrand menu.[7]

Importantly, Propositions 3 and 4 capture the entire spectrum of industry competitiveness. When v is small, competition is weak, and we obtain the sensible prediction that firms' behavior is close to that of a firm with complete market power. When v is large, equilibria approach the outcome of a perfectly competitive market.

Remark 5 *[**Vanishing Frictions**] Similarly to Proposition 3, Proposition 4 can be recast in terms of the degree of frictions of the matching technology. In the case of the Poisson-Burdett-Judd and the Generalized Butters models (where frictions can be modeled parametrically), we say that frictions vanish as $\beta \to \infty$ and $n \to \infty$, respectively. Accordingly, in the limit as frictions vanish, the distribution over menus converges to a degenerate distribution centered at the competitive (Bertrand) menu $\mathcal{M}^*$.*

3.4 Equilibrium Uniqueness

We will now discuss the important issue of equilibrium uniqueness, and identify the only possible source of equilibrium multiplicity in our model. In a nutshell, the next proposition shows that when the mass of firms v is small the ordered equilibrium is the unique equilibrium. In turn, when the mass of firms is large there are equilibria which are not ordered. As will be clear below, the uniqueness of equilibria crucially depends on whether the incentive constraint IC_h binds for all menus offered in the ordered equilibrium. In the case of multiplicity, all equilibria lead to the same distribution over contracts for each type of consumer as the ordered equilibrium. Therefore, all equilibria induce the same distribution over indirect utilities to each type of consumer, and the same ex-ante profits for firms.

Proposition 5 *[**Incentive Constraints and Equilibrium Uniqueness**] Assume that condition VM holds, and that $\lim_{v\to 0} R(1|v) = 1$ and $\lim_{v\to\infty} R(1|v) = \infty$.[8] Then there exists a threshold $\tilde{v} > 0$ on the mass of competing firms such that:*

1. *if $v \leq \tilde{v}$, the IC-threshold $\tilde{u}_h$ satisfies $\tilde{u}_h \geq \bar{u}_h$, and the downward incentive constraint (IC_h) is binding for all menus offered in the ordered equilibrium. In this case, the only equilibrium is the ordered equilibrium.*

[7]The proof is available upon request. Intuitively, in the Burdett-Mortensen matching model, there is always a positive mass of consumers that are willing to accept *any* contract (as they had their previous matches dissolved exogenously). As $v \to \infty$, these consumers receive an exploding number of new offers, and instantaneously switch away from any contract that offers less than the Bertrand contract. Firms however are indifferent at the limit between obtaining positive profits for a vanishing amount of time and getting zero profits for a longer period.

[8]This is technical condition is satisfied by the matching technologies of Examples 1 2, 3 and 4.

2. if $v > \tilde{v}$, the IC-threshold $\tilde{u}_h$ satisfies $\tilde{u}_h < \bar{u}_h$, and the downward incentive constraint (IC_h) is slack for all menus in the ordered equilibrium with $u_h > \tilde{u}_h$, and binding for $u_h \leq \tilde{u}_h$. In this case, there exist multiple equilibria that differ only on the menus for which $u_h > \tilde{u}_h$ (i.e., the efficient menus). However, all equilibria (including the non-ordered ones) lead to the same distributions over indirect utilities $F_k(\cdot)$, and the same ex-ante profits for firms.

The proof contained in the appendix shows that in *any* equilibrium, when the mass of firms is small (i.e., $v \leq \tilde{v}$), the support of utilities of type-k consumers, Υ_k, is contained in $[u_k^m, \tilde{u}_k]$. Using the increasing differences property (see Remark 3) we show that this implies that all equilibria are equal to the ordered equilibrium.

In contrast, when the mass of firms is large (i.e., $v > \tilde{v}$), some menus offered in the ordered equilibrium exhibit no binding incentive constraints. Consider such a menu $(\hat{u}_l(u_h), u_h)$, in which case $u_h \in (\tilde{u}_h, \bar{u}_h]$. For this menu the profit function $\pi(u_l, u_h)$ is locally modular, i.e. its cross-partial derivative is zero. As a result, for some (small) $\varepsilon > 0$, αboth the menus $(\hat{u}_l(u_h - \varepsilon), u_h)$ and $(\hat{u}_l(u_h), u_h - \varepsilon)$ are profit-maximizing for the firm. Based on the ordered equilibrium, we can thus construct a non-ordered equilibrium by replacing the menus $(\hat{u}_l(u_h), u_h)$ and $(\hat{u}_l(u_h - \varepsilon), u_h - \varepsilon)$ by their non-ordered counterparts $(\hat{u}_l(u_h - \varepsilon), u_h)$ and $(\hat{u}_l(u_h), u_h - \varepsilon)$. Proposition 5 confirms that this is the unique source of multiplicity of equilibria in our economy.

Remark 6 *[**Frictions and Uniqueness of Equilibrium**] The statements above can be recast in terms of the degree of frictions of the matching technology. Namely, in the case of the Poisson-Burdett-Judd and the Generalized Butters models, the uniqueness result of Proposition 5 holds if and only if the friction parameters β and n are small enough.*

4 Competition and Market Coverage with a Continuum of Types

The binary-types model presented above is useful to understand how competition affects quality distortions when firms face asymmetric information regarding costumers types. This model is silent, however, on the important issue of how competition affects market coverage, and, in particular, on how firms differentiate themselves in equilibrium regarding the breadth of types served by their menus. The aim of this section is to study this issue.

To do so, we extend the binary-types model to a continuum. For tractability, we let consumer valuations be uniformly distributed in the unit interval $[0, 1]$, and assume that firms costs are quadratic: $\varphi(q) = \frac{1}{2} \cdot q^2$. The reason for these assumptions is the following: Characterizing an ordered equilibrium with a continuum of types requires solving a nonlinear partial differential equation with nonstandard boundary conditions (as will be described below). For arbitrary distributions and cost functions, this equation does not admit a closed-form solution, and the (few) existence results available in the literature do not apply. While we believe that our results extend to environments other

than the uniform-quadratic, computing equilibria in such environments requires numerical techniques which are out of the scope of this work.[9]

The analysis proceeds analogously to that of Section 3. By the Revelation Principle, we can assume that firms post direct-revelation menus $\mathcal{M} \equiv ((q(\theta), x(\theta)) : \theta \in [0,1])$, where $q(\theta)$ is the quality, and $x(\theta)$ is the price of the contract designed for type θ. We let $u(\theta) \equiv \theta \cdot q(\theta) - x(\theta)$ be the indirect utility of type θ. By standard arguments, a menu $\mathcal{M}$ is incentive compatible (IC) if and only if the indirect utility schedule $u(\cdot)$ is absolutely continuous (with derivative $u'(\theta) = q(\theta)$ almost everywhere), and convex. The set of all menus $\mathcal{M}$ that are incentive compatible and individually rational (i.e., $u(\theta) \geq 0$ for all $\theta \in [0,1]$) is denoted by $\mathbb{I}$. For conveniency, and in light of incentive compatibility, we write $\mathcal{M} = u(\cdot)$ to describe the menu $\mathcal{M} \equiv ((q(\theta), x(\theta)) : \theta \in [0,1])$, where $q(\theta) = u'(\theta)$ and $x(\theta) = \theta \cdot u'(\theta) - u(\theta)$ for almost every θ.

As in the model with binary types, we model the heterogeneity of information possessed by consumers by means of sales functions satisfying Assumption 1. For a given cross-section distribution over menus $\tilde{F}$ (with support $\mathbb{S} \subseteq \mathbb{I}$), we denote by $F(\tilde{u};\theta)$ the marginal distribution over indirect utilities for each type θ (with support $\Upsilon(\theta)$). We can therefore write the firms profit-maximization problem as that of choosing an indirect utility schedule $u(\cdot)$ to maximize the functional

$$\pi[u] \equiv \int_0^1 \Lambda\left(F(u(\theta);\theta)|v\right) \cdot \left(\theta \cdot u'(\theta) - u(\theta) - \frac{1}{2} \cdot \left[u'(\theta)\right]^2\right) d\theta. \tag{21}$$

The expression above computes the total profits of a menu $u(\cdot)$ by integrating the product of the sales volume, $\Lambda\left(F(u(\theta);\theta)|v\right)$, with the profits per sale, $x(\theta) - \frac{1}{2} \cdot q(\theta)^2$, over all types $\theta \in [0,1]$.

Analogously to the binary type model of the previous sections, we focus on ordered equilibrium, as formally defined below.

Definition 3 ***[Ordered Equilibrium]*** *An ordered equilibrium is a distribution over menus $\tilde{F}$ (with marginal distribution over type-θ indirect utilities $F(\cdot;\theta)$) such that*

1. *$\mathcal{M} = u(\cdot) \in \mathbb{S} \subseteq \mathbb{I}$ implies that $\mathcal{M} = u(\cdot)$ maximizes (21),*

2. *if $u(\cdot), \hat{u}(\cdot) \in \mathbb{S}$, and $u(\tilde{\theta}) > \hat{u}(\tilde{\theta})$ for some $\tilde{\theta} \in [0,1]$, then $u(\theta) > \hat{u}(\theta)$ for all $\theta \in [0,1]$.*

The first condition in the definition above is the usual profit-maximization requirement. The second condition captures the "ordered" feature of our equilibrium: If a menu is "more generous" to one type of consumer, then it is more generous to all consumer types.

As in the case with binary types, it is convenient to describe the support $\mathbb{S}$ by indexing each schedule $u(\cdot) \in \mathbb{S}$ by the indirect utility received by the highest type $\theta = 1$. Accordingly, we denote by $V(\theta, \bar{u})$ the indirect utility received by type θ in the menu where the highest type $\theta = 1$ obtains utility $\bar{u}$. For a given ordered equilibrium, we refer to the bivariate function $V(\cdot, \cdot)$ as its *support*

[9] In Appendix B, we derive general necessary conditions of equilibria that might be amenable to numerical analysis.

schedule. Note that, by definition, $V(\theta, \bar{u})$ is strictly increasing in $\bar{u}$ at every type θ that is not excluded (i.e, $V(\theta, \bar{u}) > 0$).

As is usual in equilibrium models with a continuum of types, we shall impose some "smoothness" properties to our solution concept. In particular, we say that an ordered equilibrium is *smooth* if at every pair $(\theta, \bar{u})$ such that $V(\theta, \bar{u}) > 0$ the following conditions hold: (i) the support schedule $V(\theta, \bar{u})$ is twice continuously differentiable on θ, and continuously differentiable on $\bar{u}$, (ii) the distribution of type-θ indirect utilities, $F(\cdot; \theta)$, is absolutely continuous, and (iii) regarded as a function of θ, the mapping $F(u; \theta)$ is continuously differentiable in θ.

4.1 Equilibrium Characterization

The next proposition describes a smooth ordered equilibrium.

Proposition 6 ***[Equilibrium Characterization - Continuum of Types]*** *There exists a smooth ordered equilibrium. In this equilibrium, the support of indirect utilities offered by firms is described by the support schedule*

$$V(\theta, \bar{u}) = \max\left\{\frac{1}{4 \cdot \bar{u}} \cdot \theta^2 + \left(1 - \frac{1}{2 \cdot \bar{u}}\right) \cdot \theta + \bar{u} + \frac{1}{4 \cdot \bar{u}} - 1, 0\right\}, \tag{22}$$

with domain on $[0, 1] \times [\frac{1}{4}, \frac{1}{2}]$.

The equilibrium distribution over menus for the highest type solves

$$\frac{\Lambda\left(F(\bar{u}; 1)|v\right)}{\Lambda\left(0|v\right)} = \frac{\frac{1}{48}}{\int_0^1 \left(\theta \cdot q(\theta, \bar{u}) - V(\theta, \bar{u}) - \frac{1}{2} \cdot [q(\theta, \bar{u})]^2\right) d\theta}, \tag{23}$$

where the supremum point of $\Upsilon(1)$*, denoted* $\bar{\bar{u}}$*, is determined by* $F(\bar{\bar{u}}; 1) = 1$*.*

Equilibrium Construction. The equilibrium construction under a continuum of types closely mirrors that of the binary type model from the previous section. First, the ordered nature of equilibrium, together with the ranking property of kernels, implies that

$$\Lambda\left(F(\bar{u}; 1)|v\right) = \Lambda\left(F(V(\theta, \bar{u}); \theta)|v\right) \tag{24}$$

at every pair $(\theta, \bar{u})$ such that $V(\theta, \bar{u}) > 0$. Differentiating the expression above with respect to $\bar{u}$ leads to the continuous analogue of equation (18):

$$V_2(\theta, \bar{u}) = \frac{\Lambda_1\left(F(\bar{u}; 1)|v\right) \cdot f(\bar{u}; 1)}{\Lambda\left(F(\bar{u}; 1)|v\right)} \cdot \left[\frac{\Lambda_1\left(F(V(\theta, \bar{u}); \theta)|v\right) \cdot f\left(V(\theta, \bar{u}); \theta\right)\right)}{\Lambda\left(F(V(\theta, \bar{u}); \theta)|v\right)}\right]^{-1}, \tag{25}$$

which states that the partial derivative of the support schedule with respect to $\bar{u}$ at $(\theta, \bar{u})$ equals the ratio between the semi-elasticities of sales with respect to indirect utilities between the highest type and type θ.

Second, the optimality of equilibrium menus implies that the following Euler equation has to hold at any $(\theta, \bar{u})$ where $V(\theta, \bar{u}) > 0$:

$$\underbrace{\Lambda_1 \left(F(V(\theta,\bar{u});\theta)|v\right) \cdot f(V(\theta,\bar{u});\theta) \cdot \left(\theta \cdot V_1(\theta,\bar{u}) - V(\theta,\bar{u}) - \frac{1}{2}\left(V_1(\theta,\bar{u})\right)^2\right)}_{\text{sales gains}}$$

$$= \underbrace{\Lambda \left(F(V(\theta,\bar{u});\theta)|v\right)}_{\text{profit losses}} + \underbrace{\Lambda \left(F(V(\theta,\bar{u});\theta)|v\right) \cdot \frac{d}{d\theta}\left\{\theta - V_1(\theta,\bar{u})\right\}}_{\text{efficiency effect}}. \tag{26}$$

Analogously to the first-order conditions (11) and (12), the Euler equation above identifies the three effects that determine the firms' optimal choice of menus. The first term captures the effect of generosity on sales, while the second effect accounts for the effect of generosity on profits per sale. More interestingly, the third term captures the effect of increasing the indirect utility of type θ on the quality distortions of its "adjacent" types (as implied by incentive constraints). Similarly to the binary type model, the optimality condition alone is not enough to sign the efficiency effect: While increasing the indirect utility of type θ allows the firm to decrease quality distortions to its "lower neighbors", it also tightens the IC constraints of its "upper neighbors" (which leads to higher distortions).

Combining the ranking condition (25) with the optimality condition (26) leads to the following partial differential equation, that the support schedule has to satisfy in any smooth ordered equilibrium:

$$V_2(\theta,\bar{u}) = \frac{2 - V_{11}(1,\bar{u})}{2 - V_{11}(\theta,\bar{u})} \cdot \frac{\theta \cdot V_1(\theta,\bar{u}) - V(\theta,\bar{u}) - \frac{1}{2}\cdot\left(V_1(\theta,\bar{u})\right)^2}{\frac{1}{2} - \bar{u}}. \tag{27}$$

The partial differential equation above is the analogue of the ordinary differential equation (14) from Proposition 1. Guided by the binary type model, we posit that the support schedule has to satisfy the following boundary conditions:

$$V\left(\theta, \frac{1}{4}\right) = \theta^2 - \theta + \frac{1}{4}, \qquad V\left(\theta, \frac{1}{2}\right) = \frac{\theta^2}{2}, \qquad V_1(1,\bar{u}) = 1, \qquad V(1,\bar{u}) = \bar{u}. \tag{28}$$

The first boundary condition in (28) states that the Mussa-Rosen menu is the "lower" menu in the support $\mathbb{S}$ (in the sense that it provides the lower indirect utility to every type). Intuitively, the firm that offers the least generous menu is preferred to any other firm known to the consumer. Therefore, this firm must offer the monopoly menu. The second boundary condition guarantees that equilibrium menus approach the Bertrand (or efficient) menu as firms relinquish the total surplus to consumers. The third boundary condition requires that the highest type is offered the efficient quality in all menus in $\mathbb{S}$. The last boundary condition requires that the solution to (27) is consistent with the definition of the support schedule $V(\theta, \bar{u})$.

