Search Direction: Position Externalities and Position Bias*

Simon P. Anderson† and Regis Renault‡

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Abstract

A tractable model of pricing under directed search is proposed where firms have *ex ante* heterogeneous demands. Equilibrium product prices are such that the marginal consumer's surplus decreases in the order of search. Consumers always find it optimal to follow the order of search that results from whatever allocation rule used to determine firms' positions. A firm suffers from a "business stealing" externality imparted by firms that precede it and a "search appeal" externality imparted by firms that follow. Optimal rankings that achieve the maximization of joint profit, social welfare or consumer surplus are characterized by means of firm specific scores. There is typically a conflict between total industry profit and consumer welfare maximization. In a generalized second price auction, situations where the equilibrium outcome is joint profit and/or consumer welfare maximizing ar characterized. .

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†University of Virginia and Center for Economic and Policy Research. Address: Department of Economics, University of Virginia, Charlottesville VA 22904, USA. sa9w@virginia.edu.

‡Université de Cergy-Pontoise, Thema, 33 boulevard du port, F95011 Cergy, France. regis.renault@u-cergy.fr.

1 Introduction

Internet search is sequential across options, and surfer search is directed by position placement of ads. Until now, most work on consumer search and firm pricing has involved random search across options (e.g. the work following Stahl's, 1989, mixed strategy model with homogenous goods, or the search for match following Wolinsky, 1986, and Anderson-Renault, 1999: this literature is surveyed along with further developments in Anderson and Renault, 2017). *Ordered* search is quite different, and the theory has only recently started to be developed: Armstrong (2016) provides an invaluable state of the art. Research has been stymied so far by lack of tractable frameworks that can accommodate heterogeneous firms, a key ingredient for the analysis of *directed* search, in which consumers follow a suggested ordering. We propose a tractable framework that reflects a directed search environment suitable for the internet environment as well as other applications and engages Weitzman's (1979) powerful results on search behavior. It enables us to study equilibrium with optimal consumer search, product pricing, and advertiser bidding for positions. It delivers a falling surplus for the marginal consumer in the order of search, with consumers (strictly) wanting to follow the pre set order of directed search. Pricing excessively curtails search. The socially optimal order, joint profit maximizing order and consumer surplus maximizing order may each be characterized by associating a score to each firm and ranking the firms according to that score. The setting involves two positional externalities. We characterize some situations where firm bidding leads nonetheless to joint profit maximization.

An important property of our model is the externality imposed by a firm's position on other firms' profits. This effect is only partially incorporated if at all in the literature on position auctions. In our context, this also means that a firm's willingness-to-pay for a slot depends on which firm is demoted and the distribution of tastes for its product. To see these effects, note that if a firm in the ith position becomes more attractive to search, then the prices and profits of firms before it are reduced. Previous literature accounted for the negative externality from a firm selling a popular product and being searched early on firms that follow (Chen and He, 2012, and Athey and Ellison, 2012). By accounting for firm pricing we introduce an additional externality imparted by firms that are searched later, which is determined by those firm's search attractiveness. Prior studies involving endogenous prices only captured this externality with *ex ante* identical firms (Armstrong, Vickers, and Zhou, 2009, Zhou, 2011) so the externality only depends on the number if firms remaining to be searched, or in a duopoly with a very specific source of heterogeneity (Song, 2013). Yet, one key to a broader understanding of the link between equilibrium search and positions is to look at asymmetries in the other variables of the model. Fortunately, our setting is populated with several parameters that play differently and can be distributed across firms. These we break out in the model. We derive optimal ranking scores in terms of these parameters. Whereas the scores that characterize the joint profit maximizing order and the social welfare maximizing order are qualitatively similar, they both differ strongly from the scores associated with consumer surplus maximization. This is another sharp difference with the analysis in previoius work on position auctions, which assumes exogenous prices.

Consider now a position auction for slots, with slots going to firms in the order of their bids, and firms paying the bid of the next highest bidder. If the source of product heterogeneity is some measure of quality, which is private value to each firm, then there is no conflict between joint profit and consumer surplus maximization. There is then an equilibrium that implements the optimum. Furthermore, as in Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007), this equilibrium satisfies the "no-envy" refinement introduced by these authors. This means that no firm would like the position of another if it had to pay the price the other is paying for its slot. The equilibrium allocation is then the one that would be implemented by a VCG mechanism.

In more general specifications of our framework, it is still true that the "no envy" requirement ensures joint profit maximization. The corresponding ranking may however not be the consumers' preferred one. In particular, if the common value dimension of heterogeneity is the difference in "search appeal" across products, then we show that, under fairly general conditions, there exists an envy free equilibrium, which typically does not yield the ranking that maximizes consumer surplus. If, by contrast, the common value demand property is the "popularity" of the product, then no "envy free" equilibrium exists. In contrast, we provide some conditions under which there is an equilibrium that implements the ranking that is most favorable to consumers: this ranking has more popular products searched earlier, whereas joint profit maximization would prescribe the opposite. Chen and He (2011) and Athey and Ellison (2011) obtain the same ranking in equilibrium, though in their setting with exogenous prices, it maximizes both joint profit and consumer surplus.

