.

Multi-Variety Durable-Good Monopoly:

A Revisited Coase Conjecture and Product Design

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Abstract: The paper analyzes a durable-good monopoly problem in which multiple varieties are produced and sold, and offers insights on product design in this context. A revisited Coase conjecture establishes that the market eventually clears, that equilibrium profits are bounded below by static optimal market-clearing profits, and that, in any stationary equilibrium, profits converge to the lower bound as discount factors converge to unity. Equilibrium profits differ from zero (even without gaps), as static optimal market-clearing profits equal zero only when all varieties are identical. Product design results then establish why discordant (horizontally differentiated) varieties are profit maximizing for the monopolist, and why niche products can be optimal when multiple varieties are for sale.

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1 Introduction

While the incentives for firms to set up product-lines are well understood in static environments,[1] no much is known about these incentives in dynamic environments in which the monopolist faces a time consistency problem. The purpose of the paper is to give a comprehensive analysis of such incentives. The analysis delivers two main contributions to the literature: a generalization of Coasian results to multi-variety durable-good monopoly environments and a characterization of optimal product design in these environments.

Nobel Laureate Ronald Coase brought the time consistency problem of a durable-good monopolist to the attention of academic community. His work (Coase 1972) has become one of the seminal contributions to modern economic theory. Coase conjectured that a monopolist selling a durable-good could not credibly sell at a static monopoly price to rational consumers, and that "the competitive outcome may be achieved even with a single supplier". Coase's insight was that the monopolist, lacking the ability to commit to future prices, would face the competition of its own future selves thereby dissipating all of its own monopoly power. After selling the initial quantity, the monopolist would necessarily benefit by lowering prices in order to sell to consumers with lower valuations who did not yet purchase. Thus, prices would decline after each sale. Forward-looking consumers expecting prices to fall, would then be unwilling to pay the initial high price. Consequently, if the time between offers were to vanish, the opening price would converge to the marginal cost and the competitive quantity would be sold in a *twinkle of an eye*. Formal proofs of this statement appeared in seminal papers by Stokey 1981, Fudenberg, Levine and Tirole 1985, Gul, Sonnenschein and Wilson 1986, among others.

The present work considers the same environment originally studied by Coase, but presumes that monopolist can produce and sell more than one variety of the durable-good. Such a natural extension of the baseline model allows us to gain new insights on the original problem and to establish several novel results. In such settings, the monopolist regains his ability to command strictly positive profits. Our conclusions though, do not give rise to failures of classical insights on Coasian dynamics, but rather they qualify their content. Indeed, the analysis hopes to convincingly establish that the main force driving any Coasian equilibrium is market-clearing, and not competition or efficiency. This is the case as a time consistency problem only arises when the monopolist cannot commit not to clear the market. But, any market-clearing price (that is, any price at which all consumers are willing to buy) provides a credible commitment to the monopolist (as it is no longer compelled to lower prices). In multi-product settings, profit-maximizing market-clearing prices are neither

[1] See, for example, Mussa and Rosen 1978, Maskin and Riley, 1984, and Deneckere and McAfee, 1996

competitive (equal to marginal cost) nor efficient (equal to each other). Furthermore, profits at such optimal market-clearing prices bound equilibrium profits from below and identify the limit profit in any stationary equilibrium. The tight characterization of stationary limit profits is then exploited to study how different product designs affect the profitability of the monopolist.

Specifically, consider a monopolist with zero marginal costs who sells two varieties of a durable-good to a continuum of consumers with unit-demand for the product.[2] As in the classical Coasian setting, buyers' are privately informed of their value for each of the two products. Values are represented by an arbitrary distribution that can exhibit any correlation structure. The set-up thus accommodates as special cases several commonly used designs such as vertical product differentiation (when consumers' valuations for the two products are positively correlated) and horizontal product differentiation (when consumers' valuations are negatively correlated). For instance, Apple sells MacBook-Pros with hard drives of different sizes (vertical differentiation), but it also sells the more portable MacBook-Air (horizontal differentiation). The time horizon is infinite. In every period, the monopolist sets a vector of prices p_t (one for each product), while the buyers, upon observing prices, choose which variety to purchase, if any. Upon buying a product consumers exit the market never to return.[3] In these environments, the monopolist has an incentive to segment consumers between the two products so to reduce (or eliminate) its temptation to cut the prices in future and regain some of its lost market power.

The first results characterize the static monopoly problem of maximizing profits subject to supplying all consumers (or the optimal market-clearing profit). Whenever there is more than one variety for sale, optimal market-clearing prices are strictly positive as it always possible to sell one of the good at marginal cost (thereby clearing the market), while using the other to screen consumers and raise profits.[4] Coasian results establish that in any subgame perfect equilibrium (SPE) of the dynamic game the market must clear instantaneously whenever the monopolist sets static market-clearing prices. Thus, optimal market-clearing profits bound equilibrium profits in the dynamic game from below. Moreover, the market clears in a finite time in any subgame perfect equilibrium as prices converge to market-clearing in finitely many rounds. This is the case as the monopolist prefers to sell instantaneously to all consumers when the measure of buyers in the market is small. Results also establish that stationary subgame perfect equilibria exist,[5] and that stationary equilibrium profits always converge to

[2] Results easily extend to any finite number of varieties.

[3] In Section 3, we discuss how results generalize when consumers remain in the market even after purchasing one of the two varieties.

[4] Profit-maximizing market-clearing prices can exceed the lowest valuation in the support for every product (for instance with discordant products). Lemma 11 presents sufficient conditions for this to be the case.

[5] Therefore, our results do not rely on non-stationary punishments that may be hard to coordinate on.

optimal market-clearing profits when the time between offers converges to zero. Moreover, such profits accrue by selling to all consumers almost instantaneously.

The spirit of our conclusions is close to classical Coasian results obtained in the one variety case (Stokey 1981, Fudenberg, Levine and Tirole 1985, Gul, Sonnenschein and Wilson 1986, Ausubel and Deneckere 1989). Such results fall in two distinct scenarios. In the gaps case (when the lowest value for a buyer exceeds the marginal cost), the market clears in a finite time, SPE profits are positive and unique, and SPE profits converge to the lowest valuation as the time between offers converges to zero. In the no-gaps case (when the lowest value coincides the marginal cost), the market clears in an infinite time, a Folk Theorem applies to SPE profits, and stationary SPE profits converge the marginal cost. With more than one variety results are closer to the gaps case (even when the lowest values of both products coincide with marginal costs), but with some significant differences. In particular, with more than one product, the market always clears in a finite time, SPE profits are always positive, and stationary SPE profits converge to optimal market-clearing profits as the time between offers converges to zero. However, in contrast to the one variety case, even stationary equilibria may display mixing along the equilibrium path, as the seller conceals which product will be most heavily discounted. In such environments, it is also relatively straightforward to observe that increasing the number of products favours the monopolist by further segmenting the market. Essentially, intratemporal price discrimination makes up for the absence in intertemporal price discrimination caused by the lack of commitment (even when the seller immediately agrees with all the buyers).

Motivated by our pricing conclusions, we carry out a product design exercise to characterize the relationship between the distribution of buyers' values and the optimal market-clearing profits (which are a suitable approximation of equilibrium profits when the frequency of price revisions is high). Specifically, we begin by characterizing how the dependence structure among the consumers' valuations for the two products affects the monopoly profit when consumers' valuations are pinned down by given marginal distributions. The motivation for this exercise is to consider a monopolist that assembles products from a basket of characteristics[6] for whom it is costly to affect marginal distribution of values, but costless to adjust the correlation among values. For instance, we might imagine a smartphone producer deciding upon the different features of its new phone (e.g. screen resolution, battery size, or memory), or an automobile manufacturer blending performance, size and fuel efficiency into their products, or a standard monopolist choosing where to open its retail locations. The way these characteristics are assembled may generate dependence in the consumers' preferences

[6] As in Lancaster's 1966 characteristics-approach, the consumer "is assumed to derive ... satisfaction from characteristics which cannot in general be purchased directly, but are incorporated in goods".

for these products which in turn may impact the monopoly profit. For instance, a monopolist can choose to make products vertically differentiated or horizontally differentiated. In this context, we are able to show that monopoly profits are maximized when the joint distribution of valuations is most discordant (that is when the joint distribution corresponds to the Frechét-Hoeffding lower-bound). Discordance increases profits as a strong preference for one goods tends to be associated with low desire for the other (i.e. horizontally differentiated products). This favors market segmentation and profit by minimizing the value of units that are never purchased. One may conjecture that concordance (i.e. vertically differentiated products) may then be the profit-minimizing design. However, this is not the case in general, as independent products, for instance, occasionally raise lower profits that concordant ones.[7] Anecdotally observe that Apple's introduction of the iPhone 5c along with the better (in a vertical sense) iPhone 5s was admittedly a mistake remarked by the fast drop in its price.[8] In contrast to this strategy, Apple was more successful (thus far) with the two models released in 2014, the iPhone 6 and the 6+, which were arguably more horizontally differentiated compared to their predecessors.

The final substantive contribution analyzes how volatility in valuations affects the optimal market-clearing profits. In particular, we ask whether the monopolist would prefer a distribution of values or its mean. In a single-variety setting, the answer would be trivial. The monopolist would always dislike variance. As the market has to clear, selling all the units at the average value would dominate selling them at lowest value. However, when more than one variety can be sold two forces are at play: (1) variance increases the information rents of buyers, thereby hurting the monopolist; and (2) variance increases the maximum valuation for the product (that is, the value of the best variety). The analysis establishes that either of the two forces can dominate, and provides sufficient conditions for both scenarios. This shows why low volatility mass-product need not be an optimal design when the monopolist lacks the ability to commit to a price path.

A vast literature was developed either to propose tactics that a monopolist could use to avoid the time consistency problem, or to check the robustness of Coase's insight. For instance, a monopolist could relax its commitment problem and increase its profit by renting the good rather than selling it (Bulow 1982), by introducing best-price provisions (Butz 1990), or by introducing new updated versions of the durable-goods over time (Levinthaland and Purohit 1989, Waldman 1993 and 1996, Choi 1994, Fudenberg and Tirole 1998, and Lee and Lee 1998). Instead, the Coase conjecture would fail altogether when the monopolist faces

[7]If the valuations are uniformly distributed, the Frechét-Hoeffding bounds on the joint distribution induce either perfect positive or perfect negative correlation between the random variables. In this setting our result establishes that the best design displays perfect negative correlation among products.

[8]Computerworld, January 27$^{\text{th}}$ 2014.

capacity constraints (Kahn 1986, and McAfee and Wiseman 2008), when consumers use non-stationary strategies (Ausubel and Deneckere 1989), with entry of new buyers (Sobel 1991), when buyers' valuations vary stochastically over time (Biehl 2001, Deb 2011, Garrett 2016), when marginal costs vary stochastically over time (Ortner 2014), when buyers can exercise an outside option (Board and Pycia 2014), when goods depreciate over time (Bond and Samuelson 1984), and when demand is discrete (Bagnoli, Salant and Swierzbinski 1989, Fehr and Kuhn 1995, Montez 2013).

Most closely related to this paper are Board and Pycia 2014, Hahn 2006, Inderst 2008, Takeyama 2002, and Wang 1998. Hahn 2006, Inderst 2008, and Takeyama 2002, provide examples in which a multi-product durable-good monopolist is able to mitigate or eliminate the Coasian insight. Their analysis is restricted to vertically differentiated products, two types of consumers (Hahn 2006, Inderst 2008, Takeyama 2002), and two periods (Takeyama 2002). Their main focus is on the possibility of strategically changing the quality of the goods (through upgrades or downgrades). Their conclusions can be nested in our setting as vertical product differentiation is allowed. In our view however, these should not be interpreted as failures of the Coase conjecture. But rather they display the essence of the Coasian insight which is market-clearing, and not competition or efficiency. Similarly, Board and Pycia 2014 shows that a durable-good monopolist always achieves the static monopoly profit in the first period if an outside option is freely available. Again, their conclusion is nested within our framework. As in their setting any price set by the monopolist clears the market when an outside option is freely available, it is no surprise the monopolist never undercuts on its initial price. However, by pricing the outside option, the monopolist would be able to achieve an even higher profit as both varieties would be optimally sold at positive prices when there are gaps (which is the case in their setting). Similar considerations apply to Wang 1998 which establishes an instantaneous clearing result (evocative of Board and Pycia) in a two-type model. These and several other results classified as failure of the Coase conjecture are in fact no failure at all, but rather they capture features of its most general statement which entails market-clearing and not competition, or efficiency. The product design exercise carried out in the final substantive section is, to the best of our knowledge, the first model analyzing the incentives to generate particular product-lines in the context of dynamic pricing model.[9]

The rest of the paper is organized as follows. Section 2 presents the model and the relevant solution concepts. Section 3 analyses the durable-goods monopoly problem, and features our Coasian results. Section 4 discusses the optimal design of durable-goods. Section 5 concludes. All proofs are deferred to the appendix, Section 6.