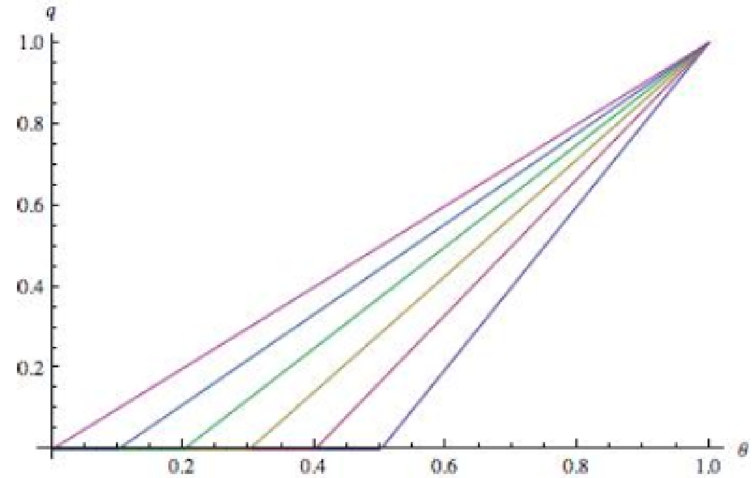

Figure 4: The quality schedules associated with $\bar{\bar{u}} = \{0.25, 0.3, 0.35, 0.4, 0.45, 0.5\}$, from the bottom to the top curves, respectively.

The support schedule $V(\theta, \bar{u})$ in equation (22) is a solution of the partial differential equation (27) subject to the boundary conditions in (28).[10] In the proof of Proposition 6, contained in the Appendix, we formalize the equilibrium construction sketched above. Most importantly, we establish that the Euler equation (26) is a necessary and sufficient condition that any menu that maximizes (21) has to satisfy, and rule out deviations to menus that offer out-of-equilibrium contracts to any type.

Finally, similarly to Proposition 1, the indifference condition (26) guarantees that all menus offered in equilibrium lead to the same total profits as the Mussa-Rosen menu. As before, the matching technology, captured by the kernel $\Lambda\left(y|v\right)$, determines the upper limit in the support of indirect utilities to the highest consumer type, $\bar{\bar{u}}$, as well as its cumulative distribution function, $F(\bar{u}; 1)$. By virtue of the ordered nature of the equilibrium, the distribution over indirect utilities of any type $\theta \in [0, 1)$ can be recovered from equation (24).

[10]We conjecture that this is the unique solution to the partial differential equation, in which case the equilibrium described in Proposition 6 is the unique smooth ordered equilibrium. It is however challenging to prove this claim, as, in contrast to the theory of ordinary differential equations, little is known about conditions under which *nonlinear* partial differential equations admit a unique solution.

4.2 Equilibrium Properties

Let us start our discussion of equilibrium properties with the relationship between generosity and distortions. To do so, let us consider the collection of quality schedules

$$q(\theta,\bar{u}) \equiv V_1(\theta,\bar{u}) = \max\left\{\frac{1}{2\cdot\bar{u}}\cdot\theta + \left(1-\frac{1}{2\cdot\bar{u}}\right), 0\right\}, \tag{29}$$

indexed by the indirect utility offered to the highest type. First, we see that whenever $q(\theta,\bar{u}) > 0$ we have that

$$\frac{\partial q(\theta,\bar{u})}{\partial \bar{u}} = \left(\frac{1-\theta}{2\bar{u}^2}\right) > 0.$$

Therefore, as in the binary type model, distortions decrease for all types as firms offer more generous menus. Figure 4 above depicts some quality schedules offered in equilibrium.

We will now turn to study the effects of competition on market coverage. From (29) we see that the range of types served by a menu with highest-type utility $\bar{u}$ is the interval $[\alpha(\bar{u}), 1]$, where

$$\alpha(\bar{u}) = (1-2\bar{u}). \tag{30}$$

It follows from (30) that $\alpha(\bar{u})$ is decreasing in $\bar{u}$. Therefore, firms segment themselves according to the range of consumer types served by their menus, which we call *inclusiveness*. As such, more generous firms, as captured by $\bar{u}$, are also more inclusive, in the sense that they serve a larger range of types. In one extreme lies the Mussa-Rosen menu, which is the least generous and the more exclusive equilibrium menu. In the other extreme, lies the menu associated with highest-type indirect utility $\bar{\bar{u}}$, which is the most generous and the more inclusive menu offered in equilibrium. This is illustrated in Figure 4 above.

Finally, and analogously to the binary-type model, firms that offer more generous contracts have their sales concentrated in consumers with low willingness to pay. Formally, for each $\theta' \in (0,1)$, the share of profits obtained from consumers with type $\theta \in [0,\theta']$ is increasing in u. We collect these findings in Proposition 7.

Proposition 7 *[**Equilibrium Properties**] The following properties hold in the ordered equilibrium of Proposition 6.*

1. ***Efficiency:*** *Menus for which customers earn higher payoffs are more efficient, i.e., $q(\theta,\bar{u}')$ is strictly increasing in $\bar{u}$ at any $(\theta,\bar{u})$ such that $\theta < 1$ and $q(\theta,\bar{u}) > 0$.*

2. ***Inclusiveness****: Firms that offer more generous contracts serve a larger set of consumers, i.e., the range $[\alpha(\bar{u}), 1]$ of types served expands as $\bar{u}$ increases.*

3. ***Profits:*** *Firms that offer more generous menus derive a greater share of profits from consumers with low willingness to pay, i.e., relative to total profits, the ratio of profits derived from consumers with types in any interval of the form $[0,\theta']$, where $\theta' < 1$, is increasing in $\bar{u}$.*

4.3 Comparative Statics

The continuous-type model of this section enables us to study how the range of types served in equilibrium, $[\alpha(\bar{\bar{u}}), 1]$, which we call *market coverage*, is affected by competition.

The next proposition shows that the equilibrium market coverage monotonically approaches its competitive level as the mass of firm increase. In one extreme, as $v \to 0$, the equilibrium market coverage approaches its monopolistic level, where only consumers with willingness to pay in the interval $\left[\frac{1}{2}, 1\right]$ are served. In the other extreme, as $v \to \infty$, the equilibrium market coverage approaches its efficient level, i.e., full market coverage.

Proposition 8 *[**Competition and Market Coverage: Comparative Statics**] Consider the smooth ordered equilibrium of Proposition 6, and assume that condition VM holds. Denote by $F(\cdot;\theta)$ and $\hat{F}(\cdot;\theta)$ the equilibrium distributions over indirect utilities of type θ when the mass of firms is v and $\hat{v}$, respectively.*

1. *If $v > \hat{v}$, then $F(\cdot;\theta)$ first-order stochastically dominates $\hat{F}(\cdot;\theta)$ for all $\theta \in [0, 1]$. In particular, the equilibrium market coverage, $[\alpha(\bar{\bar{u}}), 1]$, expands as v increases.*

2. *If $\lim_{v\to 0} R(1|v) = 1$, then, as the mass of firms converges to zero, $v \to 0$, the equilibrium distribution over menus converges to a degenerate distribution centered at the monopolistic (Mussa-Rosen) menu. In particular, the equilibrium market coverage monotonically converges to its monopoly level.*

3. *If If $\lim_{v\to\infty} R(y|v) = \infty$ for all $y \in (0, 1]$, then, as the mass of firms grows large, $v \to \infty$, the distribution over menus converges to a degenerate distribution centered at the competitive (Bertrand) menu. In particular, the equilibrium market coverage monotonically approaches $[0, 1]$, i.e., full market coverage.*

Similarly to Section 3, the results above can recast in terms of the levels of frictions of the matching technology, as discussed in Remarks 4 and 5.

5 Extensions

5.1 Fixed Costs and Free Entry

The model analyzed took the mass of firms v as exogenous. The number of firms in many markets is endogenously determined in equilibrium, as firms compare the entry cost K with the expected profit of the enterprise to make their entry decisions. Our model is flexible enough to accommodate this possibility. Letting π^m be the Mussa and Rosen profit, the profit realized by a monopolist in our model is $\Lambda(1|0) \cdot \pi^m$. Whenever the entry cost is smaller than the monopolist's profit, $K \in$

$(0, \Lambda(1|0) \cdot \pi^m)$, our model uniquely determines the competition in the market (the mass of firms v) as a function of the entry cost. In light of our comparative-static results, it is easy to show that a decrease in the entry cost decreases distortions (increasing efficiency) and increases the consumer's surplus.

5.2 Type-Specific Matching Probabilities

The model analyzed assumed that all types match with the same probability. Consumers with higher willingness to pay often obtain larger rents and hence are more eager to obtain information. In order to accommodate for this situation in a simple way, assume that type i consumer ($i \in \{l, h\}$) chooses the probability $\vartheta_i \in [0,1]$ at which he becomes informed about the availability of products in the market. The consumer who becomes informed with probability ϑ_i incurs a convex and increasing cost $\chi_i(\vartheta_i)$. In this case, the mass of informed high-type (resp. low-type) consumers is $p_h \cdot \vartheta_h$ (resp. $p_l \cdot \vartheta_l$). Therefore, firms puts probability $\left(\frac{p_h \cdot \vartheta_h}{p_h \cdot \vartheta_h + p_l \cdot \vartheta_l}\right)$ on the consumer having a high type when it observes a consumer entering their stores. Assume further that that the sales functions are homogeneous of degree zero in the measure of firms and workers. In this case, in order to calculate the equilibrium distribution over menus it suffices to consider the equilibrium of the original model in which the measure of firms is $\left(\frac{v}{p_h \cdot \vartheta_h + p_l \cdot \vartheta_l}\right)$ and the probability of a high-type consumer is $\left(\frac{p_h \cdot \vartheta_h}{p_h \cdot \vartheta_h + p_l \cdot \vartheta_l}\right)$. It is then easy to impose conditions on the primitives χ_i that guarantee the existence of equilibrium in which a positive mass of consumers is active.[11]

6 Conclusion

We study imperfect price competition in a matching market with adverse selection. On the one hand, consumers have private information about their willingness to pay for quality. On the other, consumers are imperfectly informed about the offers in the market, which is the source of market power in our model. As the market becomes more competitive, our model delivers the Bertrand equilibrium. On the other extreme, all offers are close to the the monopolistic menu when competition is weak. As we vary the mass of firms in the markets, we continuously span all competition spectrum and characterize equilibria for any degree of market imperfection. Equilibria involve dispersion over menu generosity. Firms that offer more generous menus engage less actively in price discrimination, sell to a more diverse set of consumers and obtain a higher fraction of their profits from consumers

[11] For some primitives, there are equilibria in which no consumer acquires information and in which no firm offers attractive menus. A simple assumption that rules out such equilibria is $\chi_i(0) = \chi_i(\varepsilon)$ for some $\varepsilon > 0$ (for $i \in \{l, h\}$). In this case, the equilibrium best-response of the firms depend continuously on the consumer's information acquisition strategy $(\vartheta_l, \vartheta_h) \in [0,1]^2$ of the first stage of the model. Hence, the consumer's payoff in the second stage of the model depends continuously on their action in the first stage. This final payoff, on the other hand, determines the consumer's optimal information acquisition decision. Equilibrium existence follows from Kakutani fixed-point Theorem.

with lower purchasing power.

7 Appendix A

This Appendix collects proofs of all results.

Proof. **[Proof of Lemma 1]** If the low type is offered the quality q_l, then payoffs must satisfy IC_h, i.e.,

$$u_h \geq u_l + \Delta\theta q_l. \tag{31}$$

On the other hand, IC_l requires that

$$u_l \geq u_l - \Delta\theta q_h. \tag{32}$$

The firm would like to make its offer as efficient as possible subject to the payoffs it delivers to the customer.

If $u_h - u_l < \bar{\Delta}_l \equiv \Delta\theta q_l^*$, then offering the efficient quality q_l^* for the low type is inconsistent with (31), and the firm does best to choose the highest possible value. That is, the firm chooses quality $q_l\left(u_l, u_h\right)$ which satisfies (31) with equality, or

$$q_l\left(u_l, u_h\right) \equiv \frac{u_h - u_l}{\Delta\theta}.$$

If $u_h - u_l \geq \bar{\Delta}_l$, then the constraint (31) does not bind, and the firm chooses low-type quality efficiently: $q_l\left(u_l, u_h\right) \equiv q_l^*$. Similarly, let $\bar{\Delta}_h \equiv \Delta\theta q_h^*$. If $u_h - u_l > \bar{\Delta}_h$, then asking the quality q_h^* for the high type violates (32), and so the best the firm can do is to choose $q_h\left(u_l, u_h\right)$ defined by

$$q_h\left(u_l, u_h\right) \equiv \frac{u_h - u_l}{\Delta\theta}.$$

If $u_h - u_l < \bar{\Delta}_h$, the firm offers the high-type an efficient quality: $q_h\left(u_l, u_h\right) \equiv q_h^*$. ■

For economy of notation, we define for $k \in \{l, h\}$:

$$\hat{\Phi}_k\left(u_k\right) := \Phi_k\left(F_k(u_k)|v\right).$$

Next, we prove a claim that shows that the function π satisfies increasing differences.

Claim 1 *Consider any two contracts* (u_l^1, u_h^1) *and* (u_l^2, u_h^2)*, where* $u_l^2 > u_l^1 \geq 0$ *and* $u_h^2 > u_h^1 \geq 0$ *offered in equilibrium. Then we have*

$$\pi\left(u_l^2, u_h^2\right) - \pi\left(u_l^2, u_h^1\right) \geq \pi\left(u_l^1, u_h^2\right) - \pi\left(u_l^1, u_h^1\right). \tag{33}$$

If, in addition, $u_h^1 - u_l^1 \notin [\Delta\theta q_l^*, \Delta\theta q_h^*]$ *or* $u_h^2 - u_l^2 \notin [q_l^*\Delta\theta, q_h^*\Delta\theta]$*, then the inequality in (33) is strict.*

Proof. Note that $\pi\left(u_l, u_h^2\right) - \pi\left(u_l, u_h^1\right)$ equals

$$\begin{array}{c} \hat{\Phi}_l\left(u_l\right)\left(S_l\left(u_l, u_h^2\right) - S_l\left(u_l, u_h^1\right)\right) \\ +\left(\hat{\Phi}_h\left(u_h^2\right) - \hat{\Phi}_h\left(u_h^1\right)\right)\left(S_h\left(u_l, u_h^2\right) - u_h^2\right) \\ +\hat{\Phi}_h\left(u_h^1\right)\left(\begin{array}{c} S_h\left(u_l, u_h^2\right) - u_h^2 \\ -\left(S_h\left(u_l, u_h^1\right) - u_h^1\right) \end{array}\right). \end{array} \tag{34}$$

The function $S_l\left(u_l, \cdot\right) : [u_l, \infty)$ is strictly increasing in u_h if $u_h - u_l < \Delta\theta q_l^*$ and nondecreasing otherwise. Thus the first line of (34) is increasing in u_l. The function $S_h\left(\cdot, u_h^2\right)$ is strictly increasing in u_l if $q_h\left(u_l, u_h^2\right) > q_h^*$ and nondecreasing otherwise. Thus the first second of (34) is increasing in u_l. The third term is nondecreasing in u_l if $u_h^2 - u_l \leq q_h^* \Delta\theta$. If $u_h^2 - u_l > q_h^* \Delta\theta$ we have

$$\begin{aligned} & \frac{d}{du_l}\left[S_h\left(u_l, u_h^2\right) - S_h\left(u_l, u_h^1\right)\right] \\ = & \left(\frac{\varphi'\left(q_h\left(u_l, u_h^2\right)\right) - \theta_h}{\Delta\theta}\right) - \left(\frac{\varphi'\left(q_h\left(u_l, u_h^1\right)\right) - \theta_h}{\Delta\theta}\right)\mathbf{1}\left(q_h\left(u_l, u_h^1\right) > q_h^*\right) > 0, \end{aligned}$$

where we used the fact that $\theta_h - \varphi'(q_h)$ is strictly decreasing in q_h for $q_h > q_h^*$.