There are two relevant streams of literature. Sequential ordered search has only recently been broached. A step forward is made by Armstrong, Vickers, and Zhou (2009),[1] who show that a firm which is searched first will earn more profit, and will also be more attractive to consumers to search first because it charges a lower price. However, their model has a single "prominent" firm searched first, and then the remaining firms are searched at random (without order). Moreover, they assume an independent and identical distributions of consumer tastes for the various products, which limits introducing heterogeneity in the distribution of tastes across different products. Zhou (2011) addresses some of these concerns with an ordered search model, again with symmetric firms. Song (2013) considers firms that are asymmetric regarding taste heterogeneity, but only looks at the duopoly case. The latter two papers find that earlier firms charge lower prices, which is consistent with our result that the marginal consumer's surplus goes down with firm searched later. Finally, Chen and He (2011) introduce some heterogeneity in the probability that a product is suitable for a consumer, although their model delivers monopoly pricing: hence there is no externality through prices.

The position auctions literature has made valuable progress on the auction side of the slate while suppressing the market competition side. Athey and Ellison (2011) use a setting very similar to that of Chen and He (2011) to look at auctions with asymmetric information and then optimal auction design, while assuming that consumers go on searching until a "need" is fulfilled, so they do not allow for competing products on the market-place. Their setting allows for a position externality through demand which depends on how likely are

[1] See Arbatskaya (2007) for an earlier contribution with homogenous products.

previous products in the queue to fill a consumer's need. Our setting does allow for this type of externality as well as the pricing externalities described above. Furthermore, this paper, as well as Chen and He (2011), establish that it is optimal for consumers to search in the order that emerges from the auction because firms with a higher probability to meet a consumer's need bid more. By contrast, we find that it is optimal for consumers to search in the pre specified order because of the pricing behavior they expect from firms. hence, our result holds independently of the characterization of the auction's outcome. Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007) have no position externalities between firms, and they do not engage the broader consumer search and pricing either.

Section 2 describes our search and competition environment while optimal ranking scores for maximization of total industry profit, social welfare and consumer surplus are derived in section 3. Finally we consider how allocation rules such as auctions used on internet platforms might achieve total profit maximization in Section 5.

2 Market equilibrium

2.1 Competition with ordered search

We first describe a model of oligopolistic competition with ordered consumer search, and find firms' equilibrium prices. There are n firms with firm i selling product i, with zero production costs. Consumers have unit demand with independent valuations for the n competing products. Let $F_i(v)$ denote the distribution function of a consumer's valuation with product i, $i = 1, ..., n$. For the sequel, we find it useful to break down $F_i(v)$ into three component parts.

We are thinking of situations in which consumers idiosyncratically either like the product, or they do not, but they have heterogeneous valuations if they like it. For example, a consumer may reject out-of-hand several styles of jacket, but the lowest valuation for a jacket that she will countenance taking home to take up space in the closet is quite high. Nonetheless, there may be several jackets that could interest her if she knows their details. This set-up contrasts with Athey and Ellison (2012) and Chen and He (2012), in which each consumer has at most one product that could interest her, regardless of prices.

Let then the probability of rejecting the product outright (regardless of price) be γ_i. Then lower γ_i products are more popular, per se. Second, let q_i be the lowest valuation associated to product i, conditional on it being desired. As we elaborate below, we shall assume that q_i is sufficiently large that all consumers who have some appreciation for a product end up buying it in equilibrium when they come across it. Lastly, let $\bar{F}_i(\delta)$ be the distribution function for product i of valuations net of q_i for strictly positive valuations. That is, we decompose the valuation $v_i > 0$ into two component parts, so $v_i = q_i + \delta_i$. Let the support of δ_i be S_i, with $\min S_i = 0$ and $\max S_i + \bar{\delta}$.[2]

With this break-down, we can write the distribution function of valuations for product i as

$$\begin{aligned} F_i(0) &= \gamma_i \\ F_i(v_i) &= \gamma_i + (1-\gamma_i)\bar{F}_i(v_i - q_i), \quad v_i \geq q_i, \end{aligned}$$

which shows the atom of size γ_i on a zero valuation, and then a gap of zero density up to q_i. Henceforth in this Section we will work with the F_i, and return to the decomposition when we come to describing summary statistics for welfare ranking measures.

Distribution functions F_i are common knowledge but neither consumers nor firms know the realizations of v_i. Consumers may however learn these realizations through search. Search is sequential, with search cost $s > 0$ per (additional) search. Searching a firm reveals both its actual price and the consumer's valuation of the product searched, and searching a firm is necessary for a consumer to be able to purchase its product. As is standard in sequential search settings the consumer may always purchase from any previously searched firm with no additional search cost. If a consumer ends up buying none of the n products she has an outside value utility of zero.[3]

The timing is that firms simultaneously choose prices and consumers choose their search

[2] Hence the support of valuations, v_i, is $\{0\} \cup T_i$ where $\min T_i = q_i$ and $\max T_i = \bar{v}_i \equiv q_i + \bar{\delta}_i$.