[9] The seminal reference for related static design questions is Johnson and Myatt 2016. A stylized dynamic exercise appears in House and Ozdenoren 2008 which prove the optimality of mass products with one variety.

2 Product Variety in a Dynamic Monopoly

Consider a market in which two varieties, a and b, of a durable-good can be produced and sold. A monopolist faces a unit measure of non-atomic consumers with unit-demand for the durable-good. Time is discrete, the time-horizon is infinite, and all consumers discount the future by a common factor δ. Denote by v_i the value of consuming variety i, and denote by $v = (v_a, v_b)$ the value profile. A measure $\mathcal{F}$, defined on the unit square $[0,1]^2$, describes the distribution of values among buyers. Throughout denote by F its associated cumulative distribution and by V its support.[10] To simplify parts of the discussion, we occasionally impose regularity conditions that require the measure $\mathcal{F}$ to be non-atomic.

Condition 1 (Regularity) *The market is regular if $\mathcal{F}$ is absolutely continuous on $\mathbb{R}^2$, and its density f satisfies $f(v) \in \left(\underline{f}, \overline{f}\right)$ for any $v \in V$.*

Regularity is superfluous to establish most of our results, and we will qualify its role in the analysis. Finally, let F_i denote the marginal cumulative distribution on variety i, let V_i denote its support, and let f_i denote its density when it exists.

Buyers have unit-demand for the product, and exit the market upon purchasing any one for the two varieties.[11] Value profiles are private information of buyers, and utility is quasi-linear in money. In particular, the final payoff when purchasing variety $i \in \{a, b\}$, at date t, at a price p_i simply amounts to $\delta^{t-1}(v_i - p_i)$. The monopolist discounts the future by the same discount factor δ. Its marginal cost of producing of any variety equals 0, and its payoff simply amounts to the present discounted value of future profits.

In every period: the firm sets a price for each of the two varieties produced in order to maximize the present discounted value of future profits; and consumers, who have not previously purchased any product, choose whether to buy any one of the two varieties at current prices so to maximize their present discounted utility.

Histories, Strategies, Solution Concept: Intuitively, a strategy for the monopolist specifies the prices to be set as a function of the history of play. A strategy for consumers who have yet to purchase the good, instead, specifies whether to buy anyone of the two varieties at the current prices given the history of play and the implied expected price dynamic. We impose measurability restrictions on joint consumers strategies which require the set of consumers purchasing variety $i \in \{a, b\}$ at any possible history to be a measurable set.

Let Ω denote the Borel sigma-algebra on $[0,1]^2$. A t-period seller-history, h^t, specifies the prices that were set and the subset of consumers who purchased either variety of the durable-good for every period in $\{0, ..., t-1\}$. Thus, $h^t \in \mathbb{R}^{2t} \times \Omega^{2t}$. A t-period buyer-history, $\hat{h}^t$,

[10] The support identifies the smallest closed set whose complement has probability zero.

[11] We discuss which conclusions are affected by the permanent-exit assumption at the end of Section 3.

instead, consists of a history h^t followed by the prices announced by the monopolist for date t. Thus, $\hat{h}^t \in \mathbb{R}^{2t+2} \times \Omega^{2t}$. At any history h^t, let $D_i(h^t)$ denote the set of consumers purchasing variety $i \in \{a, b\}$ at date $t-1$. A consumer with value profile $v \in V$ is said to be *active* at history h^t if he has not yet purchased a variety of the durable-good. Formally, define the set of *active buyers* $A(h^t)$ at a given history h^t as follows

$$A(h^t) = V \backslash \bigcup_{s=0}^{t-1} \left[D_a(h^s) \cup D_b(h^s)\right].$$

Occasionally, we omit the dependence on the history and we denote these sets simply by D_i^t and A^t (when clarity is not compromised).

Throughout, denote by $\mathcal{P}(X)$ the set of all probability distribution defined on the set X. A collection of strategies for consumers $\boldsymbol{\tau}$ consists of a sequence of functions $\{\tau^t\}_{t=0}^{\infty}$ satisfying: (1) $\tau^t : \mathbb{R}^{2t+2} \times \Omega^{2t} \times V \to \mathcal{P}(\{0, a, b\})$; (2) $\tau^t(h^t|\cdot)$ is measurable for any $\hat{h}^t \in \mathbb{R}^{2t+2} \times \Omega^{2t}$; (3) $\tau^t(\hat{h}^t|v) = 0$ for all $v \in \bigcup_{s=0}^{t-1} \left[D_a(h^s) \cup D_b(h^s)\right]$.[12] Action 0 is to be interpreted as a decision not to buy any product in the current period. Actions a and b indicate the decision to purchase variety a or b in the current period. Intuitively, τ^t determines the probability distribution over consumption decisions of buyers at all possible histories. A strategy $\boldsymbol{\sigma}$ for the monopolist consists of a sequence of functions $\{\sigma^t\}_{t=0}^{\infty}$ satisfying $\sigma^t : \mathbb{R}^{2t} \times \Omega^{2t} \to \mathcal{P}(\mathbb{R}^2)$. Now, σ^t determines the probability distribution over prices charged by the monopolist at date t as a function of history of play.

The strategy profile $\{\boldsymbol{\sigma}, \boldsymbol{\tau}\}$, generates a path of prices and sales which can be computed recursively. The pattern of prices and sales over time in turn determines the payoffs to the players. Let $\Pi(\boldsymbol{\sigma}, \boldsymbol{\tau})$ be the expected present value of profits generated by the strategy profile $\{\boldsymbol{\sigma}, \boldsymbol{\tau}\}$, and let $U(\boldsymbol{\sigma}, \boldsymbol{\tau}|v)$ be the expected present value of surplus derived by consumer v. The profile $\{\boldsymbol{\sigma}, \boldsymbol{\tau}\}$ is a Nash equilibrium if and only if

$$\begin{array}{lll} (1) & \Pi(\boldsymbol{\sigma}, \boldsymbol{\tau}) \geq \Pi(\boldsymbol{\sigma}', \boldsymbol{\tau}) & \text{for } \forall \boldsymbol{\sigma}' \\ (2) & U(\boldsymbol{\sigma}, \boldsymbol{\tau}|v) \geq U(\boldsymbol{\sigma}, \boldsymbol{\tau}'(v), \boldsymbol{\tau}(V \backslash v)|v) & \text{for } \forall \boldsymbol{\tau}'(v),\ v\text{-a.e.} \end{array}$$

where $\boldsymbol{\tau}(\hat{V})$ is the projection of $\boldsymbol{\tau}$ onto a subset $\hat{V} \subseteq V$. The profile $\{\boldsymbol{\sigma}, \boldsymbol{\tau}\}$ is a *subgame perfect equilibrium* if and only if the continuation strategies are a Nash equilibrium after any possible history. To guarantee the existence of an equilibrium, the monopolist is allowed to mix at any stage of the game. For convenience, when fixing an equilibrium strategy, denote by $\Pi(h^t)$ the expected present value of profits generated at a seller-history h^t. Similarly, we denote by $\Pi(p|h^t)$ the expected present value of profits at the same history h^t conditional

[12] As customary in the literature, when defining the domain of strategies, we neglect that consumers can accept at most one offer over time. More generally, the set of histories is a strict subset of $\mathbb{R}^{2t} \times \Omega^{2t}$ and should be modelled as such.

on setting a price profile p at date t, and by $U(\hat{h}^t|v)$ denotes continuation value of player v at date $t+1$ after a buyer-history $\hat{h}^t$.

As customary, we restrict attention to equilibria in which deviations by sets of measure zero of consumers change neither the actions of the remaining consumers nor those of the monopolist. Refer to these equilibria as *non-atomic subgame perfect equilibria*, or equivalently *SPE*. This restriction may affect the set of equilibrium outcomes as demonstrated in Gul, Sonnenschein and Wilson 1986. The authors, indeed, establish that in this type of extensive-form game, if the players' strategies prescribe behavior that is optimal after all histories that result from no simultaneous deviations, then the equilibrium path prescribed is the equilibrium path of a subgame perfect equilibrium.[13] With these assumptions, unilateral deviations by non-atomic consumers can change neither the actions of the remaining consumers nor the actions of the monopolist. Thus, only unilateral deviations of the monopolist can affect the course of the game. From the observation that simultaneous deviations from the equilibrium path are unimportant in checking for subgame perfection, it follows that, in order to show that a path is associated with an SPE, it is necessary and sufficient to specify actions for each agent as functions of the price histories so that: (a) these functions generate the given path; and (b) after each price history the prescribed actions are optimal.

In general, buyers' SPE strategies may depend not only on the current price profile p, but on the entire history of play (as the history may affect buyers' beliefs about future prices). Asubel and Deneckere 1989 show that a Folk Theorem holds in this class of games when the monopolist can make its pricing decision depend on the full history of play, and the lowest buyer valuation is zero. As a similar logic may occasionally apply to our environment, it is convenient to also consider stationary equilibria in which the monopolist does not exploit changes to consumer beliefs to commit to a given price path. Thus, as customary in the literature, some of our results on Coasian dynamics rely a common class of Markovian equilibria. Define a *weak Markov equilibrium*, or equivalently WME, to be an SPE in which the strategy of any active buyer depends only on the current price profile at every history that contains no simultaneous deviation by a positive measure of buyers. If so, the value function of buyers only depends on current prices at every history that contains no simultaneous deviation by a positive measure of buyers. A *strong Markov equilibrium* is a WME in which the strategy of the seller depends only on $A(h^t)$ at any history h^t that contains no simultaneous deviation by a positive measure of buyers.

[13]This follows by replacing the portion of the strategies in any subgame after some simultaneous deviation by equilibrium behavior in the subgame, and by postulating that the equilibrium action of every agent is constant on histories in which prices are the same and the sets of agents accepting at each point in time differ at most by sets of measure 0.

3 Market-Clearing and the Coase Conjecture

The section begins by defining the set of static market-clearing prices and by discussing some of its properties. It then establishes some features shared by all subgame perfect equilibria of the dynamic game, which lead to our revisited statement of the Coase's seminal result. The section concludes by discussing why several paradoxes in the durable-goods literature are indeed natural consequences of this revisited statement.

Market-Clearing Prices and Profits: A *market-clearing (MC) price* is a price profile that clears the market when the firm commits to setting such prices for the infinite future. Equivalently, it is a price profile that clears the market in the static version of the model. Formally, given any price profile p, the static demand for variety i, $d_i(p)$, satisfies

$$d_i(p) \in [\mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}), \mathcal{F}(v_i - p_i \geq \max\{v_j - p_j, 0\})] \ \text{ for } \ j \neq i.$$

The demand equation posits no tie-breaking assumptions for indifferent consumers. In any event, when the market is regular, tie-breaking assumptions are entirely inconsequential.[14] The set of market-clearing prices, M, consist of all prices at which every consumers is willing to purchase at least one of the two products,

$$M = \left\{p \in \mathbb{R}^2 | \max_{i \in \{a,b\}}(v_i - p_i) \geq 0 \text{ for all } v \in V\right\}.$$

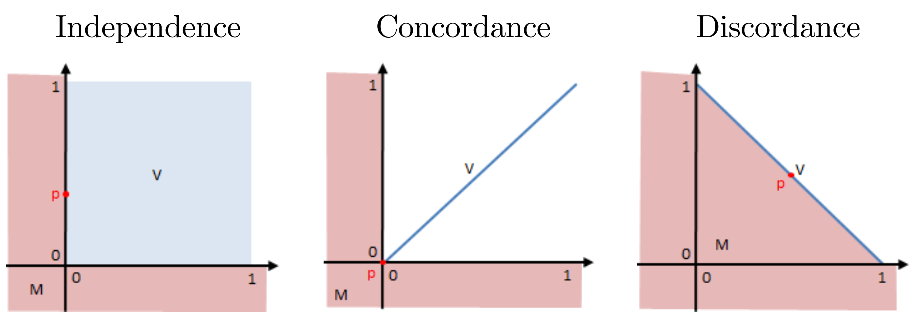

Figure 1: Market-clearing set M and the support V for three possible distributions.