It follows by the argument above that if $u_h^1 - u_l^1 \notin [\Delta\theta q_l^*, \Delta\theta q_h^*]$ (resp. if $u_h^2 - u_l^2 \notin [q_l^* \Delta\theta, q_h^* \Delta\theta]$) then $\pi\left(u_l, u_h^2\right) - \pi\left(u_l, u_h^1\right)$ is strictly increasing in u_l in a neighborhood of u_h^1 (resp. u_h^2). ■

Lemma 3 *Consider two optimal contracts* (u_l^1, u_h^1) *and* (u_l^2, u_h^2). *i) If* $u_h^1 - u_l^1 \geq \Delta\theta q_h^*$ *then* $u_h^2 > u_h^1$ *implies* $u_l^2 \geq u_l^1$. *ii) If* $u_h^1 - u_l^1 \leq \Delta\theta q_l^*$ *then* $u_h^2 > u_h^1$ *implies* $u_l^2 \geq u_l^1$.

Proof. This follows immediately from Lemma 1. For Case (i), suppose that $u_h^1 - u_l^1 \geq \Delta\theta q_h^*$ and $u_h^2 > u_h^1$ but that $u_l^2 < u_l^1$. Then $u_h^2 - u_l^1 > \Delta\theta q_h^*$, and, by (33)

$$\pi\left(u_l^1, u_h^2\right) - \pi\left(u_l^1, u_h^1\right) > \pi\left(u_l^2, u_h^2\right) - \pi\left(u_l^2, u_h^1\right)$$

or

$$\pi\left(u_l^1, u_h^1\right) + \pi\left(u_l^2, u_h^2\right) < \pi\left(u_l^1, u_h^2\right) + \pi\left(u_l^2, u_h^1\right).$$

That (u_l^1, u_h^1) and (u_l^2, u_h^2) are optimal contracts implies $\pi\left(u_l^1, u_h^1\right) = \pi\left(u_l^2, u_h^2\right)$. Thus we must have either $\pi\left(u_l^1, u_h^1\right) < \pi\left(u_l^1, u_h^2\right)$ or $\pi\left(u_l^1, u_h^1\right) < \pi\left(u_l^2, u_h^1\right)$, which contradicts optimality of $\left(u_l^1, u_h^1\right)$. ■

Proof. [Proof of Lemma 2]

Step 1. No mass points in the distribution of high-type offers.

Next, we show that F_h has no mass points. Assume towards a contradiction there is an atom of firms offering $\tilde{u}_h$.

We first show that, if a firm makes an equilibrium offer $(\tilde{u}_l, \tilde{u}_h)$, for some value $\tilde{u}_l$, then $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h > 0$. Suppose not. Then it must be that $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \leq 0$ (in case $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h \leq 0$

and $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l > 0$, offering only the option designed for the low type improve the seller's expected profit). Hence, $\pi(\tilde{u}_l, \tilde{u}_h) \leq 0$. This contradicts seller optimization. Indeed, the seller could offer a menu which yields the Mussa and Rosen utilities (u_l^m, u_h^m) and obtain a payoff at least as large as $(S_h(u_l^m, u_h^m) - u_h^m)\, \Phi_k(0)|v) > 0$.

Next, notice that $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \geq 0$. If not, the seller can profit by offering the menu $(q_l, x_l) = (0, 0)$ and $(q_h, x_h) = (q_h^*, \theta_h q_h^* - \tilde{u}_h)$. Irrespective of whether the low type finds it incentive compatible to choose the option $(0, 0)$, the seller is guaranteed an expected profit at least as high as under the original menu.

These two observations imply that $\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) > \pi(\tilde{u}_l, \tilde{u}_h)$ for $\varepsilon > 0$ sufficiently small, contradicting the optimality of $(\tilde{u}_l, \tilde{u}_h)$. To see this, note that $\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon)$ must be bounded below by

$$\begin{aligned} &\pi(\tilde{u}_l, \tilde{u}_h) - \varepsilon \left[\hat{\Phi}_h(\tilde{u}_h + \varepsilon) + \hat{\Phi}_l(\tilde{u}_l + \varepsilon)\right] \\ &+ (S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h - \varepsilon) \left[\hat{\Phi}_h(\tilde{u}_h + \varepsilon) - \hat{\Phi}_h(\tilde{u}_h)\right], \end{aligned}$$

Since $\hat{\Phi}_h(\tilde{u}_h + \varepsilon) - \hat{\Phi}_h(\tilde{u}_h)$ is bounded above zero as $\varepsilon \searrow 0$, and since $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h > 0$, the expression above is greater than $\pi(\tilde{u}_l, \tilde{u}_h)$ whenever ε is sufficiently small.

Step 2. No mass points in the distribution of low-type offers.

First, we show that there are no mass points at any $u_l > 0$. Suppose towards a contradiction that F_l has a mass point at some $\tilde{u}_l > 0$. Take a firm that offers $(\tilde{u}_l, \tilde{u}_h)$. Since, as reasoned above, $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \geq 0$, we can consider two cases.

Case 1: $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l > 0$.

As noted in Step 1, the expected profit conditional on selling to a high type must also be positive. Notice that in this case $\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon)$ is bounded below by

$$\begin{aligned} &\pi(\tilde{u}_l, \tilde{u}_h) - \varepsilon \left[\hat{\Phi}_h(\tilde{u}_h + \varepsilon) + \hat{\Phi}_l(\tilde{u}_l + \varepsilon)\right] \\ &+ (S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l - \varepsilon) \left(\hat{\Phi}_l(\tilde{u}_l + \varepsilon) - \hat{\Phi}_l(\tilde{u}_l)\right). \end{aligned}$$

Since $\hat{\Phi}_l$ has a mass point at $\tilde{u}_l$, and since $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l > 0$, the expression above is strictly greater than $\pi(\tilde{u}_l, \tilde{u}_h)$ for $\varepsilon > 0$ sufficiently small.

Case 2: $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l = 0$.

Let $\{(q_l, x_l), (q_h, x_h)\} = \{(q_l(\tilde{u}_l, \tilde{u}_h), x_l(\tilde{u}_l, \tilde{u}_h)), (q_h(\tilde{u}_l, \tilde{u}_h), x_h(\tilde{u}_l, \tilde{u}_h))\}$ be the menu offered by the firm. Consider a deviation to the menu $\{(q_l, x_l + \varepsilon), (q_h, x_h)\}$ for some small $\varepsilon > 0$. Since $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h > 0$, whether the low-type buyer chooses the option $(q_l, x_l + \varepsilon)$ or (q_h, x_h), the seller's expected profit increases strictly.

Now we show that there are no mass points at $u_l = 0$. Assume Towards a contradiction that $F_l(0) > 0$. From Step 1, there exists $\varepsilon > 0$ for which $\tilde{F}(\{0\} \times [\varepsilon, \infty)) > 0$. From (31) there is $\chi > 0$

such that $S_l(0,u_h) > \chi$ for all $(0,u_h) \in \{0\} \times [\varepsilon,\infty)$. Therefore, for small $\eta > 0$ the difference $\pi(\eta,u_h) - \pi(0,u_h)$ is

$$\begin{array}{c}\left(\hat{\Phi}_l(\eta) - \hat{\Phi}_l(0)\right)[S(\eta,u_h) - \eta] \\ -\hat{\Phi}_l(0)(S_l(0,u_h) - S(\eta,u_h) - \eta).\end{array} \tag{35}$$

We can take η^* such that $\eta \in (0,\eta^*)$ implies that the first line of (35) is at least $\left(\hat{\Phi}_l(0_+) - \hat{\Phi}_l(0)\right)\left(\frac{\chi}{2}\right) > 0$. Moreover, the second line of (35) converges to 0 as $\eta \searrow 0$, which shows a profitable deviation.

Step 3. The supports Υ_k are intervals.

Suppose for a contradiction that one or both of the supports are disconnected sets. Assume that Υ_l is disconnected. Assume that there are u_l' and u_l'' in Υ_l and that $(u_l',u_l'') \cap \Upsilon_l = \emptyset$. Let u_h' be the largest u_h such that (u_l',u_h') is optimal and let u_h'' be the smallest u_h such that (u_l'',u_h'') is optimal. From Steps 1 and 2 and Lemma 3 we may assume that $\hat{\Phi}_l(u_l') = \hat{\Phi}_l(u_l'')$, $\hat{\Phi}_h(u_h') = \hat{\Phi}_h(u_h'')$ and $u_h' \leq u_h''$.

If $u_h' < u_h''$ then there is $\varepsilon > 0$ for which $\pi(u_l'' - \varepsilon, u_h'' - \varepsilon) > \pi(u_l'',u_h'')$. Thus assume that $u_h' = u_h''$. For any $\varepsilon \in (0, u_l'' - u_l')$, optimality requires $\pi(u_l'' - \varepsilon, u_h'') \leq \pi(u_l'',u_h'')$. This implies that $q_h(u_l'',u_h'') > q_h^*$, i.e. IC_l binds. Thus $\frac{\partial^2 S_h(u_l,u_h)}{\partial u_h^2} < 0$ at (u_l'',u_h''), which implies (using $\hat{\Phi}_l(u_l') = \hat{\Phi}_l(u_l'')$ and $\hat{\Phi}_h(u_h') = \hat{\Phi}_h(u_h'')$) that there is $\lambda \in (0,1)$ for which $\pi(\lambda u_l' + (1-\lambda)u_l'', u_h'') > \lambda\pi(u_l',u_h'') + (1-\lambda)\pi(u_l'',u_h'')$, which shows that (u_l'',u_h'') is not optimal. The proof that Υ_h is connected is analogous and omitted.

Step 4. The minimum of the supports Υ_l and Υ_h are, respectively, $u_l^m = 0$ and u_h^m.

Let $\underline{u}_l$ and $\underline{u}_h$ be the minimum of the supports of Υ_l and Υ_h respectively. It follows from Steps 1 and 2 and from Lemma 3 that $(\underline{u}_l,\underline{u}_h)$ is an optimal menu. Clearly, we have $\underline{u}_l \geq 0$. Suppose that $\underline{u}_l > 0$. Notice that $\hat{\Phi}_l(0|v) = \hat{\Phi}_l\left(\frac{\underline{u}_l}{2}\right) = \hat{\Phi}_l(\underline{u}_l)$ and $\hat{\Phi}_h(0|v) = \hat{\Phi}_h\left(\underline{u}_h - \frac{\underline{u}_l}{2}\right) = \hat{\Phi}_h(\underline{u}_h)$, which implies that $\pi\left(\frac{\underline{u}_l}{2}, \underline{u}_h - \frac{\underline{u}_l}{2}\right) > \pi(\underline{u}_l,\underline{u}_h)$, establishing a contradiction. It follows that $\underline{u}_h$ maximizes

$$\hat{\Phi}_l(0|v)\,S_l(0,u_h) + \hat{\Phi}_h(0|v)(S_h(0,u_h) - u_h). \tag{36}$$

Since u_h^m is the only maximizer of (36) the claim follows.

Step 5. F_l and F_h are absolutely continuous.

We will show that F_h is Lipschitz continuous. Notice that from 1. in Assumption 1 it suffices to show that $\hat{\Phi}_h$ is Lipschitz continuous. For that, we show that there is $K > 0$ such that $\hat{\Phi}_h(u_h + \varepsilon) - \hat{\Phi}_h(u_h) < K\varepsilon$ for all $\varepsilon > 0$. For $k \in \{l,h\}$, let $\Upsilon_k := [\bar{u}_k, \underline{u}_k]$ for the remainder of this proof.

First, we claim that we may find a constant $\underline{S}_h > 0$ such that we have $S_h(u_l',u_h') - u_h' \geq \underline{S}_h$ for every optimal menu (u_l',u_h'). If not, we may find a sequence of optimal menus (u_l^n,u_h^n) such that $S_h(u_l^n,u_h^n) - u_h^n \leq \frac{1}{n}$. Taking a subsequence if necessary, assume that $(u_l^n,u_h^n) \to (u_l^*,u_h^*)$. By the continuity of $\hat{\Phi}_k$ (Steps 1 and 2) and the continuity of S_k (for $k \in \{l,h\}$) we conclude that (u_l^*,u_h^*) is optimal and that $S_h(u_l^*,u_h^*) - u_h^* = 0$. It must be that $S_l(u_l^*,u_h^*) - u_l^* > 0$, otherwise we would

have concluded that $\pi(u_l^*, u_h^*) < \pi(u_l^m, u_h^m)$. Let (x_l^*, q_l^*) be the contract purchased by l in (u_l^*, u_h^*). Therefore, offering the menu $\{(x_l^*, q_l^*), (x_l^*, q_l^*)\}$ increases the seller's profit, a contradiction.

Next, take u_h from the support of F_h. Let (u_l, u_h) be an optimal menu. Notice that $\pi(u_l, u_h + \varepsilon)$ is

$$\begin{array}{c} \hat{\Phi}_l(u_l)\left[S_l(u_l, u_h+\varepsilon) - u_l\right] + \hat{\Phi}_h(u_h+\varepsilon)\left[S_h(u_l, u_h+\varepsilon) - u_h - \varepsilon\right] \\ \geq \left[\begin{array}{c} \hat{\Phi}_l(u_l)\left[S_l(u_l, u_h) - u_l\right] + \hat{\Phi}_h(u_h)\left[S_h(u_l, u_h) - u_h - \varepsilon\right] \\ -\hat{\Phi}_l(\bar{u}_l)\xi_l - \hat{\Phi}_h(\bar{u}_h)(\xi_h+1)\varepsilon + \left[\hat{\Phi}_h(u_h+\varepsilon) - \hat{\Phi}_h(u_h)\right]\underline{S}_h \end{array}\right], \end{array}$$

where $\xi_h := \sup_{(\tilde{u}_l,\tilde{u}_h)\in\Upsilon_L\times\Upsilon_h}\left|\frac{\partial S_h(\tilde{u}_l,\tilde{u}_h)}{\partial u_h}\right|$, and $\xi_l := \sup_{(\tilde{u}_l,\tilde{u}_h)\in\Upsilon_L\times\Upsilon_h}\left|\frac{\partial S_l(\tilde{u}_l,\tilde{u}_h)}{\partial u_h}\right|$. Since $\pi(u_l, u_h+\varepsilon) \leq \pi(u_l, u_h)$ we have:

$$\frac{\hat{\Phi}_h(u_h+\varepsilon) - \hat{\Phi}_h(u_h)}{\varepsilon} \leq \left(\frac{\hat{\Phi}_l(\bar{u}_l)\xi_l + \hat{\Phi}_h(\bar{u}_h)(\xi_h+1)}{\underline{S}_h}\right),$$

which shows that $\hat{\Phi}_h$, and hence F_h, are absolutely continuous. The proof that F_l is absolutely continuous is analogous and omitted. ■

Proof. [Proof of Proposition 1] Step 1. Construction of an ordered equilibrium.