[3] It is easy to allow for a positive continuation value (searching the organic links of a search engine after searching the sponsored links, or purchasing a product off line) for consumers who have searched through all the n firms: equilibrium prices are simply scaled down by the continuation value.

rules based on match values and prices they have found out so far and the distributions at other firms and the prices expected there. We seek conditions for a Perfect Bayesian Equilibrium at which search is ordered, meaning that all consumers follow the same search order. Because, we have not specified any systematic difference between the n firms (i.e. the match distribution for firm $i = 1, ..., n$ can be any distribution satisfying the properties described above), there is no loss of generality in assuming that this order if from firm 1 to firm n and we then check that this order is indeed optimal for consumers. Furthermore, in the equilibrium we consider, firms optimally price in a manner such that each firm retains all consumers that reach it and have a positive draw for its product.

The latter pricing property means that in equilibrium, firm i renders indifferent any consumer drawing q_i with firm i so that such a consumer does not want to search further. Thus a consumer has zero willingness to pay for any product encountered before product i, and never goes back (as long as prices are strictly positive, which will hold true in equilibrium). We show below that such an equilibrium exists, provided that q_i is large enough for all i and the distribution functions $\bar{F}_i$ satisfy a limit condition as δ_i tends to zero.

To derive the equilibrium, we engage the powerful results of Weitzman (1979) to describe optimal consumer search. He shows that remaining search options can be ordered by simple myopic reservation values such that a consumer searches the option with the highest reservation value next, or else stops searching if she already holds a utility above the highest value (and buys the best option held or buys nothing). These reservation values are summary statistics for the various options, which set the expected costs and benefits of an additional search equal, and they are therefore determined independent of what has already been discovered.

A version of these reservation values is a key ingredient of our analysis of pricing below. For each product we define Δ_i as the unique solution to

$$\int_{q_i+\Delta_i}^{\bar{v}_i} (v - q_i - \Delta_i) dF_i(v) = s. \tag{1}$$

Standard search analysis shows that the left hand side strictly increases from 0 to $+\infty$ as Δ_i drops from $\bar{v}_i - q_i$ to $-\infty$. Hence a unique Δ_i always exists. Graphically, as per the first equation in footnote 2 and Figure 1 below, the value of Δ_i is determined via the value of the critical valuation $q_i + \Delta_i$ for which the area under the demand curve $(1 - F_i(v))$ equals the search cost s.[4] That is, if the consumer currently held a utility value of $q_i + \Delta_i$ then searching firm i would be a break-even prospect if it were expected to charge a zero price. With this interpretation, the values $q_i + \Delta_i$, $i = 1, ..., n$, are the reservation values that characterize consumer optimal search behavior if prices are all zero: from the analysis in Weitzman (1979), a consumer should always select to search next the remaining alternative with the highest reservation utility $q_i + \Delta_i$ or else stop searching if she already holds a higher utility. If prices were equal for all products, consumers would choose to search in the order in which we have indexed the firms only if $q_i + \Delta_i \geq q_{i+1} + \Delta_{i+1}$, $i = 1, ..., n-1$.

In the sequel we assume $\Delta_i > 0$ (and we discuss below what happens when $\Delta_i < 0$). This assumption implies that, if a consumer holds valuation q_i with product i and contemplates searching firm $i + 1$, and if price differences were to exactly reflect base quality differences, (i.e., if $p_i - p_{i+1} = q_i - q_{i+1}$), then her expected benefit from searching firm $i + 1$ would be strictly positive (as shown in Figure 1).

[Insert Figure 1.]

In our upcoming analysis it is critical that we can vary parameters γ_i, q_i, and Δ_i independently from each other. To show that this can be done in our setting, it is useful to rewrite (1) using integration by parts and then using $F_i(v) = \gamma_i + (1 - \gamma_i)\bar{F}_i(v - q_i)$ for $v \geq q_i$ as

$$(1 - \gamma_i) \int_{\Delta_i}^{\bar{\delta}} \left(1 - \bar{F}_i(\delta)\right) d\delta = s. \tag{2}$$

The left-hand side can be made arbitrarily close to $(1 - \gamma_i)(\bar{v}_i - q_i - \Delta_i)$ by moving all the weight of the distribution of δ_i in the neighborhood of $\bar{\delta}_i$. This can be made strictly larger than s by taking $\bar{\delta}_i$ large enough. It is then possible to find some specification of $\bar{F}_i$ such that Δ_i satisfies (**??**) for any value of γ_i. Hence it is possible to adjust $\bar{F}_i$ so as to obtain any desired value of Δ_i for any γ_i. Obviously, q_i can be adjusted to any level without affecting γ_i or Δ_i by keeping $\bar{F}_i$ unchanged.

[4]This is because, using integration by parts, (1) can be equivalently written $\int_{q_i+\Delta_i}^{\bar{v}_i} (v - q_i - \Delta_i) dF(v) = s$.

We now move to a characterization of the equilibrium pricing and search order.