With only one variety, the highest market-clearing price coincides with the lowest valuation in the support V. With more than one variety, any price profile in which one of the prices set to its lowest valuation in the support is a market-clearing price. In fact, when the two valuations are independently distributed, all market-clearing prices set the price of one of the

[14]When regularity holds, demand simplifies to $d_a(p) = \int_0^{\min\{1,1+p_b-p_a\}} \int_{\max\{v_b+p_a-p_b,0\}}^1 f(v)dv$, where $f(v) = 0$ for any $v \notin V$.

two varieties at its lowest valuation. But, this need not be the case in general. For instance, when the values of the two varieties display perfect negative correlation, market-clearing price profiles exist in which both prices exceed their respective lowest possible valuation.

Optimal market-clearing prices, $\bar{p}$, play a central role in the analysis of equilibrium behavior in the dynamic game. They identify those market-clearing price profiles that maximize the static monopoly profit. An optimal market-clearing price, $\bar{p}$, is defined as a solution to the following static profit maximization problem

$$\max_{p\in M}[d_a(p)p_a + d_b(p)p_b]. \tag{1}$$

Optimal market-clearing prices may fail to exist when regularity is violated as the extremum may never be attained. So, define the supremum of this problem as the *optimal market-clearing profit*, $\bar{\pi}$. Optimal profits always exist as profits are necessarily bounded above by 1 and are strictly positive in almost every market. In particular, let $\underline{v}_i$ denote the lowest value of variety i in the support V. The first result bounds optimal MC profits from below when more than one variety is for sale. The proof does not rely on any assumption on the measure, $\mathcal{F}$, and includes scenarios in which regularity is violated. We say that *varieties are identical*, if the value for one variety simply coincides to the value of the other variety (that is, if we have that $v_a = v_b$ for any $v \in V$).

Proposition 1 *Optimal MC profits always exceed* $\max_{i\in\{a,b\}} \underline{v}_i$*, and equal* $\min_{i\in\{a,b\}} \underline{v}_i$ *if and only if varieties are identical.*

The proof simply establishes that the monopolist can always sell one variety, say i, at a price that strictly exceeds its lowest valuation $\underline{v}_i$, while clearing the market by setting the price of the other variety to its lowest value $\underline{v}_j$. If so, both varieties will be sold to a positive measure of buyers provided that varieties are differentiated and that the price of variety i is sufficiently low. Thus, profits will strictly exceed $\min_{i\in\{a,b\}} \underline{v}_i$. As $\underline{v}_i \geq 0$ for any i, an immediate implication of the result is that optimal MC profits can be equal to 0 if and only if varieties are identical and $(0,0) \in V$.

Corollary 2 *Optimal MC profits equal* 0 *if and only if varieties are identical and* $(0,0) \in V$*.*

Proposition 1 hints at why the Coasian logic may not necessarily lead to zero profits when differentiated varieties can be produced and sold. Although a monopolist lacking commitment might still have to clear markets, market-clearing no longer requires that profits equal zero. Moreover, when the market is regular, optimal MC profits always strictly exceed $\min_{i\in\{a,b\}} \underline{v}_i$ as varieties cannot be identical by absolute continuity. Yet, even with regularity, optimal

MC profits will be arbitrarily close to $\min_{i\in\{a,b\}} \underline{v}_i$ in any regular market in which varieties are almost identical.

Skimming, Market-Clearing and Coasian Dynamics: This subsection extends classical Coasian results to settings with multiple varieties. It establishes that, even with more than one variety, the market for the durable-good must clear eventually. The intuition for this observation essentially coincides with Coase's seminal result. As the monopolist cannot commit to future prices, he must eventually clear the market, or else clearing the market would be a profitable deviation once prices have converged. Moreover, as with single variety and gaps, the market will clear in a finite time as the monopolist will benefit from selling to all active buyers whenever their measure is small. In a multi-variety setting, however, market-clearing will no longer imply zero profits. Indeed, SPE profits will always exceed optimal market-clearing profits, and will converge to such profits in any stationary SPE when the discount factor converges to unity. Our Coasian results thus establish why lack of commitment does not generally lead to efficiency or to competitive pricing, but only to market-clearing. Therefore, when price revisions are arbitrarily frequent, the monopolist will simply choose the profit maximizing way to supply all consumers.

We begin the analysis with a few preliminary lemmas that unveil some important features of SPE strategies in this dynamic pricing game. The first observation establishes that, as in the classical single variety setting, the measure of active buyers must be a truncation of the original measure in any SPE. To do so, we need to introduce a notion of multidimensional truncation. We say that $\mathcal{F}(\cdot|A)$ is a *truncation* of $\mathcal{F}$ on $A \subset V$ if: (i) $v \notin A$ implies $v + (\varepsilon, \varepsilon) \notin A$ for all $\varepsilon > 0$; and (ii)

$$\mathcal{F}(E|A) = \mathcal{F}(E \cap A) \text{ for any } E \cap A \in \Omega.$$

Lemma 3 *In any SPE, the measure of active buyers at any history h^t is a truncation of the original measure on the active player set $A(h^t)$.*

The proof of the lemma is intuitive and motivates our notion of truncation. If a consumer with value v is willing to purchase a variety at current prices, the same should hold for a consumer with value $v' = v + (\varepsilon, \varepsilon)$. In fact, by delaying buyer v' can capture at most $\delta\varepsilon$ on top of the continuation value of buyer v. But if so, buying now should be preferable as he captures ε more surplus than v. This naturally obtains as delay costs are higher for high value consumers. However, in general it is not the case that $v \notin A$ and $v'' > v$ together imply that $v'' \notin A$. For instance, the active player set in Figure 2 below would violate this more stringent requirement.

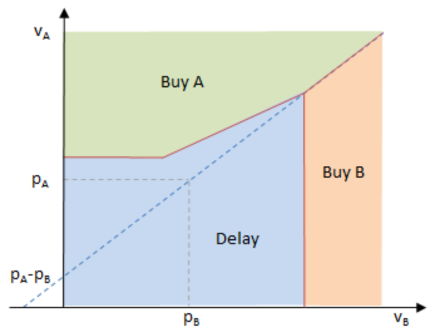

Figure 2: The active player set for a market clearing within two periods.

The next lemma relates several important features of SPE pricing to static market-clearing. The key part of the lemma establishes why any static market-clearing price must also clear the market in the dynamic model. This is a crucial observation as it immediately delivers two important conclusions. The first establishes that the monopolist never sets prices in the interior of the static market-clearing set. The second shows that the optimal market-clearing profits always bound SPE profits from below at any history and for any possible discount factor. To state the result, let $\bar{M}$ denote the boundary of the set M, or equivalently

$$\bar{M} = \left\{p \in \mathbb{R}^2 | \min_{v \in V} \max_{i \in \{a,b\}} (v_i - p_i) = 0\right\}.$$

Also, given any history h^t associated with an active player set $A(h^t) = A^t$, let $\bar{\pi}(A^t)$ denote the optimal market-clearing profit for the residual measure of buyers $\mathcal{F}(A^t)$,

$$\bar{\pi}(A^t) = \max_{p \in M} \sum_{i \in \{a,b\}} p_i \mathcal{F}(v_i - v_{-i} > p_i - p_{-i} | A^t).$$

Lemma 4 *In any SPE, at any history h^t:*
(1) The monopolist never sets a prices in the interior of M.
(2) A measure 1 of consumers buys if prices are in the interior of M.
(3) The present discounted value of profits satisfies

$$\Pi(h^t) \geq \bar{\pi}(A^t).$$

Thus in any SPE, profits exceed optimal MC profits for any $\delta < 1$. This contrasts with a classical interpretation of the Coase conjecture for settings with 1-variety and no-gaps which amounts to limit pricing being competitive and efficient. With more varieties, even when buyers with no value are in the support of the measure $(0,0) \in V$, limit pricing is neither

competitive nor efficient. However, as the next results clarify, the Coasian logic persists here in that agreement and market-clearing still dictate limit pricing. The proof of the result identifies the set of price profiles which are immediately accepted by all buyer regardless of their beliefs. As in most bargaining models, it then establishes inductively that such a set of prices must include all static market-clearing prices due to consumers' discounting.

To grasp the full connection between known Coasian results and their failures, it is useful to consider a few more observations. The first of these establishes that the market must clear in a finite time (whenever the discount factor δ is smaller than 1). The same result holds with a single variety when the smallest buyer's valuation is strictly positive (call this the *gaps* case). However, the result holds more generally with multiple varieties as it applies even when buyers with value zero for both goods are in the support of the measure, $(0,0) \in V$. In fact, what matters for this conclusion is not whether there are gaps in valuations, but rather whether it is possible to clear markets at a positive profit. The latter is always possible with more the one variety or with a single variety and gaps in valuations.

Lemma 5 *If the market is regular, there exists T such that in all SPE, a measure 1 of buyers purchases a variety of the durable-good before period $T+1$.*

When the measure of active buyers is small, the monopolist benefits from clearing the market instantaneously. This is the case as any possible price-discrimination gain is outweighed by the discounting of future profits. Thus, when few buyers remain active, the monopolist clears the market at once. Furthermore, the monopolist sells to a strictly positive measure of buyers in any time period regardless of buyers' beliefs. Or else, profits would be increased by anticipating future sales. Thus, the market clears in a finite time.

As in the single variety case, it is possible to show that SPE always exist and that at least one SPE is weakly Markovian. In particular, the next result proves directly the existence of a WME which implies SPE existence.

Proposition 6 *If the market is regular, a WME always exists.*

In contrast to classical results on the single variety case, equilibria may be in mixed strategies at any round of play. In particular, mixing may be required even in the very last stage of the game. This is the case as the monopolist may want to prevent buyers from inferring future price reductions if they would delay purchasing those varieties that are going to be more heavily discounted in the future. The proof exploits a suitable variant of the Kakutani-Fan-Glicksberg fixed point theorem to establish the result. WMEs need not be unique, however, as optimal market clearing prices are not unique in general.

The final result on the dynamic pricing game consists of a more general version of the classical Coase conjecture. As we discuss below, our more general statement will capture several phenomena that had been classified as failures of the Coase conjecture in the earlier literature. Informally, the result establishes that in environments with multiple products equilibrium profits must be close to optimal MC profits in any weakly Markovian SPE, when discount factors are close to unity.

Proposition 7 *If the market is regular, profits converge to optimal MC profits in any WME as δ converges to* 1.

The proof establishes that, when discount factors are close to 1, prices must be close to market-clearing after any real time T. Thus, profits will be close to market-clearing as patient consumers would wait any finite amount of time for a price reduction, and thus products will only ever be sold at prices that are close to market clearing. The intuition for this result is as follows. Consider a time period t in which the demand for both products is small. A possible deviation for the monopolist in period t consists of setting prices according to its mixed strategy in period $t+1$ rather than setting the price p^t. Such a deviation would have three effects on the profit of the monopolist. Firstly, it would reduce profit by lowering the price paid by those who were expecting to consume a variety i at date t and continue to do so. Secondly, it would increase profits by anticipating the stream of future revenue on all units to be sold at later stages. Thirdly, it would have an ambiguous effect on profits by inducing some consumers to change their demand from one to the other product. The first effect is, however, small as price changes must be small if a patient consumer is unwilling to wait one period to purchase the product. Similarly, the third effect must be small (if positive), because the measure of consumers buying in period t is small, and because the set of those contemplating to switch variety is a small subset of those purchasing varieties when price changes are small (by absolute continuity). Thus, for such a deviation not to be profitable profits must be arbitrarily small after any finite time T. Consequently, the result holds given that optimal MC profits bound SPE profits from below by Lemma 4, and given that optimal MC profits can be small only if the market has almost cleared by Proposition 1.

The result establishes that in any stationary equilibrium a monopolist lacking commitment extracts no more than optimal MC profits when price revisions can be arbitrarily frequent. Again, stationary monopoly-pricing without commitment simply amounts to optimal market-clearing and global agreement, and not to competition or efficiency.