Since from Lemma 2 the functions $\hat{\Phi}_h(\cdot)$ and $\hat{\Phi}_l(\cdot)$ are absolutely continuous, the following conditions are necessary for optimality:

$$\hat{\Phi}_l'(u_l)\left[S_l(u_l, u_h) - u_l\right] - \hat{\Phi}_l(u_l) + \sum_{k=l,h}\hat{\Phi}_k(u_k)\frac{\partial S_k(u_l, u_h)}{\partial u_l} = 0. \tag{37}$$

$$\hat{\Phi}_h'(u_h)\left[S_h(u_l, u_h) - u_h\right] - \hat{\Phi}_h(u_h) + \sum_{k=l,h}\hat{\Phi}_k(u_k)\frac{\partial S_k(u_l, u_h)}{\partial u_h} = 0. \tag{38}$$

In an ordered equilibrium, there is a function $\hat{u}_l(\cdot) : \Upsilon_h \to \Upsilon_l$ such that the menu (u_l, u_h) is offered if and only if $u_h \in \Upsilon_h$ and $u_l = \hat{u}_l(u_h)$. From Lemma 2, we know that $\hat{\Phi}_h(\cdot)$ and $\hat{\Phi}_l(\cdot)$ are absolutely continuous and hence $\hat{u}_l$ is differentiable. In our first step, we will show that $\hat{u}_l$ satisfies the differential equation (14).

Step 1. The differential equation.

First, notice that in any ordered equilibrium $F_l(\hat{u}_l(u_h)) = F_h(u_h)$ and hence, from Assumption 1 we have: $\Lambda(F_l(\hat{u}_l(u_h)) \mid v) = \Lambda(F_h(u_h) \mid v)$ which implies that

$$\hat{\Phi}_l(\hat{u}_l(u_h)) = \left(\frac{p_l}{p_h}\right)\hat{\Phi}_h(u_h). \tag{39}$$

Therefore, we have

$$\hat{\Phi}_l'(\hat{u}_l(u_h))\hat{u}_l'(u_h) = \left(\frac{p_l}{p_h}\right)\hat{\Phi}_h'(u_h). \tag{40}$$

As shown in Step 4 in Lemma 2, the less generous optimal menu is (u_l^m, u_h^m). Only the constraint (31) binds for menus close to (u_l^m, u_h^m). We will construct equilibria in which this property holds for every offerer menu. Then we will verify (in Step 3 below) that the constraint (32) is globally satisfied. Notice that for all $u_h \in \Upsilon_h$ we have $\pi(\hat{u}_l(u_h), u_h) = \pi^*$ for some $\pi^* > 0$. Differentiating this equality w.r.t. u_h (and using (39) and (40)) we obtain:

$$\begin{aligned} &\hat{\Phi}_h'(u_h)\left[(S_h^* - u_h) + \left(\tfrac{p_l}{p_h}\right)(S_l^* - u_l)\right] \\ &+\hat{\Phi}_h(u_h)\begin{pmatrix} \left(\frac{p_l}{p_h}\right)\hat{u}_l'(u_h)\left(\left(\frac{\partial S_l(\hat{u}_l(u_h),u_h)}{\partial u_l}\right)-1\right) \\ +\left(\frac{p_l}{p_h}\right)\left(\frac{\partial S_l(\hat{u}_l(u_h),u_h)}{\partial u_h}\right)-1 \end{pmatrix} = 0, \end{aligned} \tag{41}$$

which can be rewritten as

$$\frac{\hat{\Phi}_h'(u_h)}{\hat{\Phi}_h(u_h)} = \frac{\begin{pmatrix} \left(\frac{p_l}{p_h}\right)\hat{u}_l'(u_h)\left(1-\left(\frac{\partial S_l(\hat{u}_l(u_h),u_h)}{\partial u_l}\right)\right) \\ +\left(\frac{p_l}{p_h}\right)\left(1-\frac{\partial S_l(\hat{u}_l(u_h),u_h)}{\partial u_h}\right)-1 \end{pmatrix}}{(S_h^* - u_h) + \left(\frac{p_l}{p_h}\right)(S_l^* - u_l)}. \tag{42}$$

The first-order condition (38) can be written as

$$\frac{\hat{\Phi}_h'(u_h)}{\hat{\Phi}_h(u_h)} = \frac{1-\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(u_l,u_h)}{\partial u_h}}{S_h(u_l,u_h) - u_h}. \tag{43}$$

Thus, setting the RHS of (42) equal to the RHS of (43) and rearranging we have:

$$\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h), \tag{44}$$

where

$$h(u_l, u_h) = \frac{S_l(u_l,u_h) - u_l}{S_h^* - u_h} \cdot \frac{1 - \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(u_l,u_h)}{1-\frac{\partial S_l}{\partial u_l}(u_l,u_h)}, \tag{45}$$

i.e. (14).

Step 2. Existence and properties of solution to ODE.

We show below that $\hat{u}_l(\cdot)$ is well defined by (44) for all $u_h \in [u_h^m, S_h^*)$ given the boundary condition $\hat{u}_l(u_h^m) = 0$.

For any $\varepsilon \in (0, S_h^*)$, the function $h(\cdot,\cdot)$ is Lipschitz continuous on

$$\Lambda(\varepsilon) \equiv \{(u_l, u_h) \in [0, S_l^*] \times [u_h^m, S_h^* - \varepsilon) : u_l < u_h\}.$$

Hence, by the Picard-Lindelöf theorem, for any $\varepsilon \in (0, S_h^*)$, and for any (u_l, u_h) in the interior of $\Lambda(\varepsilon)$, there is a unique local solution to $\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h)$.

Now consider $\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h)$ with initial condition $\hat{u}_l(u_h^m) = 0$ and note the existence of $\eta > 0$ such that a unique solution exists on $[u_h^m, u_h^m + \eta]$ where $(\hat{u}_l(u_h), u_h)$ remains in $\Lambda(0)$.

We show that $h(\hat{u}_l(u_h), u_h)$ remains bounded and that $(\hat{u}_l(u_h), u_h)$ remains in $\Lambda(0)$ as u_h increases to S_h^*, implying the existence of a global solution to $\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h)$ on $[u_h^m, S_h^*)$. To do this, we first establish the existence of a value $\tilde{u}_h \in (u_h^m, S_h^*)$ such that $\tilde{u}_h - \hat{u}_l(\tilde{u}_h) = q_l^* \Delta\theta$ and such that $\hat{u}_l(u_h)$ remains in $(u_h, S_l^*]$ on $[u_h^m, \tilde{u}_h]$.

To see this, note that, provided that $u_h - \hat{u}_l(u_h)$ remains below $q_l^* \Delta\theta$, then $h(\hat{u}_l(u_h), u_h)$ remains in $[0,1)$. That $h(\hat{u}_l(u_h), u_h)$ remains below 1 follows because

$$\begin{aligned} & S_h^* - u_h - (S_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)) \\ = \;& \theta_h q_h^* - \varphi(q_h^*) - (\theta_l q_l(\hat{u}_l(u_h), u_h) - \varphi(q_l(\hat{u}_l(u_h), u_h))) - q_l(\hat{u}_l(u_h), u_h)\Delta\theta \\ = \;& \theta_h q_h^* - \varphi(q_h^*) - (\theta_h q_l(\hat{u}_l(u_h), u_h) - \varphi(q_l(\hat{u}_l(u_h), u_h))) \\ > \;& 0 \end{aligned} \tag{46}$$

whenever $u_h - \hat{u}_l(u_h) < q_l^* \Delta\theta$. That it remains non-negative follows from two observations. First, if $S_l(\hat{u}_l(u_h), u_h) = \hat{u}_l(u_h)$ for some $u_h \in [u_h^m, S_h^*)$ such that $u_h - \hat{u}_l(u_h) < q_l^* \Delta\theta$, then $\frac{d}{du_h}[S_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)] > 0$. Second, $\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(\hat{u}_l(u_h), u_h) < 1$ provided $u_h - \hat{u}_l(u_h) \geq u_h^m$, which is guaranteed by the initial condition and that $h(\hat{u}_l(u_h), u_h)$ remains less than 1.

Now, suppose with a view to contradiction that there is no value $\tilde{u}_h \in (u_h^m, S_h^*)$. Because $q_l(\hat{u}_l(u_h), u_h) \leq q_l^* < q_h^*$, (46) implies that $\hat{u}_l(u_h)$ must remain bounded below $S_l(\hat{u}_l(u_h), u_h)$ as $u_h \nearrow S_h^*$. This contradicts that $h(\hat{u}_l(u_h), u_h)$ remains below 1.

Next, consider extending the solution to $u_h \in (\tilde{u}_h, S_h^*)$. It is easily checked that $\hat{u}_l(u_h) = S_l^* - \alpha(S_h^* - u_h)$ with $\alpha = \frac{S_l^* - \hat{u}_l(\tilde{u}_h)}{S_h^* - \tilde{u}_h}$satisfies $\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h)$ and remains in $\Lambda(0)$. That $\hat{u}_l(u_h)$ remains below S_l^* follows because $\alpha < 1$, since $S_l^* - \hat{u}_l(u_h) = \alpha(S_h^* - u_h) > 0$.

Step 3. The Incentive Constraint (32) is globally satisfied.

Step 2 above showed that there is $\tilde{u}_h \in (u_h^m, S_h^*)$ for which $u_h - \hat{u}_l(u_h) < \Delta\theta q_l^*$ for all $u_h < \tilde{u}_h$. For $u_h \in [\tilde{u}_h, S_h^*)$ we have

$$u_h - \hat{u}_l(u_h) = (u_h - S_l^*) + \alpha(S_h^* - u_h). \tag{47}$$

Notice that the derivative of the RHS of (47) w.r.t. u_h is $1 + \alpha > 0$. Hence (47) achieves its maximum at $u_h = S_h^*$ and its maximum is given by

$$S_h^* - S_l^* = \Delta\theta q_l^* + \int_{q_l^*}^{q_h^*} (\theta_h - \varphi'(q))\, dq < \Delta\theta q_h^*.$$

Thus we conclude that $u_h - \hat{u}_l(u_h) \in (\Delta\theta q_l^*, \Delta\theta q_h^*)$ for all $u_h \in (\tilde{u}_h, S_h^*)$. Therefore, the incentive constraint (32) does not bind along the curve $(\hat{u}_l(u_h), u_h)$.

Step 4. Solving for the distribution $\tilde{F}$.

From Step 4 in Lemma (2) the less generous optimal menu is (u_l^m, u_h^m). In the equilibrium that we construct, all offers yield a profit equal to:

$$\pi^* := \sum_{k=l,h} p_k \cdot \Lambda(0|v) \cdot (S_k(u_l^m, u_h^m) - u_k^m).$$

Let $\bar{u}_h$ be such that:

$$\pi^* = p_l \cdot \Lambda\left(1|v\right) \cdot \left(S_l(\hat{u}_l\left(\bar{u}_h\right), \bar{u}_h) - \hat{u}_l\left(\bar{u}_h\right)\right) + p_h \cdot \Lambda\left(1|v\right) \cdot \left(S_h(\hat{u}_l\left(\bar{u}_h\right), \bar{u}_h) - \bar{u}_h\right). \tag{48}$$

To see why there exists such $\bar{u}_h$, notice first that $\frac{\Lambda(1|v)}{\Lambda(0|v)} \in (1, \infty)$. Hence, since $\lim_{\bar{u}_h \uparrow S_h^*} S_l(\hat{u}_l\left(\bar{u}_h\right), \bar{u}_h) - \hat{u}_l\left(\bar{u}_h\right) = 0$ and $\lim_{\bar{u}_h \uparrow S_h^*} \left(S_h(\hat{u}_l\left(\bar{u}_h\right), \bar{u}_h) - \bar{u}_h\right) = 0$, the existence of $\bar{u}_h$ is guaranteed by the intermediate value Theorem. We claim that $\bar{u}_h$ is unique. Indeed, using (41) it is easy to see that the derivative of (48) with respect to $\bar{u}_h$ is strictly negative.

Next, for all $u_h \in [u_h^m, \bar{u}_h]$ the function $\hat{\Phi}_h\left(u_h\right)$ is uniquely defined by (43) and the boundary condition $\hat{\Phi}_h\left(u_h^m\right) = p_h \Lambda\left(0|v\right)$. Furthermore, $\hat{\Phi}_h$ is strictly increasing. Hence, Assumption 1 implies that we may define $F_h(u_h)$ by:

$$p_h \Lambda\left(F_h(u_h)|v\right) = \hat{\Phi}_h\left(u_h\right), \tag{49}$$

for all $u_h \in [u_h^m, \bar{u}_h]$. Analogously, we construct for all $u_l \in [u_l^m, \hat{u}_l\left(\bar{u}_h\right)]$, $F_l(u_l)$ by $F_l(u_l) = F_h(\hat{u}_l^{-1}\left(u_l\right))$.

Next, we check that no firm has a deviation. By construction, all menus (u_l, u_h) such that $u_h \in [u_h^m, \bar{u}_h]$ and $u_l = \hat{u}_l\left(u_h\right)$ yield the same profit. Moreover, it is easy to show that we may restrict attention to menus $(u_l', u_h') \in [u_h^m, \bar{u}_h] \times [u_l^m, \hat{u}_l\left(\bar{u}_h\right)]$. Hence, consider a menu $(u_l', u_h') \in [u_h^m, \bar{u}_h] \times [u_l^m, \hat{u}_l\left(\bar{u}_h\right)]$ such that $u_l' < \hat{u}_l\left(u_h'\right)$. Using straightforward algebra we conclude that $\pi(\hat{u}_l\left(u_h'\right), u_h') - \pi(u_l', u_h')$ is equal to

$$\int_{\hat{u}_l\left(u_h'\right)}^{u_l'} \frac{\partial \pi\left(\tilde{u}_l, \hat{u}_l^{-1}(\tilde{u}_l)\right)}{\partial u_l} + \int_{\hat{u}_l^{-1}(\tilde{u}_l)}^{u_h'} \frac{\partial \pi\left(\tilde{u}_l, \tilde{u}_h\right)}{\partial u_l \partial u_H} d\tilde{u}_H d\tilde{u}_l \tag{50}$$

Notice that by construction $\frac{\partial \pi(u_l, u_h)}{\partial u_l} = 0$ along the curve $\left\{\left(\hat{u}_l\left(u_h\right), u_h\right) : u_h \in [u_h^m, \bar{u}_h]\right\}$. Hence the first integral in (50) is 0. On the other hand, it follows from Claim 1 that the partial derivative in the second integral is nonnegative. Thus we have $\pi(\hat{u}_l\left(u_h'\right), u_h') - \pi(u_l', u_h') \geq 0$. By a similar argument one can show that there is no profitable deviation to $u_l' > \hat{u}_l\left(u_h'\right)$, which completes the proof. ■

Proof. **[Proof of Proposition 2]** We first prove i). Consider the differential equation $\hat{u}_l'\left(u_h\right) = h\left(\hat{u}_l\left(u_h\right), u_h\right)$ with h given by (45). In Step 2 in Proposition 1 we showed that there is $\tilde{u}_h \in (u_h^m, S_h^*)$ such that $h\left(\hat{u}_l\left(u_h\right), u_h\right) < 1$ for every $u_h < \tilde{u}_h$. On the other hand, for $u_h > \tilde{u}_h$ we have $\hat{u}_l\left(u_h\right) = S_l^* - \alpha\left(S_h^* - u_h\right)$. We also showed that $\alpha = \frac{S_l^* - \hat{u}_l(\tilde{u}_h)}{S_h^* - \tilde{u}_h} < 1$. Therefore $\delta'(u_h) = 1 - \alpha > 0$.