2.2 Pricing

Consider a firm $i < n$, and suppose it prices so that even a consumer who draws a match q_i with its product chooses not to search on. Because, in equilibrium, consumers follow an optimal search order, they compare the utility they currently hold with the highest reservation value among the remaining firms. In an equilibrium where consumers search in order from 1 to n, this highest reservation value should be that for firm $i+1$. In other words, as per the Weitzman (1979) analysis, the optimal search rule is myopic and only considers the costs and benefits of searching firm $i+1$, as if it were the only firm remaining. Because consumers expect utility $v_{i+1} - p_{i+1}$ with firm $i+1$, the reservation utility, r_{i+1}, associated with firm $i+1$ is the unique solution to

$$\int_{r_{i+1}+p_{i+1}}^{\bar{v}_{i+1}} (v - p_{i+1} - r_{i+1}) dF_{i+1}(v) = s. \tag{3}$$

It is immediate from comparing the above condition to (1) that we can write the reservation valuation as $r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. From this observation, we conclude that the largest price, p_i, that firm $i = 1, ..., n-1$ can charge such that a consumer with match q_i will decline to search firm $i+1$ satisfies $q_i - p_i = r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. This equality determines the candidate equilibrium pricing rule as

$$p_i = p_{i+1} + q_i - q_{i+1} - \Delta_{i+1}, \tag{4}$$

which therefore determines a recursive relation. We now need to find a starting condition, which is the price set by firm n.

So consider now firm n's problem. It knows it is the last to be searched and that all consumers who get to it in equilibrium have zero valuation for all the other products. It is therefore in a monopoly position. As we do for the other firms, we seek an equilibrium price such that all consumers with valuations of at least q_n buy firm n's product. The largest price n can charge which is consistent with all such consumers buying is $p_n = q_n$.[5] By applying

[5] If there were a fixed continuation value, V_n, then the price would be simply $p_n = q_n - V_n$.

the recursive price relation (4) it follows by induction that equilibrium pricing is

$$p_i = q_i - \sum_{j=i+1}^{n} \Delta_j, \quad i = 1, ..., n. \tag{5}$$

We will establish below that this pricing sequence induces consumer ordered search in the specified order. We now establish that the pricing behavior described above is indeed profit maximizing as long as q_i is sufficiently large, and under a mild condition on the conditional distribution functions. In order to state and prove the next Lemma, it is useful to introduce the following notation: for an increasing function g defined on the set of real numbers and a real number a, define $g^-(a) = \lim_{x \to a, x<a} g(x)$.

Lemma 1 *Assume* $\lim_{\delta \to 0} \frac{\bar{F}_i^-(\delta)}{\delta} = \lambda > 0$ *and* $\bar{v}_i - q_i$ *is finite,* $i = 1, ..., n$*. Then it is optimal for firm* i *to charge price* p_i *defined by (5) as long as* q_i *is sufficiently large,* $i = 1, ..., n$*.*

Proof. If firm i charges its candidate equilibrium price p_i, it earns profit $(1 - \gamma_i)p_i$. At this price, it sells to all consumers with strictly positive willingness to pay for its product who have reached it. Hence, it cannot gain additional profit by charging a lower price. Assume therefore that it were to charge a price that is $\Delta p > 0$ in excess of p_i. Its corresponding profit is then at most

$$(p_i + \Delta p)(1 - \gamma_i)\left(1 - \beta_{i+1}^- \bar{F}_i^-(\Delta p)\right). \tag{6}$$

This upper bound on deviation profit is obtained as follows. First, if firm i deviates to $p_i + \Delta p$, then all consumers with valuations less than Δp in excess of q_i search firm $i+1$ (recall that at price p_i consumers holding match q_i with firm i are just indifferent between buying product i and searching on). Hence, the probability that a consumer chooses to search firm $i+1$ is $\bar{F}_i^-(\Delta p)$. Second, we assume that all consumers who do not prefer buying product $i+1$ to product i return to firm i and buy its product. Consumers who weakly prefer buying product $i+1$ are those with valuations $v_{i+1} \geq q_{i+1} + \Delta_{i+1}$ and the associated probability is $\beta_{i+1}^- \equiv 1 - F_{i+1}^-(q_{i+1} + \Delta_{i+1})$. Hence the gain from a deviation to $p_i + \Delta p$ is at most

$$(1 - \gamma_i)\left[\Delta p \left(1 - \beta_{i+1}^- \bar{F}_i^-(\Delta p)\right) - p_i \beta_{i+1}^- \bar{F}_i^-(\Delta p)\right]. \tag{7}$$

First consider a small deviation with Δp close to zero. Because $\lim_{\delta\to 0}\frac{\bar{F}_i^-(\delta)}{\delta}=\lambda>0$, there exists $\bar{\delta}$ such that if $\Delta p<\bar{\delta}$, then $\bar{F}_i^-(\Delta p)>\frac{\lambda}{2}\Delta p$. The benefit from a deviation is bounded above by

$$(1-\gamma_i)\Delta p\left[1-\beta_{i+1}^-\bar{F}_i^-(\Delta p)-\frac{\lambda}{2}p_i\beta_{i+1}^-\right], \tag{8}$$

which is negative if p_i is large enough.