Coasian Failures and Market-Clearing: The analysis is closely related to some known violations of the classical Coase conjecture. Board and Pycia 2014 considers a durable-good

monopoly problem in which buyers have the option to commit to stay out of the market by taking an outside option. The outside option amounts to a second variety of the durable-good that must be sold at a price of zero. They consider settings in which the value of the outside option is strictly positive for all players (there are gaps) and independent of the value of the durable-good. Their main contribution establishes that the monopolist sets a positive price for the durable-good, and never undercuts on such a price as the market clears at once. In our setting, their result holds immediately by Lemma 4. As the price of the outside option is zero, any price for the durable-good is a market-clearing price. Thus, the monopolist would clearly never undercut on such a price. Furthermore, this holds even without gaps and even with an arbitrary correlation structure. Of course, setting the price of the outside option to zero would be suboptimal in our environment, as any such price profile would belong to the interior of the market-clearing price set. Still in our view, Board and Pycia's novel contribution should not be classified as a failure of the Coase conjecture. Rather our results aim to highlight that the true essence of the Coasian intuition is market-clearing, and not necessarily competitive pricing, zero profits, or efficiency. Similar considerations apply to Wang 1998 who establishes a result evocative of Board and Pycia in the context of a multidimensional two type model.

Relatedly, Hahn 2006 expands on some classical conclusions to show that selling damaged products can increase the profit of a durable-good monopolist. A damaged product acts like a second variety with a lower value. In particular, his analysis considers settings in which the valuations of the two varieties are binary and perfectly correlated. Similar conclusions obtain in our setting independently of the joint measure of valuations. However, these are again no failure of the Coase conjecture, but rather its essence, as limit profits again amount to optimal market-clearing profits in our formulation of the problem. A host of similarly classified Coasian failures fit this bill.

Our conclusions hold even when the value of one of the varieties is constant. In essence, if the monopolist could pay a penny (or any small amount) to buyers for them to permanently exit the market, the Coasian profit would no longer amount to the smallest valuation in the support. Instead, the monopolist would be able to approximately extract the full static monopoly profit, as any price would again clear the market.

It may seem that our interpretation of Coase's seminal result as market-clearing relies on the assumption (implicit in this literature) which requires buyers to permanently exit the market upon purchasing a variety. Such an assumption is without loss: (i) if every buyer purchases its preferred variety; (ii) if goods are consumed when purchased thereby dissipating their need forever; or (iii) if players commit to stay out of the market upon purchasing the good. The first scenario is not so uncommon when the measure is symmetric (for instance,

for discordant symmetric distributions). In these circumstances, limit pricing in the baseline model may be efficient and clear markets despite not being competitive. The second scenario is also compelling for goods that are durable, but that are consumed once purchased (such as many services). In other markets however, it may be more plausible to assume that buyers remain active in the market until they purchase their preferred variety. If so, they may be able scrap the variety they purchased at an earlier round when their preferred variety is sufficiently cheap. This changes the shape of the market-clearing price set to

$$\underline{M} = \{p \in M \mid p_a = p_b\}.$$

Despite lacking a complete analysis of this environment, we conjecture that the key insights of our paper should not be affected. SPE profits will still be bounded below by static optimal MC profits, as setting prices in the interior of the market-clearing price set will never be optimal, by the same logic of Lemma 4. Markovian limit profits will still be uniquely pinned down by static optimal MC profits. However, trade will always be efficient in the limit as all consumers will eventually purchase the preferred variety. Furthermore, whenever $\{0, 0\} \in V$, there will be a limit WME that is efficient, competitive and that clears markets. In general though, not all limiting WME will be competitive (price goods at their lowest valuation in the support). For instance, with discordant valuations, there will be scenarios in which pricing is efficient, but not competitive (see the right panel of the plot below).

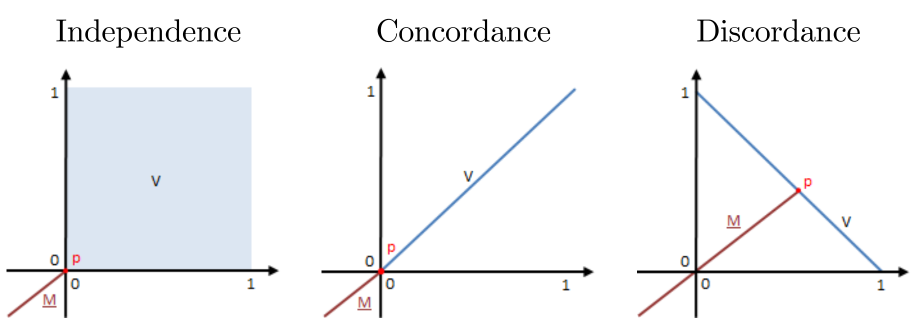

Figure 3: Market-clearing set without exit $\underline{M}$ for three possible distributions.

As a final observation on our main results, notice that the possible multiplicity of SPE contrasts with the uniqueness result obtained for the single-variety case with gaps. Multiplicity naturally arises here are optimal MC prices need not be unique when more than one variety is for sale. However, this does not imply that a Folk Theorem holds in our setting as in the Ausubel and Deneckere 1989. Their seminal contribution establishes that with one variety and no-gaps a Folk theorem obtains. The monopolist can extract the full static monopoly surplus by following a strategy with a slow price descent. Furthermore, it never wishes

to deviate from this pricing strategy if consumers' beliefs about future prices revert to the Markovian path upon observing any deviation. Such a strategy can be an equilibrium, however, only in the no-gaps case as the Markovian limit profit must equal 0, for the monopolist to comply with the slow price descent. As with multiple varieties equilibrium profits always exceed optimal MC profits (and thus necessarily differ from 0), it remains unclear whether a full-fledged SPE Folk theorem would hold in our setting. If so, our analysis would identify the lowest SPE profit and the stationary limit payoff.

4 Optimal MC Profits and Product Design

This section considers the product design problem faced by a durable-good monopolist. The monopolist assembles products from a basket of characteristics. For instance, a smartphone producer may decide which features (for instance, screen resolution, battery size, memory, camera) to include in a new line of devices, or an automobile manufacturer may choose how to blend performance, practicality, size, fuel efficiency. The way these characteristics are assembled might generate dispersion and dependence in consumers' valuations for the new products. Therefore, we ask how different product designs affect the profit of the monopolist and characterize the profit-maximizing design. For instance, would the monopolist prefer vertically differentiated products or horizontally differentiated ones? Or could products displaying both or neither of these features be optimal? Should the monopolist produce mass products with low volatility in valuations, or niche products with high dispersion in valuations?

To tackle these questions and deliver general results, we analyze our stylized environment with two varieties and unit measure of consumers. Our approach to the product design problem in a durable-good monopoly setting analyzes how optimal market-clearing profits are affected by the distribution of buyers' valuations. The approach implicitly relies on Proposition 7, and is legitimate for markets in which the monopolist can frequently revise prices (that is, when the discount factor is close to unity). We refrain from presenting an explicit model in which products are constructed by bundling components for sake for brevity.

The first result fixes the marginal distribution buyers' valuations for each variety, and asks what correlation structure between varieties maximizes the profit of the monopolist. This approach is valid when production costs depend only on the marginal distribution of valuations, but not on their correlation structure. Results exploit classical contributions on copulas (Sklar 1959) to establish that the profit-maximizing design requires full horizontal product differentiation. The second main result, instead, establishes that the monopolist does not necessarily benefit from reducing the variance of consumers' valuations, while fixing

the mean valuation for each variety. This is surprising as the monopolist would have certainly opted to minimize variance if only one product was for sale (as the product would otherwise trade at its lowest valuation in the support), and provides a novel rationale for the production of niche products.

Classical Results on Copulas: Recall that F_i denotes the marginal cumulative distribution of the consumers' valuation for variety i. For the rest of the analysis, let F_i be continuous and its support be the compact set $V_i = [0, \overline{v}_i] \subseteq [0, 1]$.[15] Before proceeding to the design problem, we briefly review the notion of copula and some of its properties. For a more detailed discussion of these topics, we refer to Nelsen 2006. A function $C : [0, 1]^2 \to [0, 1]$ is said to be a *copula* if

(1) $C(k, 0) = C(0, k) = 0$ for every $k \in [0, 1]$;

(2) $C(k, 1) = C(1, k) = k$ for every $k \in [0, 1]$;

(3) $C(x) + C(y) - C(x_a, y_b) - C(y_a, x_b) \geq 0$ for every $x, y \in [0, 1]^2$ such that $x \geq y$.

Let $\mathcal{C}$ denote the set of all copulas. Sklar's Theorem implies that for any joint distribution function F with continuous marginal distributions F_i for $i \in \{a, b\}$ there exists a unique copula C such that

$$F(v) = C(F_a(v_a), F_b(v_b)) \text{ for all } v \in V_a \times V_b.$$

A copula C can therefore be thought of as a sufficient statistic of the dependence structure between the two random variables. A commonly used copula exhibits no dependence among the random variables, and is associated to the joint distribution $I(v) = F_a(v_a)F_b(v_b)$. Two other noteworthy copulas are known as the Frechét-Hoeffding (FH) bounds, and are associated to the two joint distributions

$$\begin{aligned} K(v) &= \min\{F_a(v_a), F_b(v_b)\}; \\ L(v) &= \max\{F_a(v_a) + F_b(v_b) - 1, 0\}. \end{aligned}$$

Frechét and Hoeffding establish that these two copulas bound any other feasible copula C.

Remark 8 *Any joint distribution F consistent with the two marginals, F_a and F_b, satisfies*

$$L(v) \leq F(v) \leq K(v) \text{ for all } v \in V_a \times V_b.$$

[15] We restrict attention to the environments in which the lowest valuation is 0 to make results directly comparable to the no-gap case discussed in the literature where the Coase conjecture leads to 0 profits in the limit. The compact support assumption and the continuity assumption are not essential, but guarantee that the problem is well behaved.

Random variables v_a and v_b are said to be *concordant* if for any two profiles v and v' in the support of the joint distribution $(v_i - v'_i)(v_j - v'_j) > 0$. The random variables are said to be *discordant*, instead, if for any two profiles v and v' in the support $(v_i - v'_i)(v_j - v'_j) < 0$. The FH bounds L and K are characterized by Kendall's Tau, as well as Spearman's Rho, being 1 and -1 respectively. Both these statistics measure the association, specifically the concordance, between random variables.[16] Informally, the FH upper bound identifies the most concordant copula, as large values of one random variable are associated with large value of the other, and small values of one with small values of the other. Conversely, the FH lower bound identifies the most discordant copula as large values of one random variable are associated with small value of the other, and viceversa. For instance, if the marginal distributions are either uniform or Gaussian, the FH bounds induce either perfect positive or perfect negative correlation between the random variables.

The next remark highlights a few properties of the FH bounds that will play an important role in our product design analysis. A subset V of $\mathbb{R}^2$ is *non-decreasing* if for any $v, v' \in V$, $v_a < v'_a$ implies $v_b \leq v'_b$. Similarly, a subset V of $\mathbb{R}^2$ is *non-increasing* if for any $v, v' \in V$, $v_a < v'_a$ implies $v_b \geq v'_b$.

Remark 9 *The joint distribution F is equal to its:*

(1) FH upper bound K if and only if its support V is a non-decreasing subset of $V_a \times V_b$;

(2) FH lower bound L if and only if its support V is a non-increasing subset of $V_a \times V_b$.

Moreover, random variable v_a is almost surely a strictly monotonic function of v_b if and only if the copula coincides with one of the FH bounds.

The remark implies that it is possible to construct monotone correspondences $v_a = k(v_b)$ and $v_a = l(v_b)$ that completely pin down the support of the FH bounds. The additional assumptions imposed on marginal distributions further imply that the support of the FH bounds can have no horizontal or vertical line segments. Thus, these correspondences are

[16] For any two profiles v and v' in the support of the joint distribution, Kendall's Tau amounts to

$$\Pr[(v_i - v'_i)(v_j - v'_j) > 0] - \Pr[(v_i - v'_i)(v_j - v'_j) < 0].$$

For any three profiles v, v' and v'' in the support of the joint distribution, Spearman's Rho amounts to

$$3\left(\Pr[(v_i - v'_i)(v_j - v''_j) > 0] - \Pr[(v_i - v'_i)(v_j - v''_j) < 0]\right).$$

functions such that $k'(x) \in (0, \infty)$ and $l'(x) \in (-\infty, 0)$ almost surely.