Now we prove ii). First, assume that $u_l'(u_h)$ is given by (45) in a neighborhood of u_h. The

derivative of ratio between the firm's profits from low and high-type consumers is

$$
\begin{aligned}
& \left(\frac{p_l}{p_h}\right)\left(\frac{\left(\frac{\partial S_l(u_l,u_h)}{\partial u_l}u_l'(u_h)+\frac{\partial S_l(u_l,u_h)}{\partial u_h}\right)(S_h^*-u_h)-u_l'(u_h)\left(S_h^*-u_h\right)+\left(S_l(u_l,u_h)-u_l\right)}{\left(S_h^*-u_h\right)^2}\right) \\
> & \left(\frac{p_l}{p_h}\right)\left(\frac{-u_l'(u_h)\left(S_h^*-u_h\right)+\left(S_l(u_l,u_h)-u_l\right)}{\left(S_h^*-u_h\right)^2}\right) \\
> & \left(\frac{p_l}{p_h}\right)\left(\frac{-\left(\frac{S_l(u_l,u_h)-u_l}{S_h^*-u_h}\right)\left(S_h^*-u_h\right)+\left(S_l(u_l,u_h)-u_l\right)}{\left(S_h^*-u_h\right)^2}\right)=0,
\end{aligned}
$$

where the first inequality uses i) and the second uses $\left(\frac{1-\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(u_l,u_h)}{1-\frac{\partial S_l}{\partial u_l}(u_l,u_h)}\right)<1$ and (45).

Next, assume that $\hat{u}_l\left(u_h\right)=S_l^*-\alpha\left(S_h^*-u_h\right)$ in a neighborhood of u_h. Thus, the ratio between the profits from the low and high type is $\alpha\left(\frac{p_l}{p_h}\right)$ which is locally constant. ■

Proof. [Proof of Proposition 3]

We only prove 1. The proof of 2. is analogous and omitted. Since we are considering ordered equilibrium[12], it suffices to show that $F_h(u_h)\leq \hat{F}_h(u_h)$ for all u_h. Towards a contradiction, take $\tilde{u}_h$ such that $F_h(\tilde{u}_h)>\hat{F}_h(\tilde{u}_h)$. Without loss assume that $\tilde{u}_h\in\Upsilon_h$ (otherwise, replace $\tilde{u}_h$ with $\max\Upsilon_h$). Therefore, we have:

$$
\begin{aligned}
& \Lambda\left(0\mid v\right)\left[p_l\cdot S_l\left(0,u_h^m\right)+p_h\left(\left(S_h\left(0,u_h^m\right)-u_h^m\right)\right)\right] \\
= & \Lambda\left(F_h(\tilde{u}_h)\mid v\right)\left[p_l\cdot\left(S_l\left(\tilde{u}_h,\hat{u}_l\left(\tilde{u}_h\right)\right)-\hat{u}_l\left(\tilde{u}_h\right)\right)+p_h\cdot\left(S_h\left(\tilde{u}_h,\hat{u}_l\left(\tilde{u}_h\right)\right)-\tilde{u}_h\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \Lambda\left(0\mid \hat{v}\right)\left[p_l\cdot S_l\left(0,u_h^m\right)+p_h\left(\left(S_h\left(0,u_h^m\right)-u_h^m\right)\right)\right] \\
= & \Lambda\left(\hat{F}_h(\tilde{u}_h)\mid \hat{v}\right)\left[p_l\cdot\left(S_l\left(\tilde{u}_h,\hat{u}_l\left(\tilde{u}_h\right)\right)-\hat{u}_l\left(\tilde{u}_h\right)\right)+p_h\cdot\left(S_h\left(\tilde{u}_h,\hat{u}_l\left(\tilde{u}_h\right)\right)-\tilde{u}_h\right)\right],
\end{aligned}
$$

and hence

$$
\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)\mid\hat{v}\right)}{\Lambda\left(0\mid\hat{v}\right)}=\frac{\Lambda\left(F_h(\tilde{u}_h)\mid v\right)}{\Lambda\left(0\mid v\right)}. \tag{51}
$$

On the other hand, $F_h(\tilde{u}_h)>\hat{F}_h(\tilde{u}_h)$ implies $\frac{\Lambda(F_h(\tilde{u}_h)|v)}{\Lambda(0|v)}>\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|v\right)}{\Lambda(0|v)}$ and Condition 1 implies $\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|v\right)}{\Lambda(0|v)}>\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|\hat{v}\right)}{\Lambda(0|\hat{v})}$ and thus $\frac{\Lambda(F_h(\tilde{u}_h)|v)}{\Lambda(0|v)}>\frac{\Lambda(F_h(\tilde{u}_h)|v)}{\Lambda(0|v)}$, which contradicts (51). ■

Proof. [Proof of Proposition 4]

We first prove 1. Take $\bar{u}_h=\max\Upsilon_h$. We have:

$$
\begin{aligned}
& \left[\frac{p_l\cdot S_l\left(0,u_h^m\right)+p_h\left(\left(S_h\left(0,u_h^m\right)-u_h^m\right)\right)}{p_l\cdot\left(S_l(\tilde{u}_h,\hat{u}_l(\tilde{u}_h))-\hat{u}_l(\tilde{u}_h)\right)+p_h\cdot\left(S_h(\tilde{u}_h,\hat{u}_l(\tilde{u}_h))-\tilde{u}_h\right)}\right] \\
& =\left[\frac{\Lambda(1|v)}{\Lambda(0|v)}\right]
\end{aligned} \tag{52}
$$

[12]Using Proposition 5, it is easy to see that this result is true for any equilibrium.

Notice that $\lim_{v\to 0} R(1|v) = 1$ implies that the RHS of (52) converges to 1 as $v \to 0$. This implies that the LHS of (52) converges to 1 and hence $(\hat{u}_l(\bar{u}_h), \bar{u}_h) \to (0, u_h^m)$ as $v \to 0$. The second statement in 1. follows immediately.

Next we prove 2. Take a sequence $(v_n) \to \infty$, and let (F_n) be a sequence of respective ordered equilibria. Take $y \in (0,1]$ and let $u_h^n(y) \equiv F_n^{-1}(y)$. We have:

$$\begin{aligned}&\left[\frac{p_l \cdot S_l\left(0, u_h^m\right) + p_h\left(\left(S_h\left(0, u_h^m\right) - u_h^m\right)\right)}{p_l \cdot \left(S_l\left(u_h^n(y), \hat{u}_l\left(u_h^n(y)\right)\right) - \hat{u}_l\left(u_h^n(y)\right)\right) + p_h \cdot \left(S_h\left(u_h^n(y), \hat{u}_l\left(u_h^n(y)\right)\right) - u_h^n(y)\right)}\right] \\ &= \left[\frac{\Lambda(y|v)}{\Lambda(0|v)}\right].\end{aligned} \tag{53}$$

Notice that the RHS of (53) diverges to ∞ by assumption. Therefore, the denominator of the LHS of (53) converges to 0, which implies that $(\hat{u}_l(u_h^n(y)), u_h^n(y))$ converge to the Bertrand menu. The second statement in 2. follows immediately. ■

Proof. [Proof of Proposition 5]

Let $\tilde{F}^0$ denote the ordered distribution established in Proposition 1, F_k^0 the marginal for type k with density f_k^0. Let π^0 be the profit function obtained from that equilibrium. Consider a possibly non-ordered equilibrium $\tilde{F}$ and recall that from Lemma 2 $\Upsilon_h = [u_h^m, \bar{u}_h]$ for some $\bar{u}_h$ and $\Upsilon_h = [0, \bar{u}_l]$ for some $\bar{u}_l$.

Step 1. There is $\varepsilon > 0$ for which $\tilde{F}(A) = \tilde{F}^0(A)$ for each (Borel-measurable) set $A \subset \mathbb{R}_+^2 \setminus [u_h^m + \varepsilon, \infty) \times [\hat{u}_l(u_h^m + \varepsilon), \infty)$.

Let $B := \mathbb{R}_+^2 \setminus [u_h^m + \varepsilon, \infty) \times [\hat{u}_l(u_h^m + \varepsilon), \infty)$. Take $\varepsilon > 0$ such that $u_h^m + \varepsilon < \Delta\theta \left(\frac{q_l\left(u_h^m, u_l^m\right) + q_l^*}{2}\right)$ and $F_h(u_h^m + \varepsilon) < 1$. Let ε_l be such that $F_l(\varepsilon_l) = F_h(u_h^m + \varepsilon)$. Take $(u_h', u_h'') \in [u_h^m, u_h^m + \varepsilon]^2$ and consider two optimal contracts (u_h', u_l') and (u_h'', u_l''). Notice that $\max\{|u_h' - u_l'|, |u_h'' - u_l''|\} < \Delta\theta \left(\frac{q_l\left(u_h^m, u_l^m\right) + q_l^*}{2}\right)$ and thus if $u_h'' > u_h'$ then Claim 1 implies $u_l'' > u_l'$. Therefore, since from Lemma 2 F_k is continuous (for $k \in \{l, h\}$) and since $\tilde{F}$ puts probability 1 on optimal contracts, $F_h(u_h'') > F_h(u_h')$ implies $F_l(u_l'') > F_l(u_l')$. This shows that there exist a strictly increasing function $\breve{u}_l$ such that $F_h(u_h) = F_l(\breve{u}_l(u_h))$ for all $u_h \in [u_h^m, u_h^m + \varepsilon]$. From Lemma 2, we see that $\breve{u}_l$ is absolutely continuous and from the same argument as the one from Proposition 1 we see that $\breve{u}_l$ satisfies (45). From Picard–Lindelöf Theorem we conclude that $\breve{u}_l$ and $\hat{u}_l$ agree on $[u_h^m, u_h^m + \varepsilon]$. Moreover, since from Lemma 2 $(0, u_h^m)$ is on the bottom of the support of every equilibrium, we conclude that every equilibrium leads to the same profit: $\pi^0(u_l^m, u_h^m)$. Using an analogous argument to the one from Step 4 in Proposition 1, we conclude that the densities f_l and f_h are (a.e.) equal to f_l^0 and f_h^0 respectively. Therefore, the measures $\tilde{F} \cdot 1_B$ and $\tilde{F}^0 \cdot 1_B$ agree on every rectangles $[u_l^1, u_l^2] \times [u_h^1, u_h^2] \subset \mathbb{R}_+^2$ and hence $\tilde{F}$ and $\tilde{F}^0$ agree on each set $A \subset \mathbb{R}_+^2 \setminus [u_h^m + \varepsilon, \infty) \times [\hat{u}_l(u_h^m + \varepsilon), \infty)$ (see Billingsley 1995, Theorem 3.3).

Step 2. $\tilde{F}(A) = \tilde{F}^0(A)$ for each (Borel-measurable) set $A \subset \mathbb{R}_+^2 \setminus [\tilde{u}_h, \infty) \times [\hat{u}_l(\tilde{u}_h), \infty)$.

By construction, $u_h^m + \varepsilon - \tilde{u}_l(u_h^m + \varepsilon) < \Delta\theta q_l^*$. Therefore, we can find $\varepsilon_2 > 0$ such that $u_h^m + \varepsilon + \varepsilon_2 - \tilde{u}_l(u_h^m + \varepsilon) < \Delta\theta q_l^*$. Thus we can extend the argument from Step 1 to $\mathbb{R}_+^2 \setminus [u_h^m + \varepsilon + \varepsilon_2, \infty) \times$

$[\hat{u}_l(u_h^m+\varepsilon+\varepsilon_2),\infty)$. Letting

$$u_h^{\#} := \sup\left\{\begin{array}{c} u_h \le \tilde{u}_h : \tilde{F}(A) = \tilde{F}^0(A) \text{ for all Borel measurable set} \\ A \subset \mathbb{R}_+^2 \setminus [u_h,\infty)\times[\hat{u}_l(u_h),\infty) \end{array}\right\},$$

we conclude that $u_h^{\#} = \tilde{u}_h$. Indeed, if $u_h^{\#} < \tilde{u}_h$ then we could find $\varepsilon > 0$ for which $u_h^{\#} + \varepsilon - \tilde{u}_l(u_h^{\#}) < \Delta\theta q_l^*$. Applying the same logic as in Step 1, we would conclude that $u_h^{\#}$ is not a supremum of the set above. This argument proves the first part of Proposition 5. Assume for the remainder of this proof that $\bar{u}_h > \tilde{u}_h$.

Step 3. There is $\varepsilon > 0$ such that if (u_l', u_h') is an optimal menu and $u_h' \in (\tilde{u}_h, \tilde{u}_h+\varepsilon)$ then $u_h' - u_l' \ge \Delta\theta \cdot q_l^*$.

From Lemma 2 for each $u_h \in (\tilde{u}_h, \tilde{u}_h+\varepsilon)$ we can find an optimal Lemma (u_l, u_h) such that $F_l(u_l) = F_h(u_h)$. Taking $\varepsilon < \Delta\theta\cdot(q_h^* - q_l^*)$ we assure that $u_h - u_l < \Delta\theta\cdot q_h^*$. Thus the menu (u_l, u_h) satisfies the following necessary conditions:

$$p_l\Lambda'(F_l(u_l)|v)\,f_l(u_l)\,[S_l(u_l,u_h)-u_l] - p_l\Lambda(F_l(u_l)|v) + p_l\Lambda(F_l(u_l)|v)\frac{\partial S_l(u_l,u_h)}{\partial u_l} = 0 \quad (54)$$

$$p_h\Lambda'(F_h(u_h)|v)\,f_h(u_h)\,[S_h(u_l,u_h)-u_h] - p_h\Lambda(F_h(u_h)|v) + p_l\Lambda(F_l(u_l)|v)\frac{\partial S_l(u_l,u_h)}{\partial u_h} = 0. \quad (55)$$

Using $F_l(u_l) = F_h(u_h)$ and straightforward algebra we obtain:

$$\frac{f_h(u_h)}{f_l(u_h)} = \left(\frac{1-\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(u_l,u_h)}{\partial u_l}}{1-\frac{\partial S_l(u_l,u_h)}{\partial u_l}}\right)\left(\frac{S_l(u_l,u_h)-u_l}{S_h(u_l,u_h)-u_h}\right). \quad (56)$$

Notice that $\frac{\partial S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)}{\partial u_l} = \frac{\partial S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)}{\partial u_h} = 0$. Thus, since F_h and F_l are continuous and $S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)$ is C^1, for each $\eta > 0$ we may choose $\varepsilon_1 > 0$ such that $u_h \in (\tilde{u}_h, \tilde{u}_h+\varepsilon_1)$ implies that the first term on the RHS of (56) is at most $(1+\eta)$:

$$\frac{f_h(u_h)}{f_l(u_h)} \le (1+\eta)\left(\frac{S_l(u_l,u_h)-u_l}{S_h(u_l,u_h)-u_h}\right).$$

Similarly, since $\left(\frac{S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\hat{u}_l(\tilde{u}_H)}{S_h(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\tilde{u}_H}\right) < 1$ we can take $\varepsilon_2 > 0$ such that $u_h \in (\tilde{u}_h, \tilde{u}_h+\varepsilon_1)$ implies

$$\frac{f_h(u_h)}{f_l(u_h)} \le (1+\eta)\left(\frac{1+\left(\frac{S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\hat{u}_l(\tilde{u}_H)}{S_h(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\tilde{u}_H}\right)}{2}\right).$$

Using the findings above, we can take $\varepsilon > 0$ such that $u_h \in (\tilde{u}_h, \tilde{u}_h+\varepsilon)$ implies

$$\frac{f_h(u_h)}{f_l(u_h)} \le \left(\frac{3+\left(\frac{S_l(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\hat{u}_l(\tilde{u}_H)}{S_h(\hat{u}_l(\tilde{u}_H),\tilde{u}_H)-\tilde{u}_H}\right)}{4}\right) < 1. \quad (57)$$

Hence assume towards a contradiction there is $u_h' \in (\tilde{u}_h, \tilde{u}_h+\varepsilon)$ for which there is an optimal menu (u_h', u_l') such that $u_h' - u_l' < \Delta\theta\cdot q_l^*$. From Lemma 2 we have $F_l\left(u_l'\right) = F_h\left(u_h'\right)$. Notice also

that since F_k are absolutely continuous we may define their inverses F_k^{-1}. Therefore, using (57) we have

$$
\begin{aligned}
\Delta\theta \cdot q_l^* &> u_h' - u_l' = \Delta\theta \cdot q_l^* + u_h' - \tilde{u}_h - \left(u_l' - \hat{u}_l(\tilde{u}_h)\right) \\
&= \Delta\theta \cdot q_l^* + F_h^{-1}\left(F_h(u_h')\right) - F_h^{-1}\left(F_h(\tilde{u}_h)\right) - \left(F_l^{-1}\left(F_l(u_h')\right) - F_l^{-1}\left(F_l(\hat{u}_l(\tilde{u}_h))\right)\right) \\
&= \Delta\theta \cdot q_l^* + \int_{F_h(\tilde{u}_h)}^{F_h(\tilde{u}_h)} \left(\frac{1}{f_h(F_h^{-1}(s)} - \frac{1}{f_l(F_l^{-1}(s)}\right) ds \geq \Delta\theta \cdot q_l^*,
\end{aligned}
$$

which is an absurd.