Now consider a large deviation, $\Delta p>\bar{\delta}$. If follows that $\bar{F}_i^-(\Delta p)\geq\bar{F}_i^-(\bar{\delta})>0$. An upper bound for the deviation gain is then

$$(1-\gamma_i)\left[\left(\bar{v}_i-q_i+\sum_{j>i}\Delta_j\right)\left(1-\beta_{i+1}^-\bar{F}_i^-(\Delta p)\right)-p_i\beta_{i+1}^-\bar{F}_i^-(\bar{\delta})\right]. \tag{9}$$

Again, this is negative for p_i large enough. Hence firms do not deviate from their candidate equilibrium price as long as it is large enough. From the pricing expression (5), it is immediate that prices can be made as large as required by taking q_i large enough. ■

Lemma 1 establishes that firms do not wish to deviate from the candidate equilibrium prices. In order to establish that this is an equilibrium, we merely need to verify that the specified search order is optimal for consumers, that is, $r_i\geq r_{i+1}$ for $i=1,...,n-1$. Recall that $r_i=q_i+\Delta_i-p_i$ which, from the pricing expression (5), implies $r_i=\sum_{j\geq i}\Delta_j$. So r_i indeed monotonically decreases as i increases. We summarize with the following Proposition.

Proposition 1 *Under the assumptions of Lemma 1, there exists an equilibrium where consumers search firms in the order of the firm index, $i=1,...,n$ and firm i charges a price given by (4).*

The pricing sequence in (5) bears the hallmark property that firms that are searched early on extract less surplus from consumers than firms that are searched later. Specifically, for any draw δ_i of net valuation for a product i, the consumer's surplus is higher if a firm higher in the order is searched earlier. This is a generalization of the finding in previous studies involving products with symmetric match distributions (Armstrong *et al.*, 2009, and Zhou, 2011) that firms that come first in the order charge lower prices.[6]

[6]This result is obtained with a uniform distribution of prices. Other articles also find this result in contexts with a very specific form of asymmetry: merged and not merged products in Moraga-Gonzalez and Petrikaite (2012), or products with different degrees of heterogeneity in matches in Song (2013).

The equilibrium price has two components: a private value measured by q_i and an externality from the remaining firms in the search order measured by $\sum_{j>i} \Delta_j$. The private value is a surplus associated with the consumption of the firm's product, for which it captures any additional dollar through its price. The firm cannot however capture the entirety of this surplus because of the downward pressure on its price resulting from the option consumers have to search on to the next firms down the line. The amount by which price is lowered may be interpreted as the total search appeal associated with the remaining products to be checked out by the consumer. Because of the "myopic" search rule used by the consumer, only the search appeal of the next product down, Δ_{i+1}, is directly relevant for firm i's pricing. However, firm i must also take into account the pricing behavior of firm $i+1$, which is dependent on the search appeal of firm $i+2$. This is why, in the end the total search appeal externality imparted on a firm takes into account the cumulative search appeal of all the firms remaining. The search appeal externality also arises in the duopoly setting with uniform match distributions of Song (2013) where this externality is all the more substantial that the next products match distribution is more dispersed in the sense of second order stochastic dominance.

In the benchmark case where firms would all have identical product match distributions so that, $\Delta_i = \Delta$ for all $i = 1, ..., n$, the search appeal externality would merely be $(n-i)\Delta$ for firm i, which only depends on the number of firms following firm i. In our setting where products are *ex ante* heterogenous, the externality also depends on the identity of the remaining firms. This property is key to the welfare analysis of the optimal ordering of firms for total profit or consumer surplus. It also has important implications for the firms' willingness to pay to be searched earlier rather than later.

The search appeal externality highlights the competition that a firm faces from the following firms. By contrast, a firm does not directly compete in price with the preceding firms. This is because the search behavior of consumers only takes into account the expected price at remaining firms. Hence, a firm has no way of stealing customers from its predecessors by committing to a lower price than expected. However, preceding firms do affect firm i's profit because they price in such a way that all consumers who have a positive valuation with at least one of those firms stop searching before getting to firm i. As a result, only a fraction $\Pi_{j<i}\gamma_j$ reach firm i. This constitutes a business stealing externality that has previously been accounted for by Chen and He (2012) and Athey and Ellison (2012) in settings where prices are exogenous. Our analysis introduces novel insights regarding the interaction between business stealing and pricing, as illustrated in section 3. As for the search appeal externality, if firms were *ex ante* symmetric, with $\gamma_i = \gamma$ for all i, market stealing for firm i would only depend on the number of predecessors and the fraction of consumers reaching firm i would merely be γ^{i-1}. Again, the identity of firms that are searched prior to firm i become relevant, once match distribution are allowed to differ across products.