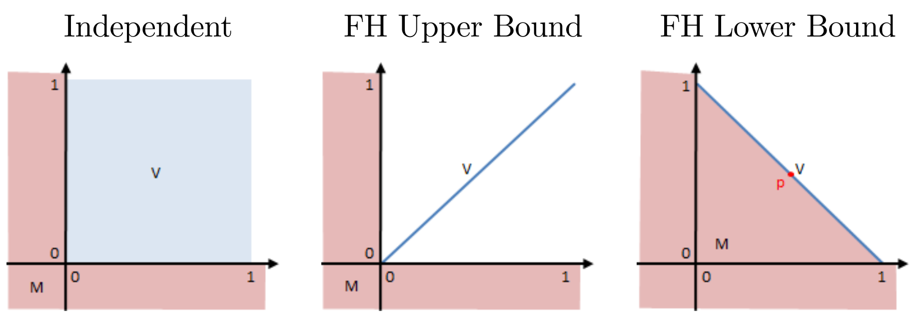

Figure 4: Three copulas for uniform marginals $F_a(x) = F_b(x) = 0$.

Optimal Product Design: Our product design exercise begins by analyzing how dependence, or association, between the random variables v_a and v_b affects the optimal MC profit. By the analysis carried out in section 3, our conclusions also identify how dependence affects the limiting WME profits of the dynamic pricing game. The first result fixes the marginal distributions F_i (postulating that any costs incurred in generating value depends only such marginals) and asks which joint distribution compatible with such marginals maximizes optimal MC profits. Formally, let V^F denote the support of a joint distribution F, and let $\bar{\pi}^F$ denote the optimal MC profit for the joint distribution is F. Specifically, we solve the following problem

$$\max_{C \in \mathcal{C}} \bar{\pi}^F \quad \text{s.t.} \quad F = C(F_a, F_b).$$

The next result establishes that optimal MC profits for any joint distribution F are bounded above by the profits at the FH lower bound. Profits are maximized when selling varieties that are most discordant. Intuitively, selling discordant varieties maximizes profits as segmenting the market guarantees that no value is waisted on varieties that are never purchased. Thus, full horizontal product differentiation is profit maximizing.

Proposition 10 *For any joint distribution F consistent with the two marginals, F_a and F_b,*

$$\bar{\pi}^F \leq \bar{\pi}^L.$$

The result implies that independent outside-option products are suboptimal since discordance would further increase profits. As the monopolist loses its ability to intertemporally discriminate buyers (when the discount factor converges to unity), statically screening consumers by selling different varieties is the only option available to extract surplus. But such a task is most profitable when the market is segmented and values are discordant. As discordance benefits profits, one may conjecture that a lower bound on optimal MC profits may then

be pinned down by the other FH bound. This would be natural, as separating high value consumers for the two varieties could be most difficult when varieties are concordant. But, such conclusion does not always hold, as we show with a simple example in the conclusive subsection.

We say that varieties are *partially discordant* if $v \in V$ and $v_i = 0$ together imply that $v_j = \bar{v}_j$ for $j \neq i$. That is, varieties are partially discordant if any player having the lowest value for one product has the highest value for the other. Clearly, varieties are partially discordant whenever they are discordant; but the converse does not hold. The next result establishes that the optimal MC price of both varieties is strictly positive whenever varieties are partially discordant. The result holds as any MC price profile on the boundary of the market-clearing price set that is not strictly positive raises zero profits.

Lemma 11 *If the market is regular and varieties are partially discordant, the optimal MC price of both varieties is strictly positive.*

Indeed, when varieties are partially discordant, any MC price in $\bar{M}$ such that $p_i = 0$ for some $i \in \{a, b\}$ would raise zero profits as a measure zero of buyers would purchase the variety sold at a positive markup. The very same conclusion would hold here even if consumers stayed in the market and did not commit to exit upon purchasing a variety of the durable-good.

The final substantive contribution looks at how variance in valuations affects the profitability of the monopolist. When a single product is for sale the monopolist always prefers to minimize variance, as the good necessarily trades at the lowest valuation in the support. However, when multiple varieties can be sold a trade-off emerges as variance increases both buyers' information rents (which hurts profits) and total surplus (which benefits profits) as the maximal value grows. Thus, we ask whether the monopolist prefers a distribution F to the distribution $\hat{F}$ in which all buyers value products at the mean of F (that is, $\hat{F}$ is a degenerate distribution with unit measure at $E^F(v)$). The result establishes the answer to this question depends on the details of the measure F whenever multiple varieties are for sale.

Remark 12 *Minimizing variance does not necessarily increase profits.*

Examples, in the following subsection, prove the remark and establish that variance is most beneficial for profits whenever values are discordant and the support of the distribution is a concave map.

Product Differentiation Examples: The first example of the section establishes why maximal concordance does not necessarily minimize profits. Consider two products with uniform marginal distributions, respectively, on $[0, 1]$ and $[0, x]$ for some $x \in (0, 1]$. Three

possible product designs are: (1) perfect concordance, when the joint distribution of consumers' preferences is K with support

$$V^K = \{v \in V_a \times V_b | v_b = xv_a\};$$

(2) independence, when the joint distribution of consumers' preferences is I with support $V(I) = V_a \times V_b$; (3) perfect discordance, when the joint distribution of consumers' preferences is L with support

$$V^L = \{v \in V_a \times V_b | v_b = x(1 - v_a)\}.$$

For convenience, for any joint distribution F, let $\bar{d}_a^F = d_a^F(\bar{p}^F)$. The first scenario corresponds to classical models of vertical price differentiation. The set of market-clearing prices, M^K, coincides with the set of price profiles for which at least one of the two prices is non-positive. When varieties are independent, the set of MC prices does not change, $M^I = M^K$. In both these scenarios, the optimal MC profits are found by setting the price of the worse variety to zero, $\bar{p}_b = 0$, and by maximizing profits. When varieties are discordant, optimal MC prices live in the support of the distribution V^L. The problem again reduces to a 1-dimesional problem as the support is a decreasing set. The solution in each of the three cases respectively satisfies:

C	Upper Bound	Independent	Independent	Lower Bound
x	$[0, 1]$	$[0, 2/3]$	$[2/3, 1]$	$[0, 1]$
$\bar{p}_a$	$(1 - x)/2$	$(2 - x)/4$	$1/3$	$1/2$
$\bar{p}_b$	0	0	0	$x/2$
$\bar{d}_a$	$1/2$	$(2 - x)/4$	$2/(9x)$	$1/2$
$\bar{\pi}$	$(1 - x)/4$	$(2 - x)^2/16$	$2/(27x)$	$(1 + x)/4$

As proven in Proposition 10, the monopolist maximizes profits by choosing a design that induces perfect discordance, as $\bar{\pi}^L \geq \max\{\bar{\pi}^I, \bar{\pi}^K\}$. Moreover, optimal MC profits increase in x when varieties are discordant, but decrease in x otherwise (see Figure 5). With vertically differentiated products, optimal MC profits decrease to zero when products become less differentiated (that is, when $x \to 1$) as expected by Proposition 1. Surprisingly, however, perfectly concordant varieties raise more revenue than independently distributed varieties when x is small. If so, the monopolist is able sell the expensive product to a larger measure of buyers at a given MC price than with independence.

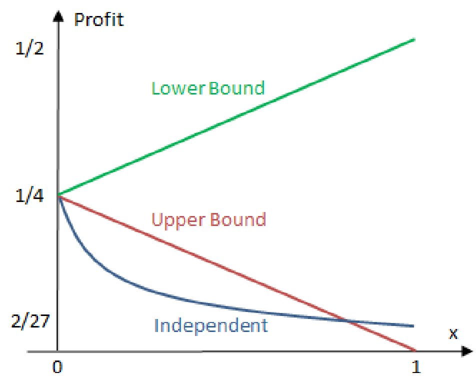

Figure 5: Optimal MC profits for the three copulas as a function of x.

The second example establishes why the effect of variance in valuations on optimal MC profits is ambiguous. The insight is important as it provides a rationale for a monopolist to sell products with volatile valuations, thus establishing why low volatility mass-products are not necessarily a profit maximizing design. First, consider the discordant distribution

$$L(v) = \max\{v_a^2 + v_b^2 - 1, 0\}.$$

The support of L is given by the decreasing set $V^L = \{v \mid v_a^2 + v_b^2 = 1\}$. The degenerate distribution $\hat{L} = [E^L(v)]$ has support $V^{\hat{L}} = \{(2/3, 2/3)\}$. The optimal MC profit associated to L can be found by solving

$$\bar{\pi}^L = \arg\max_p p_a(1 - p_a^2) + p_b(1 - p_b^2) \;\text{ s.t. }\; p_a^2 + p_b^2 = 1.$$

The monopolist clears the market by selling all units at a price of $1/\sqrt{2}$. In this scenario the monopolist benefits from the variance as

$$\bar{\pi}^L = 1/\sqrt{2} > 2/3 = \bar{\pi}^{\hat{L}}.$$

More volatile niche-products are optimal. When the measure of high value buyers is large, the smallest valuation for the preferred variety exceeds the average valuation, and thus niche-products are preferred.

Next, consider the discordant distribution

$$L(v) = \max\{v_a^{1/2} + v_b^{1/2} - 1, 0\}.$$

The support of L is given by the decreasing set $V^L = \left\{v \mid v_a^{1/2} + v_b^{1/2} = 1\right\}$. The degenerate distribution $\hat{L} = \left[E^L(v)\right]$ has support $V^{\hat{L}} = \{(1/3, 1/3)\}$. The optimal MC profit associated

to L can be found by solving

$$\bar{\pi}^L = \arg\max_p p_a(1 - p_a^{1/2}) + p_b(1 - p_b^{1/2}) \text{ s.t. } p_a^{1/2} + p_b^{1/2} = 1.$$

The monopolist clears the market by selling all units at a price of $1/4$. The monopolist is hurt by the variance as

$$\bar{\pi}^L = 1/4 < 1/3 = \bar{\pi}^{\hat{L}}.$$

Less volatile mass-products are optimal. When the measure of low value buyers is large, the average valuation exceeds the smallest valuation for the preferred variety, and thus mass-products are preferred.

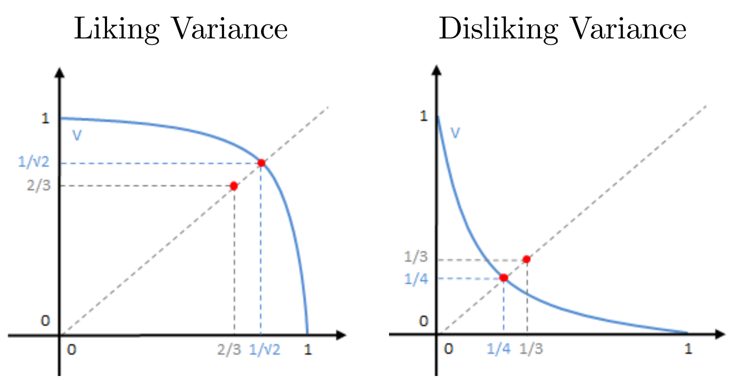

Figure 6: Effects of volatility on optimal MC prices in two markets.

One may postulate that the monopolist prefers variance whenever varieties are perfectly discordant and the support is a concave map. However, this is not true in general. For instance, when $L(v) = \max\{v_a^2 + v_b - 1, 0\}$, the monopolist dislikes variance despite the support being concave.

5 Conclusions

The paper discussed a dynamic monopoly problem in which multiple varieties of a product could be produced and sold. It extended classical conclusions on equilibrium pricing, and developed product design exercises to compare the limit stationary profits associated to different product lines. The main punch-line was that, with more than one variety, intratemporal price discrimination would make up for absence of intertemporal price discrimination caused by the lack of commitment. Although the paper was presented for two varieties analogous conclusions would obtain with more than two varieties.[17] The table below summarizes

[17] However, we would lose the uniqueness of the FH lower bound.

classical contributions on dynamic monopoly pricing with one variety and highlights which conclusions are specific to this scenario.