Step 4. Conclusion.

Define:

$$
u_h^{\#\#} := \sup \left\{ \begin{array}{c} u_h \leq \bar{u}_h : \text{if } \left(u_l', u_h'\right) \text{ is optimal and } u_h \in (\tilde{u}_h, u_h') \\ \text{then } u_h' - u_l' \in [\Delta\theta \cdot q_l^*, \Delta\theta \cdot q_h^*] \end{array} \right\}.
$$

If we show that $u_h^{\#\#} = \bar{u}_h$ we will conclude that neither IC_l nor IC_h binds for contracts (u_l', u_h') such that $u_h' > \tilde{u}_h$. It is then straightforward to show using (54) and (55) (with $\frac{\partial S_l}{\partial u_k} = 0$ for $k \in \{l, h\}$) that $F_k^0 = F_k$ for $k \in \{l, h\}$. Thus we would have concluded the proof of the proposition. Hence, suppose towards a contradiction that $u_h^{\#\#} < \bar{u}_h$. Define $u_l^{\#\#}$ such that $F_l(u_l^{\#\#}) = F_h(u_h^{\#\#})$. From Lemma 2 $(u_l^{\#\#}, u_h^{\#\#})$ is an optimal menu for which $u_h^{\#\#} - u_l^{\#\#} \in \{\Delta\theta \cdot q_l^*, \Delta\theta \cdot q_h^*\}$. Assume that $u_h^{\#\#} - u_l^{\#\#} = \Delta\theta \cdot q_h^*$. It follows from (56) that

$$
\begin{aligned}
\frac{f_h(u_h^{\#\#})}{f_l(u_h^{\#\#})} &= \left(\frac{S_l^* - u_l^{\#\#}}{S_h^* - u_l^{\#\#} - \Delta\theta \cdot q_h^*}\right) \\
&= \left(\frac{S_l^* - u_l^{\#\#}}{S_l^* - u_l^{\#\#} + \int_{q_l^*}^{q_h^*} (\theta_l - \varphi'(s))ds}\right) > 1.
\end{aligned}
$$

Using and argument similar to the one from Step 3, one can show that there is $u_h' < u_h^{\#\#}$ for which the optimal contract $\left(F_l^{-1}\left(F_h(u_h')\right), u_h'\right)$ satisfies $u_h' - F_l^{-1}\left(F_h(u_h')\right) > \Delta\theta \cdot q_h^*$, which contradicts the definition of $u_h^{\#\#}$. The case that $u_h^{\#\#} - u_l^{\#\#} = \Delta\theta \cdot q_l^*$ can be handled by a similar argument and its proof is omitted. ■

8 Continuum of Types

In this note we will derive the analogous for a continuum of types of the support function obtained in the case of a binary type space. The firms' problem is to choose an indirect utility schedule $u(\theta)$ to maximize

$$
\int_{\underline{\theta}}^{\bar{\theta}} \Lambda(G[u(\theta)|\theta]) \cdot (\theta \cdot \dot{u}(\theta) - \varphi(\dot{u}(\theta)) - u(\theta)) \cdot f(\theta) d\theta, \tag{58}
$$

where $\Lambda(y)$ is the kernel of the matching technology, $\varphi(q)$ is the cost of producing a good of quality q, and $f(\theta)$ is the density of type θ (with support $[\underline{\theta}, \bar{\theta}]$).

We posit that in equilibrium each firm is indifferent between choosing any schedule in some support $\mathbb{S}$. Each of the schedules $u(\theta)$ in $\mathbb{S}$ are strictly increasing and weakly convex in θ (that is, there is an implementable direct-revelation mechanism delivering an indirect utility $u(\theta)$).

We can conveniently describe the support $\mathbb{S}$ by indexing each schedule by the indirect utility received by type $\bar{\theta}$. Denote by $V(\theta, u)$ the indirect utility received by type θ when type $\bar{\theta}$ obtain utility u. Note that $V(\bar{\theta}, u) = u$. The set of indirect utility schedules is then

$$\mathbb{S} = \left\{ V(\theta, u) : u \in [\bar{u}^m, \bar{S}^*] \right\},$$

where $\bar{u}^m$ is the Mussa-Rosen indirect utility of type $\bar{\theta}$, and $\bar{S}^*$ is the Bertrand indirect utility of type $\bar{\theta}$: $\bar{S}^* \equiv \max_q \left\{ \bar{\theta} \cdot q - \varphi(q) \right\}$ Let us consider a differentiable equilibrium where $V(\theta, u)$ is twice continuously differentiable at every point where $V(\theta, u) > 0$.

It is a property of the ordered equilibrium that for any two types $\theta, \tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$ such that $V(\theta, u), V(\tilde{\theta}, u) > 0$

$$\Lambda(G\left[V(\theta, u)|\theta\right]) = \Lambda(G\left[V(\tilde{\theta}, u)|\tilde{\theta}\right]). \tag{59}$$

It is an implication of (59) that

$$\frac{d}{d\theta}\left\{\Lambda(G\left[V(\theta, u)|\theta\right])\right\} = 0. \tag{60}$$

Note that $V(\bar{\theta}, u) = u$, what implies that

$$\Lambda(G\left[V(\theta, u)|\theta\right]) = \Lambda(G\left[u|\bar{\theta}\right]).$$

Differentiating with respect to u leads to

$$\Lambda'(G\left[V(\theta, u)|\theta\right]) \cdot g\left[V(\theta, u)|\theta\right] \cdot V_2(\theta, u) = \Lambda'(G\left[u|\bar{\theta}\right]) \cdot g\left[u|\bar{\theta}\right]. \tag{61}$$

Optimality implies that every menu $V(\theta, u) \in \mathbb{S}$ has to satisfy the following Euler equation at any θ where $V(\theta, u) > 0$:

$$\Lambda'(G\left[V(\theta, u)|\theta\right]) \cdot g\left[V(\theta, u)|\theta\right] \cdot (\theta \cdot V_1(\theta, u) - \varphi\left(V_1(\theta, u)\right) - V(\theta, u)) \cdot f(\theta) - \Lambda(G\left[V(\theta, u)|\theta\right]) \cdot f(\theta)$$

$$= \frac{d}{d\theta}\left\{\Lambda(G\left[V(\theta, u)|\theta\right]) \cdot \left(\theta - \varphi'(V_1(\theta, u))\right) \cdot f(\theta)\right\}. \tag{62}$$

Because of (60), it follows that

$$\frac{d}{d\theta}\left\{\Lambda(G\left[V(\theta, u)|\theta\right]) \cdot \left(\theta - \varphi'(V_1(\theta, u))\right) \cdot f(\theta)\right\}$$

$$= \Lambda(G\left[V(\theta, u)|\theta\right]) \cdot \frac{d}{d\theta}\left\{\left(\theta - \varphi'(V_1(\theta, u))\right) \cdot f(\theta)\right\}. \tag{63}$$

Plugging (63) into (62) and manipulating leads to:

$$\frac{\Lambda'(G\left[V(\theta,u)|\theta\right])}{\Lambda(G\left[V(\theta,u)|\theta\right])}\cdot g\left[V(\theta,u)|\theta\right]=\frac{f(\theta)+\frac{d}{d\theta}\left\{(\theta-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}}{\left(\theta\cdot V_1(\theta,u)-\varphi\left(V_1(\theta,u)\right)-V(\theta,u)\right)\cdot f(\theta)}. \tag{64}$$

Let us choose $\theta=\bar{\theta}$ in (64) to obtain that:

$$\frac{\Lambda'(G\left[u|\bar{\theta}\right])}{\Lambda(G\left[u|\bar{\theta}\right])}\cdot g\left[u|\bar{\theta}\right]=\frac{f(\bar{\theta})+\frac{d}{d\theta}\left\{(\theta-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}_{\theta=\bar{\theta}}}{\left(\bar{\theta}\cdot V_1(\bar{\theta},u)-\varphi\left(V_1(\bar{\theta},u)\right)-u\right)\cdot f(\bar{\theta})}. \tag{65}$$

Note that

$$\left(\frac{\Lambda'(G\left[V(\theta,u)|\theta\right])}{\Lambda(G\left[V(\theta,u)|\theta\right])}\cdot g\left[V(\theta,u)|\theta\right]\right)^{-1}\cdot\frac{\Lambda'(G\left[u|\bar{\theta}\right])}{\Lambda(G\left[u|\bar{\theta}\right])}\cdot g\left[V(\theta,u)|\bar{\theta}\right]=V_2(\theta,u), \tag{66}$$

where the equality follows from (61).

Dividing (64) by (65), and using the relation (66), we then obtain that

$$V_2(\theta,u)=\frac{\left(\theta\cdot V_1(\theta,u)-\varphi\left(V_1(\theta,u)\right)-V(\theta,u)\right)\cdot f(\theta)}{\left(\bar{\theta}\cdot V_1(\bar{\theta},u)-\varphi\left(V_1(\bar{\theta},u)\right)-V(\bar{\theta},u)\right)\cdot f(\bar{\theta})}\cdot\frac{f(\bar{\theta})+\frac{d}{d\theta}\left\{(V_1(\theta,u)-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}_{\theta=\bar{\theta}}}{f(\theta)+\frac{d}{d\theta}\left\{(V_1(\theta,u)-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}}.$$

We posit that in the ordered equilibrium the highest type $\bar{\theta}$ is always assigned the efficient quality level. The PDE is then

$$V_2(\theta,u)=\frac{\left(\theta\cdot V_1(\theta,u)-\varphi\left(V_1(\theta,u)\right)-V(\theta,u)\right)\cdot f(\theta)}{\left(\bar{S}^*-u\right)\cdot f(\bar{\theta})}\cdot\frac{f(\bar{\theta})+\frac{d}{d\theta}\left\{(V_1(\theta,u)-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}_{\theta=\bar{\theta}}}{f(\theta)+\frac{d}{d\theta}\left\{(V_1(\theta,u)-\varphi'(V_1(\theta,u)))\cdot f(\theta)\right\}}. \tag{67}$$

Denote by $\bar{u}^m$ the indirect utility of type $\bar{\theta}$ in the Mussa-Rosen schedule. The PDE (67) has to be solved in the range $u\in[\bar{u}^m,\bar{S}^*]$, $\theta\in[\underline{\theta},\bar{\theta}]$ with boundary conditions

$$V(\theta,\bar{S}^*)=\max_q\ \ \theta\cdot q-\varphi(q), \tag{68}$$

$$V(\theta,\bar{u}^m)=\max_q\ \ \left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)\cdot q-\varphi(q), \tag{69}$$

$$V_1(\bar{\theta},u)=V_1(\bar{\theta},\bar{S}^*) \tag{70}$$

and

$$V(\bar{\theta},u)=u. \tag{71}$$

The boundary condition (68) states that the Bertrand schedule is the "supremum" contract in the support $\mathbb{S}$. The boundary condition (69) states that the Mussa-Rosen schedule is the "infimum" contract in the support $\mathbb{S}$. The boundary condition (70) requires that the type $\bar{\theta}$ receives the same quality (which is the efficient one) in all contracts in $\mathbb{S}$. The boundary condition (71) requires that the solution to (67) is consistent with the definition of $V(\theta,u)$.

8.1 Uniform-Quadratic Case

Assume that production costs are quadratic, $\varphi(q) = \frac{1}{2} \cdot q^2$, and types are uniformly distributed, $\theta \sim U[0,1]$.

The PDE (67) becomes

$$V_2(\theta, u) = \frac{2 - V_{11}(\bar{\theta}, u)}{2 - V_{11}(\theta, u)} \cdot \frac{\theta \cdot V_1(\theta, u) - \frac{1}{2} \cdot (V_1(\theta, u))^2 - V(\theta, u)}{\frac{1}{2} - u}, \tag{72}$$

with domain on $[0,1] \times [\frac{1}{4}, \frac{1}{2}]$.

The proposed solution to the PDE above subject to the boundary conditions (68), (69), (70), and (71) is

$$V(\theta, u) = \frac{1}{4 \cdot u} \cdot \theta^2 + \left(1 - \frac{1}{2 \cdot u}\right) \cdot \theta + u + \frac{1}{4 \cdot u} - 1.$$

Verification

Let us first compute partial derivatives:

$$V_1(\theta, u) = \frac{1}{2 \cdot u} \cdot \theta + \left(1 - \frac{1}{2 \cdot u}\right), \tag{73}$$

and

$$V_2(\theta, u) = -\frac{1}{4 \cdot u^2} \cdot \theta^2 + \frac{1}{2 \cdot u^2} \cdot \theta + 1 - \frac{1}{4 \cdot u^2}.$$

Let us first verify the boundary conditions. To verify (68), note that $\bar{S}^* = \frac{1}{2}$. Therefore,

$$V(\theta, \bar{S}^*) = V\left(\theta, \frac{1}{2}\right) = \frac{1}{2} \cdot \theta^2 = \max_q \;\; \theta \cdot q - \frac{1}{2} \cdot q^2.$$

To verify (69), note that $\bar{u}^m = \frac{1}{4}$. Therefore,

$$V(\theta, \bar{u}^m) = V\left(\theta, \frac{1}{4}\right) = \theta^2 - \theta + \frac{1}{4} = \max_q \;\; (2 \cdot \theta - 1) \cdot q - \frac{1}{2} \cdot q^2.$$

To verify (70), note that

$$V_1(1, u) = \frac{1}{2 \cdot u} + \left(1 - \frac{1}{2 \cdot u}\right) = 1.$$

To verify (71), note that

$$V(1, u) = \frac{1}{4 \cdot u} + 1 - \frac{1}{2 \cdot u} + u + \frac{1}{4 \cdot u} - 1 = u.$$

To verify that (72) is satisfied, note that

$$\theta \cdot V_1(\theta, u) - \frac{1}{2} \cdot (V_1(\theta, u))^2 - V(\theta, u)$$

$$= \theta \cdot \left(\frac{1}{2 \cdot u} \cdot \theta + \left(1 - \frac{1}{2 \cdot u} \right) \right) - \frac{1}{2} \cdot \left(\frac{1}{2 \cdot u} \cdot \theta + \left(1 - \frac{1}{2 \cdot u} \right) \right)^2$$

$$- \frac{1}{4 \cdot u} \cdot \theta^2 - \left(1 - \frac{1}{2 \cdot u} \right) \cdot \theta - u - \frac{1}{4 \cdot u} + 1,$$

which, after some algebra, can be shown to be equal to

$$\left(\frac{1}{2} - u \right) \cdot \left(- \frac{1}{4 \cdot u^2} \cdot \theta^2 + \frac{1}{2 \cdot u^2} \cdot \theta + 1 - \frac{1}{4 \cdot u^2} \right) = \left(\frac{1}{2} - u \right) \cdot V_2(\theta, u).$$

Because $V_{11}(\bar{\theta}, u) = V_{11}(\theta, u)$, it follows that (72) holds.