Finally, a critical property of the equilibrium in Proposition 1 is that pricing induces too little search on the part of consumers as compared to the social optimum. The relevant search problem for social welfare maximization is to achieve the best possible gross valuation net of search costs. This is equivalent to the consumer's search problem if all prices are zero. We have defined Δ_i so that the reservation value associated with searching firm i is $q_i + \Delta_i$. Hence a consumer at firm i should select to search firm $i+1$ if and only if her match is less than $q_{i+1} + \Delta_{i+1}$. In equilibrium, the consumer searches firm $i+1$ if and only if her match is zero. If $q_i \geq q_{i+1} + \Delta_{i+1}$, the behavior of the consumer is optimal. However, in the reverse case, then there is not enough search in equilibrium because a consumer holding q_i at firm i should search but does not do so because of firm i's pricing. Also note that the socially optimal search order has consumers searching in the decreasing order of the reservation values $q_i + \Delta_i$, where as Proposition 1 shows that for any ordering of the n firms, there is an equilibrium where consumers follow that order and firms price accordingly.

3 Optimal rankings

The results of the previous section indicate that ANY order of search can be sustained as an equilibrium to the game in which consumers follow their optimal search protocol and firms set their prices. Prices though differ across these equilibrium search orders when firms are asymmetric, and so the search order matters to various measures of market performance. Typically, the optimal order varies by market performance measure. We here determine the optimal orders, given equilibrium search and pricing, for total industry profit, social welfare, and consumer surplus. For short, call these TIP, W, and CS respectively.

A priori, this is a complicated problem because position order affects all prices and search probabilities: with n active firms there are $n!$ configurations to compare. Nevertheless, the structure of our model delivers a simple and clean characterization for the optimal order under each criterion. The optimal order is described by ordering firms according to a simple summary statistic, which is different for each surplus criterion.

The idea is as follows. For any neighboring pair of firms, A and B, in slots i and $i+1$ respectively, (and for each surplus criterion), we can find a summary statistic Φ_k for firm k such that the maximand (CS, W, or TIP) evaluated in these two slots is higher if $\Phi_A \geq \Phi_B$. Crucially, the summary statistics are derived solely from parameters of the match distribution of the corresponding product, F_A for Φ_A and F_B for Φ_B. Hence they do not depend on which two slots are flipped (e.g., first and second or fifteenth and sixteenth). The key property of our model is that such a flip only affects the welfare objective through the joint impact in the two consecutive slots: the welfare in all the other slots only depends on the joint externality that the two firms exert, either because they are in front (the business stealing externality) or because they come later in the search order (the search appeal externality that affects prices in those earlier slots). Thus, with $\phi_A \geq \phi_B$, A being in front of B (rather than the reverse) yields a higher welfare criterion computed over all the n slots. Clearly, a necessary condition for a maximum is that flipping the order of the two firms in each successive pair does not strictly increase the desired objective function. Because the flipping rule is independent of the positions i and $i+1$ to be flipped, this criterion induces an ordering of firms based on the indices Φ_k as claimed above. Put another way, any alternative order, with at least one pair of consecutive firms violating the pairwise flip condition, cannot be an optimum. Thus the ranking of firms by the size of their summary statistics is a necessary condition for optimality. It is also sufficient because, if there are no ties among firms in the sufficient statistics ϕ_k, there is only one such order out of a finite set of possible configurations, and if there are ties, flipping two consecutive firms that are tied leaves the objective unchanged so that the multiple solutions obtained by ranking according to ϕ_k are all optimal.

We now derive the specific summary statistics for the different criteria. We also give the intuition for the various orders.

3.1 Total Industry Profit

The profit for the firm in position i is

$$\pi_i = (q_i - \kappa_i)\,\lambda_i\,(1-\gamma_i)\,, \quad i = 1, ...n, \tag{10}$$

where we have defined $\kappa_i = \Sigma_{j>i}\Delta_j$ as the sum of all later price steps (where κ_n is taken to be zero), and $\lambda_i = \Pi_{j<i}\gamma_j$ for $i > 1$ as the as the probability that a consumer has no interest in any of the previous products (and we let $\lambda_1 = 1$). The term in the first parenthesis in (10) is the equilibrium price (5) and it is multiplied by the probability that the consumer ends up searching firm i, λ_i, and then buying product i, $1-\gamma_i$.

As explained above, we just need to look at the change in profit from switching firms A and B between slots i and $i+1$. Thus we have A before B as long as

$$\pi_A^i + \pi_B^{i+1} \geq \pi_B^i + \pi_A^{i+1}, \tag{11}$$

where π_k^i denotes the profit of firm k when it is in slot i. For our model, we can write this out to yield:

$$(1-\gamma_A)\,(1-\gamma_B)\,(q_A - q_B) + (1-\gamma_B)\,\Delta_A - (1-\gamma_A)\,\Delta_B > 0. \tag{12}$$

To derive this, first notice that we can divide through by the total number of consumers who search up to slot i, i.e., λ_i, and then the terms in all prices after $i+1$ (i.e., κ_{i+1}) cancel out. Importantly, the condition is independent of the position in the overall order of the two slots that are switched.