Varieties	Gaps	MC	MC Time	Efficiency	Competitive	Unique SPE	LWME Profit
1	No	Yes	Infinite	Yes	WME	Folk Thm	Optimal MC
1	Yes	Yes	Finite	Yes	Yes	Yes	Optimal MC
Multiple	Any	Yes	Finite	No	No	No	Optimal MC

As usual, it is possible to consider our setting as a model of bargaining with one-sided incomplete information in which the uninformed party always proposes. In this interpretation varieties consist of alternative projects that the monopolist can offer to the agent to screen his type. If so, our conclusions establish that the uninformed party regains some bargaining power by screening consumers in such a manner, as it can extract surplus even if it has to agree with every possible type of the informed player. The bargaining interpretation of the Coase conjecture as immediate agreement in stationary equilibria holds even in our setting, but essentially coincides with optimal market clearing here.

In our view, most robust theme to results in the area (including that of many, so called, failures of the Coase conjecture) is market clearing. Moreover, the approximation of stationary equilibrium profits (when the frequency of price revisions is high) by optimal MC profits may be useful to motivate applied research in the area and to carry our more elaborate product design exercises in durable-goods markets.

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6 Appendix

Proof Proposition 1. Without loss of generality assume that $\underline{v}_a \leq \underline{v}_b$. If so, by setting a MC prices $p = (1, \underline{v}_b)$, the monopolist achieves a profit of $\underline{v}_b$. Thus, optimal MC profits exceed $\underline{v}_b$. Consequently, if $\underline{v}_a < \underline{v}_b$, optimal MC profit strictly exceed $\underline{v}_a$.

So suppose that $\underline{v}_a = \underline{v}_b = \underline{v}$. Consider the price profile $p^\varepsilon = (\underline{v} + \varepsilon, \underline{v})$ for some small number ε. Clearly, such a price belongs to M, as $v_b \geq p_b^\varepsilon = \underline{v}$ for all $v \in V$. Moreover, by the definition of the static demand, $\lim_{\varepsilon \to 0} d_i(p^\varepsilon) \geq \mathcal{F}(v_i > v_j)$ for any $i \in \{a, b\}$. If at p^ε some consumers were to purchase variety a, optimal MC profits would strictly exceed $\underline{v}$. If instead optimal MC profits were equal $\underline{v}$, $\lim_{\varepsilon \to 0} d_a(p^\varepsilon) = \mathcal{F}(v_a > v_b) = 0$, and thus $v_a \leq v_b$ for any $v \in V$. A symmetric argument would then establish that $v_b \leq v_a$ for any $v \in V$. Therefore, if optimal market-clearing profits were to equal $\underline{v}$, $v_a = v_b$ for any $v \in V$.

Next suppose that there is a single variety in the market. If so, any market-clearing price would set one of the two prices to $\underline{v}$ as $(\underline{v}, \underline{v}) \in V_a \times V_b$. But if so, optimal MC profits would amount $\underline{v}$ as all players would necessarily purchase the cheapest variety. ■

Proof Lemma 3. We establish the result by showing that at any history h^t, if a buyer with valuation v is willing to purchase a variety with positive probability, then for any $\varepsilon > 0$ a buyer with valuations $v' = v + (\varepsilon, \varepsilon)$ is willing to purchase a variety with probability 1. To prove the latter observe that since v purchases a variety

$$\max_i \{v_i - p_i\} \geq \delta U(h^t | v),$$

where $U(h^t|v)$ denotes continuation value of player v at date $t+1$ after history h^t. We establish the desired conclusion by showing that

$$\max_i \{v_i' - p_i\} > \delta U(h^t | v').$$

As buyer v can mimic the strategy of buyer v' from period $t+1$ onwards – by accepting and rejecting the very same offers – it follows that

$$U(h^t|v') - U(h^t|v) \leq \sum_{s=0}^{\infty} \delta^s \left[\sum_{j \in \{a,b\}} \alpha_j^s(h^t|v') \left(v_j' - v_j\right) \right],$$

where $\alpha_j^s(h^t|v')$ denotes the probability conditional on h^t that variety j is purchased by v' at time $t+s+1$. Thus,

$$U(h^t|v') - U(h^t|v) \leq \max \{v_a' - v_a, v_b' - v_b\} = \varepsilon.$$

But if so, the result follows as

$$\max_i\{v_i' - p_i\} > \max_i\{v_i - p_i\} + \delta\varepsilon \geq \delta\left[U(h^t|v) + \varepsilon\right] \geq \delta U(h^t|v').$$

■

Proof Lemma 4. It is sufficient to observe that in any SPE all consumers accept any price in M. Suppose this were not the case and for any selected equilibrium let $P \subsetneq M$ denote the set of prices that will be accepted by all consumers (except possibly for a set of measure zero) after any history. Clearly, P must satisfy

$$p \in P \text{ and } p' \leq p \quad \Rightarrow \quad p' \in P. \tag{2}$$

As $P \neq M$, there exist prices $\bar{M}\backslash P \neq \emptyset$ by (2). Consider any price $\hat{p} \in \bar{M}\backslash P$. Construct a corresponding price p satisfying

$$\sup_{p\in P} p_a \text{ subject to } p_a - p_b = \hat{p}_a - \hat{p}_b.$$

Such a price necessarily exist because for any $\gamma \in \mathbb{R}$ there exists $p \in P$ such that $p_a - p_b = \gamma$, as all consumers buy immediately when $\max_{i\in\{a,b\}} p_i < -1$. The latter holds because the proof of Lemma 3 establishes that $U(h^t|v)$ is non-decreasing in v, and because $U(h^t|v) \leq 1$, as buyers cannot extract more than the full surplus in an SPE. But, in turn this implies that all buyers purchase a variety of the durable-good when $\max_{i\in\{a,b\}} p_i < -1$, as

$$\max\{v_a - p_a, v_b - p_b\} > 1 > \delta U(h^t|v).$$

Observe that every consumer buys when prices are set to $\tilde{p}_i = p_i - \eta$ for any $\eta > 0$ as $\tilde{p} \in P$ by definition of p. Next, construct the price

$$q = (1-\delta)\,\hat{p} + \delta\tilde{p}$$

By construction, for any $\varepsilon > 0$, every consumer prefers the offer $q - \varepsilon$ to an anticipated offer p tomorrow as

$$\max_{i\in\{a,b\}}\{v_i - (1-\delta)\,\hat{p}_i - \delta\tilde{p}_i + \varepsilon\} \geq \max_{i\in\{a,b\}}\{\delta(v_i - \tilde{p}_i)\},$$

where the inequality holds as $v_a - \tilde{p}_a > v_b - \tilde{p}_b$ if and only if $v_a - q_a > v_b - q_b$. But, $q - \varepsilon \notin P$ for any small values for ε and η, and for any $\hat{p} \in \bar{M}\backslash P$. This contradicts the definition of P, and consequently establishes (1) and (2). As (2) establishes that every consumer purchases

a variety when prices belong to $M\backslash\bar{M}$, the monopolist can always secure a payoff arbitrarily close to the optimal market-clearing profits $\bar{\pi}(A^t) > 0$, given the active measure $\mathcal{F}(A^t)$, by choosing a price in $M\backslash\bar{M}$. Part (3) then follows. ■

Proof Lemma 5. We begin by establishing that, whenever the measure of active players $\mathcal{F}(A^t)$ is in an ε-neighborhood of 0 for $\varepsilon > 0$ sufficiently small (that is, $\mathcal{F}(A^t) \in \mathcal{N}_\varepsilon(0)$),[18] then $p^{t+1} \in M$ in any SPE. To do so, fix h^t and the corresponding state A^t. Consider any market-clearing price $q \in M$ that raises strictly positive profits when clearing the market. Such a price exists as the market is regular. Define the the set of prices at which a measure of active buyers extracts a positive surplus by purchasing one of the two varieties

$$Q^t = \left\{p \in \mathbb{R}^2 | \mathcal{F}\left(\max_{i\in\{a,b\}}\{v_i - p_i\} > 0 | A^t\right) > 0\right\}.$$

Next, we construct an upper bound $R(p|q)$ on equilibrium profit, and establish that it satisfies the following three properties when $\mathcal{F}(A^t) \in \mathcal{N}_\varepsilon(0)$:

(a) $R(p|q) > \Pi(p|h^t)$ for all $p \gneqq q$ such that $p \in Q^t$;

(b) $R(q|q) = \Pi(q|h^t)$;

(c) $R(q|q) > R(p|q)$ for all $p \gneqq q$ such that $p \in Q^t\backslash M$.

If so, for all $p \gneqq q$ such that $p \in Q^t\backslash M$,

$$\Pi(p|h^t) - \Pi(q|h^t) < R(p|q) - R(q|q) < 0.$$

But in turn, this would establish the first part of the claim since: setting a price outside Q^t could not be optimal given that no consumer would buy; setting a price below q would also not be optimal given that the monopolist could sell to every active buyer at q immediately. For convenience, for any $i \in \{a, b\}$ define

$$\begin{aligned}
\mathcal{K}_i(p) &= \mathcal{F}(v_i - v_j > p_i - p_j | A^t);\\
\mathcal{H}_i(p) &= \mathcal{F}(v_i - v_j > p_i - p_j \cap v_i \geq p_i | A^t);\\
\mathcal{Y}_i(p) &= \mathcal{H}_i(p)/\mathcal{K}_i(p).
\end{aligned}$$

and consider the following map

$$\begin{aligned}
R(p|q) &= \sum\nolimits_{i\in\{a,b\}} \left[\mathcal{H}_i(p)p_i + \delta(\mathcal{K}_i(p) - \mathcal{H}_i(p))p_i\right]\\
&= \sum\nolimits_{i\in\{a,b\}} \mathcal{K}_i(p)\left[\mathcal{Y}_i(p)p_i + \delta p_i(1 - \mathcal{Y}_i(p))\right].
\end{aligned}$$

[18]For any $\varepsilon \in \mathbb{R}$, any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^k$ let $\mathcal{N}_\varepsilon(x) = \left\{y \in \mathbb{R}^k \mid \max_{i\in\{1,\dots,k\}} |x_i - y_i| \leq \varepsilon\right\}$.

Next observe that (a) must obviously hold as $R(p|q)$ denotes the profit if: (i) every buyer with value $v_i > p_i$ for some variety i buys the preferred variety at the current price in spite of any future price reduction; (ii) every buyer with value $v \leq p$ buys variety i tomorrow at price p_i in spite any future price reduction. Moreover, (b) also holds as

$$\Pi(q|h^t) = \sum\nolimits_{i\in\{a,b\}} \mathcal{H}_i(q)q_i,$$

as in an SPE the monopolist necessarily sells to a measure 1 of buyers when setting $q \in M$ (given that all buyers purchase the preferred variety at price $q - \eta$ for any small value of $\eta > 0$). To establish (c) observe that, if $\mathcal{F}(A^t)$ is in an ε-neighborhood of 0, so is $\mathcal{K}_i(p)$ – formally, $\mathcal{F}(A^t) = \mathcal{K}_a(p) + \mathcal{K}_b(p) \in \mathcal{N}_\varepsilon(0)$ implies $\mathcal{K}_i(p) \in \mathcal{N}_\varepsilon(0)$. Furthermore, by regularity $\mathcal{F}(A^t) \in \mathcal{N}_\varepsilon(0)$ implies that $p \in \mathcal{N}_\eta(q)$ for any $p \in Q^t$ and for some small $\eta > 0$.[19] But if so, given that $\delta < 1$, for all $p \gneqq q$ such that $p \in Q^t$

$$\begin{aligned} R(p|q) - R(q|q) &= \sum\nolimits_{i\in\{a,b\}} \mathcal{K}_i(p)\left[\mathcal{Y}_i(p)p_i + \delta p_i(1-\mathcal{Y}_i(p)) - q_i\right] \\ &\leq \sum\nolimits_{i\in\{a,b\}} \mathcal{K}_i(p)\left[\eta - (1-\delta)p_i(1-\mathcal{Y}_i(p))\right] < 0, \end{aligned}$$

where the first inequality holds as $p_i - q_i \leq \eta$, and where the second inequality holds for η sufficiently small since $p \gneqq q$ and since $q_i > 0$ for some $i \in \{a, b\}$. The latter follows in particular, as η can be made arbitrarily small for sufficiently small values of ε. This establishes that in any SPE $p^{t+1} \in M$ whenever $\mathcal{F}(A^t) \in \mathcal{N}_\varepsilon(0)$ for $\varepsilon > 0$ sufficiently small.