Checking Indifference

Denote

$$\Pi(u) \equiv \int_{\alpha(u)}^{\bar{\theta}} \Lambda(G\left[V(\theta, u)|\theta\right]) \cdot \left(\theta \cdot V_1(\theta, u) - \frac{1}{2} \cdot (V_1(\theta, u))^2 - V(\theta, u) \right) d\theta,$$

where $\alpha(u)$ solves

$$V_1(\alpha(u), u) = 0.$$

It is easy to verify that $V(\alpha(u), u) = 0$.

To simplify notation, let

$$F\left(V(\theta, u), V_1(\theta, u), \theta\right) \equiv \Lambda(G\left[V(\theta, u)|\theta\right]) \cdot \left(\theta \cdot V_1(\theta, u) - \frac{1}{2} \cdot (V_1(\theta, u))^2 - V(\theta, u) \right).$$

Then

$$\Pi'(u) \equiv \int_{\alpha(u)}^{\bar{\theta}} \left\{ F_1\left(V(\theta, u), V_1(\theta, u), \theta\right) \cdot V_2(\theta, u) \right\} d\theta$$

$$+ \int_{\alpha(u)}^{\bar{\theta}} \left\{ F_2\left(V(\theta, u), V_1(\theta, u), \theta\right) \cdot V_{12}(\theta, u) \right\} d\theta$$

$$- F\left(V(\alpha(u), u), V_1(\alpha(u), u), \alpha(u)\right) \cdot \alpha'(u). \qquad (74)$$

Integration by parts delivers that

$$\int_{\alpha(u)}^{\bar{\theta}} \left\{ F_2\left(V(\theta, u), V_1(\theta, u), \theta\right) \cdot V_{12}(\theta, u) \right\} d\theta$$

$$= F_2\left(V(\theta, u), V_1(\theta, u), \theta\right) \cdot V_2(\theta, u) \Big|_{\alpha(u)}^{\bar{\theta}}$$

$$- \int_{\alpha(u)}^{\bar{\theta}} V_2(\theta, u) \cdot \left\{ \frac{d}{d\theta} \left[F_2\left(V(\theta, u), V_1(\theta, u), \theta\right) \right] \right\} d\theta.$$

Plugging the above into (74) and using the fact $V(\theta,u)$ solves the Euler equation for every (θ,u) with $\theta>\alpha(u)$ leads to

$$\Pi'(u)=F_2\left(V(\theta,u),V_1(\theta,u),\theta\right)\cdot V_2(\theta,u)|_{\alpha(u)}^{\overline{\theta}}-F\left(V(\alpha(u),u),V_1(\alpha(u),u),\alpha(u)\right)\cdot\alpha'(u).$$

Note that

$$F_2\left(V(\overline{\theta},u),V_1(\overline{\theta},u),\overline{\theta}\right)=\Lambda(G\left[V(\overline{\theta},u)|\overline{\theta}\right])\cdot\left(\overline{\theta}-V_1(\overline{\theta},u)\right)=0.$$

Recall that $V(\alpha(u),u)=0$. Total differentiation yields

$$V_1(\alpha(u),u)\cdot\alpha'(u)+V_2(\alpha(u),u)=0.$$

Because by construction $V_1(\alpha(u),u)=0$, it follows that $V_2(\alpha(u),u)=0$. This implies that

$$F_2\left(V(\theta,u),V_1(\theta,u),\theta\right)\cdot V_2(\theta,u)|_{\alpha(u)}^{\overline{\theta}}=0.$$

Finally, because $V(\alpha(u),u)=0$ and $V_1(\alpha(u),u)=0$,

$$F\left(V(\alpha(u),u),V_1(\alpha(u),u),\alpha(u)\right)=0.$$

This establishes that $\Pi'(u)=0$ for all $u\in[\frac{1}{4},\frac{1}{2}]$.

9 Existence

We write $\Phi(u,\theta)$ for the sales function from offering a utility u to the type $\theta:\Phi(u,\theta)\equiv\Lambda(G\left[u|\theta\right])$.For every $\bar{u}\in\left[\frac{1}{4},\bar{\bar{u}}\right]$ write $(u_{\bar{u}}(\theta))$ for the curve $(V(\theta,\bar{u}))_{\theta\in[0,1]}$.

We consider the relaxed problem in which $q(\theta)\in\mathbb{R}$ and $q(\theta)$ need not be monotonic.

Lemma 4 *There exists a solution in the class of absolutely continuous functions* $AC[0,1]$..

Proof. We define:

$$\Pi(\theta,u,\dot{u}):=\Phi(u,\theta)\left[\theta\cdot\dot{u}-\frac{1}{2}\cdot(\dot{u})^2-u\right].$$

Let $A:=\sup_{u,\theta}\Phi(u,\theta)$. It is straightforward to see that we may restrict attention to allocations such that $\left[\theta\cdot\dot{u}-\frac{1}{2}\cdot(\dot{u})^2-u\right]\geq 0$ for almost all θ. Therefore

$$\begin{aligned}\Pi(\theta,u,\dot{u}) &\leq A[\dot{u}-\frac{1}{2}\cdot(\dot{u})^2]\leq A\dot{u}\left(1-\frac{1}{4}\cdot\dot{u}\right)-\left(\frac{A}{4}\right)\cdot(\dot{u})^2\\ &\leq A-\left(\frac{A}{4}\right)\cdot(\dot{u})^2\end{aligned}$$

Hence, we may assume that Π is coercive of degree 2: $\Pi(\theta, u, \dot{u}) \leq A - \left(\frac{A}{4}\right) \cdot (\dot{u})^2$ for (almost) every $(\theta, u, \dot{u})$. Furthermore, notice that $\Pi(\theta, u, \dot{u})$ is continuous in $(\theta, u, \dot{u})$ and it is concave in $\dot{u}$. The existence of an absolutely continuous solution follows from Theorem 16.2 in Clarke (2013). ∎

Proof. [Proof of Proposition 6]. Write $(u^* (\theta))$ for an optimal allocation.

We claim that for every $\bar{u} \in \Upsilon(1)$ the allocation $(u_{\bar{u}} (\theta))$ is optimal. First, consider two curves $(u_\beta (\theta)), (u_\gamma (\theta)) \in AC[0, 1]$. Let $K > 0$ be such that $|\Phi(u_\beta (\theta), \theta) - \Phi(u_\gamma (\theta), \theta)| \leq K |u_\beta (\theta) - u_\gamma (\theta)|$. Furthermore, since $\max_{q,\theta} \left|\theta \cdot q - \frac{1}{2} \cdot (\dot{u} (\theta))^2\right| \leq \frac{1}{2}$ we have:

$$
\begin{aligned}
& |\Pi(\theta, u_\beta (\theta), \dot{u}_\beta (\theta)) - \Pi(\theta, u_\gamma (\theta), \dot{u}_\gamma (\theta))| \\
= & \left|\Phi(u_\beta (\theta), \theta) \left[\theta \cdot \dot{u}_\beta (\theta) - \frac{1}{2} \cdot (\dot{u}_\beta (\theta))^2 - u_\beta (\theta)\right] - \Phi(u' (\theta), \theta) \left[\theta \cdot \dot{u}_\gamma (\theta) - \frac{1}{2} \cdot (\dot{u}_\gamma (\theta))^2 - u_\gamma (\theta)\right]\right| \\
\leq & K |u_\beta (\theta) - u_\gamma (\theta)| + KA |\dot{u}_\beta (\theta) - \dot{u}_\gamma (\theta)|,
\end{aligned}
$$

hence the Lipschitz condition (LH) in page 348 of Clarke (2013) is satisfied. Therefore, we may apply Theorem 18.1 in Clarke (2013).

We have to deal with 5 cases.

Case 1: $u^* (1) \in \left(u_{\frac{1}{4}} (1), u_{\bar{u}}(1)\right)$.

Notice that since $u^* (\theta)$ is absolutely continuous there is $\varepsilon_1 > 0$ and a neighborhood of

$\{(\theta, u^* (\theta), \dot{u}^* (\theta)) : \theta \in [1 - \varepsilon_1, 1]\}$ for which Π is C^2. Thus condition (E) for a (locally C^1 function) in Theorem 18.1 assures the existence of and arc $p^* : [0, 1] \to \mathbb{R}$ for which

$$
\begin{aligned}
\dot{p}^*(\theta) &= \frac{\partial\Pi(\theta, u^* (\theta), \dot{u}^* (\theta))}{\partial u^* (\theta)} && (75) \\
p^*(\theta) &= \frac{\partial\Pi(\theta, u^* (\theta), \dot{u}^* (\theta))}{\partial \dot{u}^* (\theta)}. && (76)
\end{aligned}
$$

From the transversality condition (T) in Theorem 18.1 we conclude that $p^*(1) = 0$ and thus there is $\varepsilon_2 \in (0, \varepsilon_1)$ for which $\dot{u}^* (\theta) \in \left[\frac{1}{2}, 2\right]$ for (almost) all $\theta \in [1 - \varepsilon_2, 1]$. It follows from (76) that for (almost) all $\theta \in [1 - \varepsilon_2, 1]$ we have

$$\dot{u} (\theta) = \theta - \left(\frac{p^*(\theta)}{\Phi(u (\theta), \theta)}\right),$$

and hence $\dot{u} (\theta)$ is a Lipschitz function for this interval. Since $\Pi(\theta, u (\theta), \dot{u} (\theta))$ is strictly concave in $\dot{u} (\theta)$ for this interval and Φ is smooth for this interval, we may apply Theorem 15.7 in Clarke (2013) to conclude that $u (\theta)$ is smooth function for this interval. Therefore, the Euler Equation

$$\frac{d}{d\theta} [\Phi(u (\theta), \theta) (\theta - \dot{u} (\theta))] = \frac{\partial}{\partial u (\theta)} \left[\Phi(u (\theta), \theta) \left[\theta \cdot \dot{u} (\theta) - \frac{1}{2} \cdot (\dot{u} (\theta))^2 - u (\theta)\right]\right] \qquad (77)$$

holds for this interval. Let $(y (\theta), y'(\theta))$ be a solution of the Euler equation(77) subject to $y(1) = \bar{u}(1)$.Picard–Lindelöf theorem establishes that this solution is unique and $y(\theta) = u^*(\theta)$ for all $\theta \in [1 - \varepsilon_2, 1]$. Let $\underline{\theta} (u^* (\theta))$ be the greatest point in $[0, 1]$ such that $y(\underline{\theta} (u^* (\theta))) = 0$. One may thus use

a similar argument to show that the Euler equation (77) holds for all θ for which $u(\theta) > 0$. It follows that $u^*(\theta) = y(\theta)$ for all $\theta \in [\underline{\theta}(u^*(1)), 1]$ and 0 elsewhere.

Case 2: $u^*(1) > u_{\bar{\bar{u}}}(1)$.

Using a similar argument to the one from **Case 2** one can show that there exists $\varepsilon > 0$ and a neighborhood of $\{(\theta, u^*(\theta), \dot{u}^*(\theta)) : \theta \in [1-\varepsilon, 1]\}$ for which an increase in $u(\theta)$ does not increase sales, that is $\Phi(u^*(\theta), \theta) = A$, and the Euler equation holds. Thus, the Euler equation (77) implies $\ddot{u}^*(\theta) = 2$. Since from the transversality condition (T) in Theorem 18.1 we have $\dot{u}^*(1) = 1$ we conclude that there is an interval for which $u^*(\theta) = \theta^2 - \theta + c$. Let $[\theta_a, 1]$ be the largest interval with this property.

Next, consider the Euler equation (77) evaluated at the curve $(u_{\bar{\bar{u}}}(\theta))$. Since $\Phi(u_{\bar{\bar{u}}}(\theta), \theta) = A$ we have:

$$\ddot{u}_{\bar{\bar{u}}}(\theta) = 2 - \frac{\Phi_1(u_{\bar{\bar{u}}}(\theta)_-, \theta)}{A}\left[\theta \cdot \dot{u}_{\bar{\bar{u}}}(\theta) - \frac{1}{2} \cdot (\dot{u}_{\bar{\bar{u}}}(\theta))^2 - u_{\bar{\bar{u}}}(\theta)\right] < \ddot{u}^*(\theta) = 2,$$

thus $\dot{u}_{\bar{\bar{u}}}(\theta) > \dot{u}^*(\theta)$ for all $\theta \in (\theta_a, 1)$. Therefore $\theta_a < \underline{\theta}(u_{\bar{\bar{u}}}(1))$, however since $\dot{u}_{\bar{\bar{u}}}(\alpha(u_{\bar{\bar{u}}}(1))) = 0$ we conclude that for all $\theta \in (\theta_a, \alpha(u_{\bar{\bar{u}}}(1)))$ we have $\dot{u}^*(\theta) < 0$, which shows that there is a profitable deviation.

Case 3: $u^*(1) < u_{\frac{1}{4}}(1)$.

Proceeding exactly as in **Case 2**, we conclude that there is an interval $(\hat{\theta}, 1)$ for which u^* is smooth. Furthermore, we have $\dot{u}^*(1) = 2$ we have $\ddot{u}^*(\theta) = 2$ for every θ in this interval. We see that:

$$\ddot{u}_{\frac{1}{4}}(\theta) = 2 - \frac{\Phi_1(u_{\frac{1}{4}}(\theta)_+, \theta)}{\Phi(u_{\frac{1}{4}}(\theta)_+, \theta)}\left[\theta \cdot \dot{u}_{\frac{1}{4}}(\theta) - \frac{1}{2} \cdot \left(\dot{u}_{\frac{1}{4}}(\theta)\right)^2 - u_{\frac{1}{4}}(\theta)\right],$$

which shows that $\ddot{u}_{\frac{1}{4}}(\theta) \leq \ddot{u}^*(\theta)$ for every $\theta \in (\hat{\theta}, 1)$.

First, assume that $u_{\frac{1}{4}}(\theta) > u^*(\theta)$ for all θ for which $u^*(\theta) > 0$. Write $B := \Phi(u_{\frac{1}{4}}(1)_+, 1)$. In this case, the firm sells to each consumer for which $u^*(\theta) > 0$ with probability B. Therefore, the profit from this contract is weakly lower than the profit from a monopolist who faces a constant sales function equal to B. Since the unique solution to the later problem is given by the curve $\left(u_{\frac{1}{4}}(\theta)\right)$ we conclude that there is a profitable deviation.