Dividing through (12) by $(1-\gamma_A)\,(1-\gamma_B)$ delivers the TIP summary statistics such that A should be before B (in any consecutive pair, and hence in the global maximum) as long as

$$\Phi_A^\pi \equiv q_A + \frac{1}{(1-\gamma_A)}\Delta_A > q_B + \frac{1}{(1-\gamma_B)}\Delta_B \equiv \Phi_B^\pi.$$

The TIP summary statistic is readily apparent from this inequality, and is given next:

Proposition 2 *An order of firms maximizes Total Industry Profit if and only if it follows the ranking of the summary statistic*

$$\Phi_k^\pi \equiv q_k + \frac{1}{(1-\gamma_k)}\Delta_k \tag{13}$$

and firms should follow a decreasing order of the Φ_k^π. *Ceteris paribus, higher* q_k, Δ_k, *and* γ_k *should go earlier in the order.*

To understand this result, recall that, in equilibrium, firms that are early in the search order sell more but extract less consumer surplus (they have deeper quality discounts), whereas firms that come later sell less but extract more consumer surplus because their prices are closer to their qualities. TIP maximization is achieved by ensuring that firms that sell more extract as much surplus as possible and firms that extract the most surplus sell as much as possible. The first objective is achieved by having firms with a large quality q_k and a large search appeal Δ_k searched early. Having firms with least popular products (low γ_k) searched first serves the second goal.

One way to see these effects clearly in isolation is by looking at each as the sole source of heterogeneity (so the other parameters are set the same for all firms). A large quality ensures that there is much potential consumer surplus to be extracted by early firms, which should therefore have the most consumers sampling them. Notice that the quality effect is NOT an externality on the other firms. The other two parameters imbue externalities of opposite impact on other firms.

High Δ firms cause low prices on all those which precede them. Switching a high-Δ firm with a low-Δ one that was initially earlier, raises the prices for all the firms in between the two slots, and so raises total profits. The idea of stacking up early all the high-Δ firms is to "clear-the-decks" of them to suppress their shadow on all prices that come earlier, which they would otherwise bring down. Put another way, having the firms that are most appealing to search early mitigates the search appeal externality imparted by these firms: they can keep their own prices relatively high because consumers are not too eager to search the remaining firms.

Finally, it may seem surprising that firms with less popular products (large γ_k) should be presented first to consumers because these firms are less likely to make a sale. However, early slots have low prices, so the ranking uses up these slots on less likely prospects.[7] Firms that extract the most surplus from their customers have larger sales because the business stealing externality from earlier firms is limited. Both this feature and the search appeal externality already suggest that consumer surplus may run the opposite way from TIP, a property that is confirmed in broad-brush terms, modulo the differences in the exact order statistics, in the analysis below.

3.2 Social Welfare

We next consider the pairwise ranking condition for Welfare (given equilibrium firm pricing). First note that, because pricing ensures that all consumers who are interested in any of the products buy one of them, prices are just a straight transfer between firms and consumers, and so do not enter the calculus. We divide up the problem into the surplus generated on each product.

Conditional on reaching firm k, the expected social welfare when a consumer searches firm k may be written as

$$-s + (1 - \gamma_k)q_k + \int_{q_k}^{\bar{v}_k} (v - q_k)\, dF_k(v). \tag{14}$$

Substituting in the definition of Δ_k in equation (1) to cancel out s, the above expression may be rewritten

$$(1 - \gamma_k)q_k + [1 - F_k^-(q_k + \Delta_k)]\Delta_k + \int_{q_k}^{q_k+\Delta_k} (v - q_k)\, dF_k(v). \tag{15}$$

[7] For example, suppose there were two firms, and $\gamma_A = 0.1$ while $\gamma_B = 0.9$. Then the number of consumers who buy constitute 91 percentage points, regardless of the order of search. Having A first entails 89% buying at the high price, while B first means only 1% do.

Now define $\beta_k = 1 - F_k^-(q_k + \Delta_k)$ and $\alpha_k = \int_{q_k}^{q_k+\Delta_k} (v - q_k)\, dF_k(v)$, so that this expected social welfare may be written as

$$(1 - \gamma_k)q_k + \beta_k\Delta_k + \alpha_k. \tag{16}$$

Consider now the surplus on searching A then B (conditional on having reached A at some position i). Then, comparing it with the converse while using the analogous expression (switching subscripts) for the opposite order yields the condition for the sequence AB (for any consecutive pair) to be profitable than BA as:

$$\begin{aligned} & q_A\,(1-\gamma_A) + \beta_A\Delta_A + \alpha_A + \gamma_A\,(q_B\,(1-\gamma_B) + \beta_B\Delta_B + \alpha_B) \\ \geq\ & q_B\,(1-\gamma_B) + \beta_B\Delta_B + \alpha_B + \gamma_B\,(q_A\,(1-\gamma_A) + \beta_A\Delta_A) + \alpha_A, \end{aligned}$$

and hence

$$\Phi_A^W \equiv q_A + \frac{\beta_A\Delta_A + \alpha_A}{(1-\gamma_A)} \geq q_B + \frac{\beta_B\Delta_B + \alpha_B}{(1-\gamma_B)} \equiv \Phi_B^W.$$

The summary statistic is thus the one given next:

Proposition 3 *An order of firms maximizes Social Welfare if and only if it follows the ranking of the summary statistics*

$$\Phi_k^W \equiv q_k + \frac{\beta_k\Delta_k + \alpha_k}{(1-\gamma_k)} \tag{17}$$

and firms should follow a decreasing order of the Φ_k^W*. Ceteris paribus, higher* q_k*,* Δ_k*,* γ_k*, should go earlier in the order.*

As can be seen by comparing (17) with (13) the criterion for determining the search order that maximizes social welfare is remarkably similar to that which we derived to optimize total industry profit: larger values of q_i, Δ_i and γ_i should come first. If all else is the same, the two criteria are perfectly aligned regarding base qualities q_i. This is easily understood: any additional dollar of surplus induced by an increase in q_i is entirely captured by firm i as can be seen from the equilibrium pricing expression (5).