Part (3) of Lemma 4 further implies that, if the monopolist ever sets a price in M, it must set a price that is arbitrarily close to $\bar{p}(A^t)$ (or else a profitable deviation would exist). Moreover, the monopolist profits must be positive as $\bar{\pi}(A^t) > 0$ for any A^t by regularity and Lemma 3.

Next we show that there exists T such that in all SPE, a measure 1 of buyers purchases a variety of the durable-good before period $T + 1$. We do so by showing that in any SPE after T period the support of the measure of active buyers must satisfy $\mathcal{F}(A^{t+T}) \in \mathcal{N}_\varepsilon(0)$, for $\varepsilon > 0$ sufficiently small. Let $\mathcal{L}(\cdot)$ denote the Lesbegue measure of a set. We claim that there exists constants w and s such that $\mathcal{L}(A^t \backslash A^{t+s}) \geq w$. Suppose not. If so, the measure of consumers with whom the seller trades in the next s periods must be bounded above by

$$\mathcal{F}(A^t \backslash A^{t+s}) \leq \mathcal{L}(A^t \backslash A^{t+s}) \max\nolimits_{v\in V} f(v) < w\bar{f}.$$

[19]The latter holds as $\mathcal{F}(A^t) \in \mathcal{N}_\varepsilon(0)$ implies $\mathcal{L}(A^t) \in \mathcal{N}_\kappa(0)$ for $\kappa = \varepsilon\bar{f}/\underline{f}$, where $\mathcal{L}(\cdot)$ denotes the Lesbegue measure of a set. If so however, for any $p \in Q^t \subseteq A^t$, $p \in \mathcal{N}_\eta(q)$ as A^t is a compact subset of V by Lemma 3.

The equilibrium profit of the seller must consequently be bounded by

$$\Pi(h^t) < w\bar{f} + \delta^s,$$

as the monopolist never sells anyone of the two varieties at price higher than 1 (which is the highest valuation in the initial support). However, for w sufficiently small and s sufficiently large, a contradiction emerges since

$$w\bar{f} + \delta^s < \bar{\pi}(A^t), \tag{3}$$

given that $\bar{\pi}(A^t) > 0$ by the multi-variety condition. Thus, T can be defined as

$$T = \left\lceil \frac{s}{w} \right\rceil,$$

for any values of s and w satisfying condition (3), where $\lceil x \rceil$ denotes the ceiling of x. ■

Proof Proposition 6. We proceed by induction on the number of periods which it takes for the game to end $s \in \{0, ..., T\}$. We define several objects recursively first. Then, we prove existence. Let $X^0 = M$ denote the set of prices which the seller must charge in equilibrium to guarantee that the game ends immediately. Let $U^0(p|v) = \max_i \{v_i - p_i\}$ denote the continuation value of the buyer with value v when prices equal p. Similarly, for any mixed strategy of the seller $\rho \in \mathcal{P}(X^0)$, let $U^0(\rho|v)$ denote the expected continuation value,

$$U^0(\rho|v) = \int_{X^0} \max_i \{v_i - p_i\} \, d\rho(p).$$

For any variety $i \in \{a, b\}$, let

$$d_i^0(p|A) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}|A).$$

The probability distribution $\sigma^0(A) \in \mathcal{P}(X^0)$ denotes an equilibrium mixed strategy of the seller when the market clears instantaneously. Specifically, for any set A

$$\sigma^0(A) \in B^0(A) \equiv \arg\max_{\rho \in \mathcal{P}(X^0)} \int_{X^0} p d^0(p|A) d\rho(p),$$

where we adopted the convention that $pd^0(p|A) = p_a d_a^0(p|A) + p_b d_b^0(p|A)$. Let $\Pi^0(A)$ denote the value of this program, or profit that the seller would make if the market had to clear and only buyers in A were active, clearly $\Pi^0(A) = \bar{\pi}(A)$. The best response correspondence $B^0(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values by Berge's

maximum theorem.[20] The theorem applies here because the objective function is continuous in ρ and in A (as $d_i^0(p|A)$ is continuous in A by regularity), and because the solution belongs to $\mathcal{P}\left(X^0 \cap [0,1]^2\right)$ which is non-empty and compact.[21] The convexity of the correspondence $B^0(A)$ follows instead by the linearity in ρ. Finally for any $\rho \in \mathcal{P}\left(X^0\right)$ and $p \notin X^0$, let $\mathcal{A}^0(p|\rho)$ identify the subset of valuations for which a buyer weakly prefers purchasing a variety of the durable-good tomorrow at ρ than today at prices p

$$\mathcal{A}^0(p|\rho) = \left\{v \in V \mid \max_i \{v_i - p_i\} \leq \delta U^0(\rho|v)\right\}.$$

We begin by showing that for any $p^1 \notin X^0$ there exists $\bar{\sigma}^0 \in \mathcal{P}\left(X^0\right)$ such that

$$\bar{\sigma}^0 \in B^0(\mathcal{A}^0(p^1|\bar{\sigma}^0)). \tag{4}$$

To do so, observe that $\mathcal{A}^0(p|\rho)$ is continuous in ρ since $U^0(\rho|v)$ is linear and thus continuous in ρ. Moreover, $\mathcal{A}^0(p|\rho)$ is single-valued, and thus convex-valued, in ρ. Therefore, the correspondence $B^0(\mathcal{A}^0(p^1|\rho))$ has closed graph and convex values, since $B^0(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values. As $\mathcal{P}\left(X^0\right)$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem applies.[22] Thus, equation 4 has a non-empty compact set of fixed points to which $\bar{\sigma}^0(p^1) \in \mathcal{P}(X^0)$ belongs.

Next we proceed by induction. For $s \in \{1, ..., T\}$, suppose that an equilibrium exists whenever it takes no more than $s-1$ periods for the game to end. If so, we show that an equilibrium exists when it takes s periods for the game to end. Let X^s denote the set of prices which the seller must charge in equilibrium to guarantee that the game ends in no more than s periods. Denote the continuation value of the buyer when prices equal p by

$$U^s(p|v) = \max\{\max_i \{v_i - p_i\}, \delta U^{s-1}(\bar{\sigma}^{s-1}(p)|v)\}.$$

Similarly, for any mixed strategy of the seller $\rho \in \mathcal{P}\left(X^s\right)$, let $U^s(\rho|v)$ denote the expected continuation value,

$$U^s(\rho|v) = \textstyle\int_{X^s} U^s(p|v)d\rho(p).$$

For any variety $i \in \{a, b\}$, let

$$d_i^s(p|A) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, \delta U^{s-1}(\sigma^{s-1}(p))\}|A).$$

[20] For the relevant statement of the Maximum Theorem see Aliprantis and Border 2006 page 570.

[21] $\mathcal{P}(X^0 \cap [0,1]^2)$ is compact, because $X^0 \cap [0,1]^2$ is compact. See Aliprantis and Border 2006 page 513.

[22] For the statement of the relevant Fixed Point Theorem see Aliprantis and Border 2006 page 583.

The probability distribution $\sigma^s(A) \in \mathcal{P}(X^s)$ denotes an equilibrium mixed strategy of the seller when the market clears in no more than s periods. Specifically, for any set A

$$\sigma^s(A) \in B^s(A) \equiv \arg\max_{\rho \in \mathcal{P}(X^s)} \int_{X^s} p d^s(p|A) + \delta \Pi^{s-1}(\bar{\sigma}^{s-1}(p)) d\rho(p).$$

Let $\Pi^s(A)$ denote the value of this program, or the present discounted value of the seller's profit if the market has to clear in s periods. The best response correspondence $B^s(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values by Berge's maximum theorem. The theorem applies here because the objective function is continuous in ρ and in A (as $d_i^s(p|A)$ is continuous in A by regularity), and because the solution belongs to $\mathcal{P}\left(X^s \cap [0,1]^2\right)$ which is non-empty and compact. The convexity of the correspondence $B^s(A)$ follows instead by the linearity in ρ. Finally, for any $s < T$, any $\rho \in \mathcal{P}(X^s)$ and any $p \notin X^s$, let $\mathcal{A}^s(p|\rho)$ identify the subset of valuations for which a buyer weakly prefers purchasing a variety of the durable-good tomorrow at ρ than today at prices p

$$\mathcal{A}^s(p|\rho) = \{v \in V \mid \max_i \{v_i - p_i\} \leq \delta U^s(\rho|v)\}.$$

As before, next we argue that for any $s < T$ and any $p^{s+1} \notin X^s$ there exists $\bar{\sigma}^s \in \mathcal{P}(X^s)$ such that

$$\bar{\sigma}^s \in B^s(\mathcal{A}^s(p^{s+1}|\bar{\sigma}^s)). \tag{5}$$

To do so, observe that $\mathcal{A}^s(p|\rho)$ is continuous ρ since $U^s(\rho)$ is linear and thus continuous in ρ. Moreover, $\mathcal{A}^s(p|\rho)$ is single-valued, and thus convex-valued, in ρ. Therefore, the correspondence $B^s(\mathcal{A}^s(p^{s+1}|\rho))$ has closed graph and convex values, since $B^s(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values. As $\mathcal{P}(X^s)$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem applies. Thus, equation 5 has a non-empty compact set of fixed points to which $\bar{\sigma}^s(p^{s+1})$ belongs. This completes the proof, as in the initial period T the monopolist simply chooses $\bar{\sigma}^T \in B^T(V)$, which is non-empty by the previous arguments. ■

Proof Proposition 7. Fix an WME $(\boldsymbol{\sigma}, \boldsymbol{\tau})$. We shall omit the dependence on $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ whenever possible to simply notation. Let $\delta = e^{-r\Delta}$ and consider what happens when Δ converges to 0. As the buyers' strategy is stationary, let $\hat{U}(p|v)$ denote the WME expected payoff of a buyer with value v when p was the last price quoted by the monopolist. As in the proof of Proposition 6, for any quoted price p define the set buyers who remain active on the equilibrium path,

$$\mathcal{A}(p) = \left\{v \in V \mid \max_i \{v_i - p_i\} \leq \delta \hat{U}(p|v)\right\}.$$

Because of stationarity of buyers' strategies, the monopolist's WME present discounted value

of profits depends only on the distribution of active buyers (which is summarized by its support A^t) and not on the entire history of play h^t. Let $\hat{\Pi}(p)$ denote the present discounted value of WME profits when the active player set is $\mathcal{A}(p)$. Finally, define the WME demand for variety $i \in \{a, b\}$ when the active buyer set is A and the price quoted is p as

$$d_i(p|A) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, \delta \hat{U}(p|v)\} \mid A).$$

Let $\hat{\sigma}(p^t)$ denote the seller WME strategy when the set of active buyers is $\mathcal{A}(p^t)$, and let $\hat{E}[p^t]$ denote the expectation with respect to such distribution.

Fix any real time $\hat{T} > 0$. For any $\eta > 0$ there exist a Δ sufficiently small and a period $t < (\hat{T}/\Delta - 2)$ such that

$$\max_i d_i(p^t|A^t) < \eta.$$

That is, as the number of periods between 0 and $\hat{T}$ diverges to infinity, it is always possible to find a period in which the total quantity sold is bounded above by a given number.