Next, assume that there is θ' for which $0 < u_{\frac{1}{4}}(\theta') = u^*(\theta')$ and let θ^* be the greatest θ' satisfying this condition. Recall that the curve $\dot{u}_{\frac{1}{4}}(\theta)$ solved:

$$\begin{array}{l} \max_{(\dot{u}(\theta))} B \int_{\frac{1}{2}}^{1} \left[\theta \cdot \dot{u}(\theta) - \frac{1}{2} \cdot (\dot{u}(\theta))^2 - u(\theta)\right] d\theta \\ \text{s.t.: } u(\theta) = \int_{\frac{1}{2}}^{1} \dot{u}(z) dz \text{for all } \theta \in \left[\frac{1}{2}, 1\right] \end{array} \tag{78}$$

which implies that $\left(\dot{u}_{\frac{1}{4}}(\theta)\right)_{\theta \in [\theta^*, 1]}$ solves:

$$\begin{array}{l} \max_{\left(\dot{u}_{\frac{1}{4}}(\theta)\right)_{\theta \in [\theta^*, 1]}} B \int_{\theta^*}^{1} \left[\theta \cdot \dot{u}(\theta) - \frac{1}{2} \cdot (\dot{u}(\theta))^2 - u(\theta)\right] d\theta \\ \text{s.t.: } u(\theta) = u_{\frac{1}{4}}(\theta^*) + \int_{\theta^*}^{1} \dot{u}(z) dz \text{for all } \theta \in [\theta^*, 1]. \end{array} \tag{79}$$

Since $\left(\dot{u}_{\frac{1}{4}}(\theta)\right)_{\theta\in[\theta^*,1]}$ was (a.e.) unique and $\left(\dot{u}_{\frac{1}{4}}(\theta)\right)_{\theta\in[\theta^*,1]}$ is different from $(\dot{u}^*(\theta))_{\theta\in[\theta^*,1]}$ in a subset of positive measure, we conclude that $(u^*(\theta))$ is not optimal.

Case 4: $u^*(1) = u_{\bar{\bar{u}}}(1)$.

First we claim that $u^*(\theta) \leq u_{\bar{\bar{u}}}(\theta)$ for every θ. Suppose that there is $\theta' \in (0,1)$ such that $u^*(\theta') > u_{\bar{\bar{u}}}(\theta')$. Let $\theta^* \in (\theta', 1]$ be the smallest $\theta > \theta'$ such that $u^*(\theta) = u_{\bar{\bar{u}}}(\theta)$. Thus notice that $\Phi(u^*(\theta), \theta) = A$ for all $\theta \in (\theta', \theta^*)$. Therefore, using an argument similar to the one from Case 1, we conclude that u^* is smooth in (θ', θ^*). Thus, we have $\dot{u}^*(\theta^*) \leq \dot{u}_{\bar{\bar{u}}}(\theta^*)$ and by the same argument as in Case 2, $\ddot{u}^*(\theta) > \ddot{u}_{\bar{\bar{u}}}(\theta)$ for all $\theta \in (\theta', \theta^*)$. Therefore, $\dot{u}^*(\theta) < \dot{u}_{\bar{\bar{u}}}(\theta)$ for all $\theta \in (\alpha(u_{\bar{\bar{u}}}(1)), \theta^*)$ and thus there is $\hat{\theta} \in (\alpha(u_{\bar{\bar{u}}}(1)), \theta^*)$ such that $\theta \in \left(\alpha(u_{\bar{\bar{u}}}(1)), \hat{\theta}\right)$ implies $u^*(\theta) > 0$ and $\dot{u}^*(\theta) < 0$, a contradiction.

The remaining of this proof of this case is complicated because the function $\Phi(\cdot, \theta)$ is not differentiable along the curve $(\theta, u_{\bar{\bar{u}}}(\theta))$. Indeed, we have $\partial_1 \Phi(u_{\bar{\bar{u}}}(\theta), \theta) = \left[0, \Phi_1(u_{\bar{\bar{u}}}(\theta)_-, \theta)\right]$. In this case, we have to write Condition (E) in Theorem 18.1 in Clarke (2013) in its general form, which implies[13]:

$$\begin{aligned} \dot{p}^*(\theta) &\in -\Phi(u^*(\theta), \theta) + \partial_1 \Phi(u^*(\theta), \theta)\left[\theta \cdot \dot{u}^*(\theta) - \frac{1}{2} \cdot (\dot{u}^*(\theta))^2 - u^*(\theta)\right] && (80) \\ p^*(\theta) &= \Phi(u^*(\theta), \theta)\left[\theta - \dot{u}^*(\theta)\right]. && (81) \end{aligned}$$

Condition (T) implies $p^*(1) = 0$ and hence we can find an interval $\theta \in [1-\varepsilon, 1]$ for which $\dot{u}^*(\theta)$ is Lipschitz. We fix this interval in the analysis below. We claim that

$$\dot{p}^*(\theta) = -\Phi(u^*(\theta), \theta) + \max_{\xi \in \partial_1 \Phi(u^*(\theta), \theta)} \xi \left[\theta \cdot \dot{u}^*(\theta) - \frac{1}{2} \cdot (\dot{u}^*(\theta))^2 - u^*(\theta)\right] \tag{82}$$

for almost every Lebesgue point of this interval. Thus, for almost all $\theta \in [1-\varepsilon, 1]$ for which

$$0 = \lim_{\varepsilon \downarrow 0} \sup \frac{\int_{\theta-\varepsilon}^{\theta+\varepsilon} |f(\theta) - f(x)|\, dx}{2\varepsilon}$$

we have (82). Since p^* is integrable, Theorem 7.7 in Rudin (1987) implies almost every point is a Lebesgue point. Hence (82) holds almost everywhere. Therefore, since non-differentiabilities hold in a zero measure set, since we have established in the first paragraph of the analysis of this Case that $u^* \leq u_{\bar{\bar{u}}}$, the rest of the analysis is identical to Case 1. Thus we will conclude that proof by showing that (82) holds at almost every Lebesgue point of p^*.Consider a Lebesgue point for which

$$\dot{p}^*(\theta) < -\Phi(u^*(\theta), \theta) + \max_{\xi \in \partial_1 \Phi(u^*(\theta), \theta)} \xi \left[\theta \cdot \dot{u}^*(\theta) - \frac{1}{2} \cdot (\dot{u}^*(\theta))^2 - u^*(\theta)\right]. \tag{83}$$

[13] The condition (E) from Theorem 18.1 in Clarke reads: $p'(\theta) \in co\omega : (\omega, p(\theta)) \in \partial_L \Pi(\theta, u^*(\theta), \dot{u}^*(\theta))$ where $\partial_L \Pi$ is the limiting subdifferential of Π with respect of $(u, \dot{u})$ (see Definition 11.10 in Clarke (2013)). The necessary condition above follows from Exercise 18.4 in Clarke (2013) that states that the condition above implies: $(p'(\theta), p(\theta)) \in \partial_C \Pi(\theta, u^*(\theta), \dot{u}^*(\theta))$, where $\partial_C \Pi$ is the generalized subgradient (see Definition 10.3 in Clarke (2013)) with respect to $(u, \dot{u})$.

Clearly $u^*(\theta) = u_{\bar{\bar{u}}}(\theta)$. First, assume that $\dot{u}^*(\theta) \neq \dot{u}_{\bar{\bar{u}}}(\theta)$. From (81) we have $\dot{u}^*(\theta) = \theta - \frac{p^*(\theta)}{\Phi(u^*(\theta),\theta)}$ and hence (since $p^*(\theta)$ and $\Phi(u^*(\theta),\theta)$ are continuous) we can find an interval $(\theta-\epsilon,\theta+\epsilon)$ for which $\left|\dot{u}^*\left(\tilde{\theta}\right) - \dot{u}_{\bar{\bar{u}}}\left(\tilde{\theta}\right)\right| > 0$ for all $\tilde{\theta} \in (\theta-\epsilon,\theta+\epsilon)$ which implies that θ is the only point that $u^*\left(\tilde{\theta}\right) = u_{\bar{\bar{u}}}\left(\tilde{\theta}\right)$ in this interval. Hence, (82) holds (a.e.) in this interval. Next, assume towards a contradiction that we can find a lebesgue point θ of $\dot{p}^*$ such that $\dot{u}^*(\theta) = \dot{u}_{\bar{\bar{u}}}(\theta)$ and $\zeta > 0$ for which

$$\dot{p}^*(\theta) = -\Phi(u^*(\theta),\theta) + \Phi_1(u_{\bar{\bar{u}}}(\theta)_-,\theta)\left[\theta \cdot \dot{u}^*(\theta) - \frac{1}{2}\cdot(\dot{u}^*(\theta))^2 - u^*(\theta)\right] - \zeta. \tag{84}$$

Let $\dot{p}_{\bar{\bar{u}}}$ be the (smooth) arc associated with the curve $(u_{\bar{\bar{u}}}(\theta))$. Since θ is a Lebesgue point we can find $\epsilon > 0$ such that for every $\tilde{\theta} \in (\theta-\epsilon,\theta)$ we have:

$$\frac{\int_{\tilde{\theta}}^{\theta}\dot{p}^*(z)dz}{\theta-\tilde{\theta}} < \frac{\int_{\tilde{\theta}}^{\theta}\dot{p}_{\bar{\bar{u}}}(z)dz}{\theta-\tilde{\theta}} - \frac{\zeta}{2}. \tag{85}$$

Notice that (85) implies that there is $\hat{\theta} \in (\theta-\epsilon,\theta)$ such that $u^*\left(\hat{\theta}\right) = u_{\bar{\bar{u}}}\left(\hat{\theta}\right)$. Using (81) we have for all $\tilde{\theta} \in \left(\hat{\theta},\theta\right)$

$$\begin{aligned}
\dot{u}^*(\theta) - \dot{u}^*\left(\tilde{\theta}\right) &= \theta - \tilde{\theta} + \left(\frac{p^*(\tilde{\theta})}{\Phi(u^*\left(\tilde{\theta}\right),\tilde{\theta})}\right) - \left(\frac{p^*(\theta)}{\Phi(u^*(\theta),\theta)}\right) \\
&\geq \theta - \tilde{\theta} - \left(\frac{p^*(\theta) - p^*(\tilde{\theta})}{\Phi(u^*(\theta),\theta)}\right) \\
&= \theta - \tilde{\theta} - \left(\frac{\int_{\tilde{\theta}}^{\theta}\dot{p}^*(z)dz}{\Phi(u^*(\theta),\theta)}\right) \\
&\geq \theta - \tilde{\theta} - \left(\frac{\int_{\tilde{\theta}}^{\theta}\dot{p}_{\bar{\bar{u}}}(z)dz}{\Phi(u^*(\theta),\theta)}\right) + \frac{\zeta\left(\theta-\tilde{\theta}\right)}{A} \\
&= \dot{u}_{\bar{\alpha}}(\theta) - \dot{u}_{\bar{\alpha}}\left(\tilde{\theta}\right) + \frac{\zeta\left(\theta-\tilde{\theta}\right)}{A},
\end{aligned}$$

where the first inequality uses $\Phi(u^*\left(\tilde{\theta}\right),\tilde{\theta}) \leq \Phi(u^*(\theta),\theta) = A$ and the second inequality uses (85).

Therefore, since $\dot{u}^*(\theta) = \dot{u}_{\bar{\bar{u}}}(\theta)$ we have $\dot{u}^*\left(\tilde{\theta}\right) \leq \dot{u}_{\bar{\bar{u}}}\left(\tilde{\theta}\right) - \frac{\zeta(\theta-\tilde{\theta})}{A}$. Therefore,

$$\begin{aligned}
& u_{\bar{\bar{u}}}(\theta) - u_{\bar{\bar{u}}}\left(\hat{\theta}\right) \\
= \; & u^*(\theta) - u^*\left(\hat{\theta}\right) \\
= \; & \int_{\hat{\theta}}^{\theta} \dot{u}^*(z)\,dz \\
\leq \; & \int_{\hat{\theta}}^{\theta} \left[\dot{u}_{\bar{\bar{u}}}(z) - \frac{\zeta\left(z-\tilde{\theta}\right)}{A}\right] dz \\
= \; & u_{\bar{\bar{u}}}(\theta) - u_{\bar{\bar{u}}}\left(\hat{\theta}\right) - \left(\frac{\zeta}{A}\right)\int_{\hat{\theta}}^{\theta}(\theta - z)\,dz \\
< \; & u_{\bar{\bar{u}}}(\theta) - u_{\bar{\bar{u}}}\left(\hat{\theta}\right) - \left(\frac{\zeta}{2A}\right)\left(\theta-\hat{\theta}\right)^2,
\end{aligned}$$

which is an absurd.

Case 5: $u^*(1) = u_{\frac{1}{4}}(1)$.

For all $\theta \in \left(\frac{1}{2}, 1\right)$ the numerator on the RHS of (64) is $1 + \frac{d}{d\theta}\{\theta - V_1(\theta, u)\}$. Using (73), the last expression is $1 + \frac{d}{d\theta}\left\{\theta - \frac{1}{2\cdot\frac{1}{4}}\cdot\theta - \left(1 - \frac{1}{2\cdot\frac{1}{4}}\right)\right\} = 0$. Hence, from (64) we conclude that $\Phi_1(u_{\frac{1}{4}}(\theta)_+, \theta) = \Phi_1(u_{\frac{1}{4}}(\theta)_-, \theta) = 0$. Thus we can find a neighborhood of $(\theta, u_{\frac{1}{4}}(\theta), \dot{u}_{\frac{1}{4}}(\theta))_{\theta\in[0,1]}$ in which Π is (a.e.) C^1. Therefore, the argument in Case 1 applies, *mutatis mutandis*, to this case. ■

9.1 Equilibrium Properties and Comparative Statics

Proof. [Proof of Proposition 7]

We only need to prove iii. Take θ and u such that the $q(\theta, u) > 0$. The profit obtained from this type as a function of u is $\Phi(u,\theta)\cdot\hat{\pi}(u,\theta)$, where

$$\begin{aligned}
& \hat{\pi}(u,\theta) \\
\equiv \; & \left[\begin{array}{c} \theta\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right) - \frac{1}{2}\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right)^2 \\ -\left(\frac{1}{4\cdot u}\cdot\theta^2 + \left(1-\frac{1}{2\cdot u}\right)\cdot\theta + u + \frac{1}{4\cdot u} - 1\right) \end{array}\right].
\end{aligned} \tag{86}$$

Therefore, we have:

$$\begin{aligned}
& \frac{d}{du}\left[\frac{\int_{\alpha(u)}^{\hat{\theta}}\hat{\pi}(u,\theta)d\theta}{\int_{\hat{\theta}}^{1}\hat{\pi}(u,\theta)d\theta}\right] \\
= \; & \frac{d}{du}\left[\frac{\int_{1-2u}^{\theta}\left(\begin{array}{c} \theta\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right) - \frac{1}{2}\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right)^2 \\ -\left(\frac{1}{4\cdot u}\cdot\theta^2 + \left(1-\frac{1}{2\cdot u}\right)\cdot\theta + u + \frac{1}{4\cdot u} - 1\right)\end{array}\right)d\theta}{\int_{\theta}^{1}\left(\begin{array}{c} \theta\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right) - \frac{1}{2}\left(\frac{1}{2\cdot u}\cdot\theta + \left(1-\frac{1}{2\cdot u}\right)\right)^2 \\ -\left(\frac{1}{4\cdot u}\cdot\theta^2 + \left(1-\frac{1}{2\cdot u}\right)\cdot\theta + u + \frac{1}{4\cdot u} - 1\right)\end{array}\right)d\theta}\right] \\
= \; & 48\frac{u^2}{(1-\theta)}\frac{4u^2-(\theta-1)^2}{\left(12u^2-\theta^2+2\theta-1\right)^2}
\end{aligned}$$

First, we claim that $12u^2 - (\theta - 1)^2 > 0$ for all $\theta \geq \alpha(u)$. Indeed, since the expression is decreasing in θ we have $12u^2 - (\theta - 1)^2 \geq 12u^2 - (\alpha(u) - 1)^2 = 8u^2 > 0$. Therefore, the denominator is always strictly positive. Second notice that $4u^2 - (\theta - 1)^2$ is strictly decreasing in θ and $q(\theta, u) > 0$ implies $\theta > \alpha(u) = 1 - 2u$. Therefore $4u^2 - (\theta - 1)^2 > 4u^2 - (\alpha(u) - 1)^2 = 0$, which establishes that the term above is strictly positive. ■

Proof. [Proof of Proposition 8]. The proof is analogous to the proof of Proposition 4 and is omitted for brevity. ■

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