For the monopolist to charge prices p^t at date t and the distribution of prices $\hat{\sigma}(p^t)$ at date $t+1$, profits must be higher than by pricing according to $\hat{\sigma}(p^t)$ directly at date t. Formally,

$$p^t d(p^t|A^t) + \delta \hat{E}\left[pd(p|\mathcal{A}(p^t)) + \delta\hat{\Pi}(p)|p_t\right] \geq \hat{E}\left[pd(p|A^t) + \delta\hat{\Pi}(p)|p_t\right].$$

The condition can be rewritten as follows,

$$\hat{E}\left[p^t d(p^t|A^t) + p\delta d(p|\mathcal{A}(p^t)) - pd(p|A^t)|p_t\right] \geq \delta(1-\delta)\hat{E}\left[\hat{\Pi}(p)|p_t\right]. \tag{6}$$

Its left-hand side immediately simplifies to

$$\hat{E}\left[p^t d(p^t|A^t) + p\delta d(p|\mathcal{A}(p^t)) - pd(p|A^t)|p_t\right] = \tag{7}$$
$$= \hat{E}[\underbrace{(p^t - p)d(p^t|A^t)}_{\text{(I) Discrimination Gains}} + \underbrace{p\left(d(p^t|A^t) + d(p|\mathcal{A}(p^t)) - d(p|A^t)\right)}_{\text{(II) Substitution Effect}} - \underbrace{(1-\delta)pd(p|\mathcal{A}(p^t))}_{\text{(III) Deferral Loss}}|p_t].$$

Clearly, for such a deviation not to be profitable when $d(p^t|A^t) = (0,0)$ it must be that $\hat{\Pi}(p^t) = 0$. So, suppose that $d(p^t|A^t) \neq (0,0)$. Recall that D_i^t denotes the set of buyers who purchase variety i at prices p^t, and let $S_i^t(p)$ denote those buyers who in D_i^t would purchase variety j at p,

$$S_i^t(p) = \left\{v \in D_i^t \mid v_j - p_j > \max\{v_i - p_i, \delta\hat{U}(p|v)\}\right\}.$$

If $d_i(p^t|A^t) > 0$, there exists an active buyer $v \in A_t$ who purchases variety i at p^t. For such a buyer it must be that

$$v_i - p_i^t \geq \delta\left(v_i - \hat{E}\left[p_i|p_t\right]\right),$$

or equivalently

$$(1-\delta)v_i \geq \hat{E}\left[p_i^t - \delta p_i | p_t\right] \geq \hat{E}\left[p_i^t - p_i | p_t\right]. \tag{8}$$

If $d_i(p^t|A^t) > 0$, let $\hat{v}_i$ be such a value. If instead $d_i(p^t|A^t) = 0$, let $\hat{v}_i = 0$. But if so, the first term in 7 is bounded above by

$$\hat{E}[(p^t - p)d(p^t|A^t)|p_t] \leq (1-\delta)\hat{v}d(p^t|A^t) \leq (1-\delta)2\eta. \tag{9}$$

Next, observe that the third term in 7 is obviously bounded above by 0. Finally, the second term in 7 is bounded above by

$$\begin{gathered}\hat{E}[p\left(d(p^t|A^t) + d(p|\mathcal{A}(p^t)) - d(p|A^t)\right)|p_t] = \\ = \hat{E}[(p_i - p_j)\left(d_i(p^t|A^t) + d_i(p|\mathcal{A}(p^t)) - d_i(p|A^t)\right)|p^t] \leq \hat{E}[\max_i \mathcal{F}\left(S_i^t(p)\right)|p^t],\end{gathered}$$

where the equality holds because stationarity implies that

$$\textstyle\sum_{i\in\{a,b\}}\left[d_i(p^t|A^t) + d_i(p|\mathcal{A}(p^t))\right] = \sum_{i\in\{a,b\}} d_i(p|A^t),$$

and where the inequality holds because $(p_i - p_j) \leq 1$ and because

$$\left|d_i(p^t|A^t) + d_i(p|\mathcal{A}(p^t)) - d_i(p|A^t)\right| = \max_i \mathcal{F}\left(S_i^t(p)\right).$$

Observe next that regularity implies that: (i) $\mathcal{F}\left(S_i^t(p)\right) \leq \overline{f}\mathcal{L}(S_i^t(p))$; (ii) $\mathcal{F}\left(D_i^t\right) \geq \underline{f}\mathcal{L}(D_i^t)$; and (iii) $\mathcal{L}(D_i^t) > 0$ whenever $\mathcal{F}\left(D_i^t\right) > 0$ (by absolute continuity of $\mathcal{F}$). Also, notice that

$$S_i^t(p) \subseteq \hat{S}_i^t(p) = \left\{v \in D_i^t \;\middle|\; p_i - p_j > v_i - v_j \geq p_i^t - p_j^t\right\}.$$

Suppose that $\max_i \mathcal{F}\left(S_i^t(p)\right) > 0$. Let i denote the variety satisfying $\mathcal{F}\left(S_i^t(p)\right) > \mathcal{F}(S_j^t(p)) = 0$. If so, we have that $\mathcal{F}\left(D_i^t\right) > 0$, that $p_i - p_j > p_i^t - p_j^t$, and that, for some $\gamma > 0$,

$$\frac{\mathcal{L}(S_i^t(p))}{\mathcal{L}(D_i^t)} \leq \frac{\mathcal{L}(\hat{S}_i^t(p))}{\mathcal{L}(D_i^t)} \leq \left(p_j^t - p_j - p_i^t + p_i\right)/\gamma.$$

The latter holds by absolute continuity as $\mathcal{L}(\hat{S}_i^t(p)) \leq \left(p_i - p_j - p_i^t + p_j^t\right)w$, while $\mathcal{L}(D_i^t) = \gamma w$ for some $\gamma > 0$, where w denotes the maximal width of D_i^t in direction $(1,1)$.[23] That follows since $\hat{S}_i^t(p)$ is a subset of D_i^t with height bounded by $\left(p_i - p_j - p_i^t + p_j^t\right)$. By collecting the last few observations, we obtain an upper-bound on the substitution effect in 7 as by

[23] That is, $w = \max_{(g,z)\in\mathbb{R}\times D_i^t} g$ subject to $z + (g,g) \in D_i^t$.

equation 8

$$\begin{aligned}\hat{E}[\max_i \mathcal{F}\left(S_i^t(p)\right)|p^t] &= (\overline{f}/\underline{f})/\gamma\hat{E}[\max_{i\in\{a,b\}}\{p_j^t - p_j - p_i^t + p_i\}\mathcal{F}\left(D_i^t\right)|p^t] \leq \\ &\leq (\overline{f}/\underline{f})/\gamma E\left[p_i^t - p_i + p_j^t - p_j|p^t\right]\left(\mathcal{F}\left(D_i^t\right) + \mathcal{F}(D_j^t)\right) \leq \\ &\leq (1-\delta)\,2\hat{\gamma}(d_i(p^t|A^t) + d_j(p^t|A^t)) \leq (1-\delta)\,4\hat{\gamma}\eta, \qquad (10)\end{aligned}$$

for $\hat{\gamma} = (\overline{f}/\underline{f})/\gamma$ since we have that $\mathcal{F}\left(D_i^t\right) \leq \mathcal{F}\left(D_i^t\right) + \mathcal{F}(D_j^t)$ and that

$$E\left[\max_{i\in\{a,b\}}\{p_j^t - p_j - p_i^t + p_i\}|p^t\right] \leq E\left[p_i^t - p_i + p_j^t - p_j|p^t\right],$$

as $p_i^t \geq p_i$ and $p_j^t \geq p_j$. Putting 6-10 together then yields

$$(2+4\hat{\gamma})\eta - E[pd(p|\mathcal{A}(p^t))|p^t] \geq \delta E[\hat{\Pi}(p)|p^t].$$

Thus, by choosing η sufficiently small, the expected WME profit of the monopolist $E[\hat{\Pi}(p)|p^t]$ must be arbitrarily small after any real time $\hat{T}$. This follows because η is small and because $\gamma(\eta) > 0$ for all $\eta \geq 0$, by absolute continuity and since $v \in D_i^t$ implies $(v_i, v_j') \in D_i^t$ for all $(v_i, v_j') \in A^t$ such that $v_j' < v_j$ by an argument equivalent to Lemma 3.

But, by Lemma 4 such profits exceed optimal MC profits. Thus, they can be small only if few players are expected to remain active in period $t+2$ by Proposition 1. But if almost all players purchase before time $\hat{T}$, prices must be close to market clearing by period $\hat{T}$ (as consumers never pay more than their value for a product). But, as the cost of delaying consumption for any real time $\hat{T}$ goes to zero, no consumer would purchase a variety until prices are close to market-clearing, as for δ sufficiently close to 1

$$\max_i\left\{v_i - p_i^t\right\} < \delta\max_i\left\{v_i - p_i\right\} \text{ for any } p^t > p.$$

Thus, all consumers purchase products only when prices prices are close to market-clearing, as the monopolist must almost clear the market after any real time. But if so, the monopolist profit must be close to a static market-clearing profit. ■

Proof Proposition 10. For clarity let $d_i^F(p)$ denote the static demand for product i when prices are p and the measure of buyers is F. Recall that for joint distribution F, optimal MC profit satisfy

$$\begin{aligned}\bar{\pi}^F &= d_a^F(\bar{p}^F)\bar{p}_a^F + (1 - d_a^F(\bar{p}^F))\bar{p}_b^F \\ &\geq d_a^F(p)p_a + (1 - d_a^F(p))p_b \text{ for any } p \in M(F),\end{aligned}$$

where $\bar{p}^F \in M^F$ denotes the optimal market-clearing price (which exists by regularity). We want to establish that $\bar{\pi}^F \leq \bar{\pi}^L$. To do so, it suffices to find prices $p \in M^L$ such that $p \geq \bar{p}^F$ and $d_a^F(\bar{p}^F) = d_a^L(p)$. For any market-clearing price $p \in M^F$ such that $p_a \geq p_b$, we have by regularity that

$$d_a^F(p) = \mathcal{F}^F\left(v_a - v_b \geq p_a - p_b\right).$$

Let V^L denote the support of the distribution L. Since V^L is a non-increasing set by remark 9, $v_a < p_a$ implies $v_b \geq p_b$, for any $v, p \in V^L$. Thus, $V^L \subseteq M^L$. Remark 9 also implies that for any $p \in V^L$

$$d_a^L(p) = \mathcal{F}^L\left(v_a - v_b \geq p_a - p_b\right) = \mathcal{F}^L\left(v_a - l(v_a) \geq p_a - l(p_a)\right) = \mathcal{F}^L\left(v_a \geq p_a\right) = 1 - F_a(p_a),$$

where the second equality holds as V^L is a non-increasing set, and the third as $l' \leq 0$. As marginal distributions are continuous by regularity, they admit an inverse. Thus, it is possible to find a price p_a satisfying

$$p_a = F_a^{-1}(1 - d_a^F(\bar{p}^F)).$$

To conclude, we establish that $p_a \geq \bar{p}_a^F$. By construction it must be that

$$\mathcal{F}^F\left(v_a - v_b \geq \bar{p}_a^F - \bar{p}_b^F\right) = 1 - F_a(p_a).$$

However, since $\bar{p}^F \in M^F$, it is also the case that

$$\mathcal{F}^F\left(v_a - v_b \geq \bar{p}_a^F - \bar{p}_b^F\right) \leq 1 - F_a(\bar{p}_a^F),$$

as $v_a - \bar{p}_a^F \geq v_b - \bar{p}_b^F$ implies $v_a - \bar{p}_a^F \geq 0$, for any $\bar{p}^F \in M^F$. The last two observation together imply that $F_a(p_a) \geq F_a(\bar{p}_a^F)$, and thus $p_a \geq \bar{p}_a^F$. Similar argument applies for variety b. ■

Proof Lemma 11. Consider partially discordant varieties with joint distribution F. Let $Y = \left\{v \in [0,1]^2 \,|\, \min_i v_i = 0\right\}$. First, fix any price $p \in V^F \cap Y$ if such a price exists. Without loss of generality assume that $p = (\overline{v}_a, 0)$. Such a price profile cannot be an optimal MC price as profits are 0 given that

$$d_a^F(p) = \mathcal{F}(v_a - v_b \geq \overline{v}_a) = 0,$$

where the last equality holds by regularity.

Next consider any other price $p \in [M^F \cap Y] \backslash V^F$. Again, without loss of generality, suppose that $p_a \geq p_b = 0$. We show that such price cannot be optimal. Observe that price profile $p' = (p_a + \varepsilon, \varepsilon)$, also, clears the market for $\varepsilon > 0$ sufficiently small. This follows as

$p \in Y \backslash V^F$ implies

$$\min_{v \in V^F \backslash (0,0)} \max_{i \in \{a,b\}} (v_i - p_i) > 0$$

which in turn implies that $p' \in M^F$ for ε sufficiently small. However, profits at p' are higher than at p. This follows as demand for the two varieties does not change,

$$d_a^F(p') = \mathcal{F}^F(v_a \geq p_a + v_b) = d_a^F(p),$$

given that the measure of indifferent consumers is small by regularity. But, if so, profits must increase as

$$p'_a d_a^F(p') + p'_b(1 - d_a^F(p')) > p_a d_a^F(p).$$

■