The fact that both criteria call for having products with large values of Δ_k searched early is somewhat misleading, because the underlying economic reason is quite different. As was explained above, the rationale for having products with large search appeal early in the order when maximizing TIP is to mitigate the search appeal externality imparted on the first firms. By contrast, when considering social welfare, Δ_k is relevant to the extent that it enters into the measure of the total surplus generated in excess of the base quality q_k, for which the expression is $\beta_k \Delta_k + \alpha_k$. Generally, social welfare maximization puts less weight on Δ_k than TIP maximization. There can even be an extreme configuration where Δ_k is essentially irrelevant to social welfare: this happens when there is only a small probability that the match with product k exceeds $q_k + \Delta_k$ so that β_k is nearly zero (and $\bar{\delta}_k$ is then large so that a consumer can draw very large matches and (1) is satisfied).[8]

Finally the requirement that less popular products (low values of γ_k) should come first to achieve maximum social welfare may seem even more surprising than the analogous result for TIP maximization. Indeed, if the probability that consumers are not at all interested in the first products they encounter, they will keep searching longer, which seems wasteful. However, as we noted in the previous Section, consumers actually do not search enough in equilibrium. They stop searching as soon as they have a strictly positive valuation with a product whereas the social optimum would have them search as long as their valuation is below $q_k + \Delta_k$. This inefficiency is more severe when consumers draw a strictly positive match early on.

3.3 Consumer Surplus

The consumer surplus case proceeds analogously to the welfare one, except now prices feature explicitly. A second key difference is that the q_k's do not enter because they are priced out.

Letting p_k^i be firm k's price when in slot i, consumer surplus with product k in slot i is

$$-s + (1 - \gamma_k)(q_k - p_k^i) + \int_{q_k}^{\bar{v}_k} (v - q_k) dF_k(v), \tag{18}$$

[8] Also note that α_k could be anywhere between 0 and arbitrarily close to $\Delta_k(1 - \beta_k - \gamma_k)$: the former happens when q_k is drawn with probability $1 - \beta_k - \gamma_k$ and the latter when all the weight of the distribution of matches between q_k and $q_k + \Delta_k$ is concentrated just below $q_k + \Delta_k$.

The pricing rule (5) gives $p_k^i = q_k - \Delta_{i+1} - \kappa_{i+1}$ and $p_k^{i+1} = q_k - \kappa_{i+1}$ where $\kappa_{i+1} = \Sigma_{j>i+1}\Delta_j$ denotes the sum of later price steps. Using the search identity (1) and the pricing equation to substitute out the search cost and the price, consumer surplus can be expressed in a manner similar to the social welfare expression (16), while defining β_k and α_k in the same way so we have

$$(1-\gamma_k)\left(\Delta_{i+1} + \kappa_{i+1}\right) + \beta_k \Delta_k + \alpha_k, \tag{19}$$

with firm k in slot i and

$$(1-\gamma_k)\kappa_{i+1} + \beta_k \Delta_k + \alpha_k, \tag{20}$$

with firm k in slot $i+1$.

Hence the consumer surplus associated with AB exceeds that of BA (which is found by transposing subscripts again) if

$$(1-\gamma_A)\,\Delta_B + \beta_A\Delta_A + \alpha_A + \gamma_A\left(\beta_B\Delta_B + \alpha_B\right) \geq (1-\gamma_B)\,\Delta_A + \beta_B\Delta_B + \alpha_B + \gamma_B\left(\beta_A\Delta_A + \alpha_A\right)$$

where the κ_{i+1} terms all cancel out because they are common to both firms' prices: hence the same calculus applies regardless of which slot i is the base one.

Rearranging yields

$$\Phi_B^{CS} \equiv \frac{1}{(1-\gamma_B)}\left((\beta_B - 1)\Delta_B + \alpha_B\right) \geq \frac{1}{(1-\gamma_A)}\left((\beta_A - 1)\Delta_A + \alpha_A\right) \equiv \Phi_A^{CS},$$

Because $\beta_k\Delta_k + \alpha_k < \Delta_k$, all the terms above are negative and the implication for the summary statistic is given next:

Proposition 4 *An order of firms maximizes Consumer Surplus if and only if it follows the ranking of the summary statistics*

$$\Phi_k^{CS} \equiv \frac{1}{(1-\gamma_k)}\left((\beta_k - 1)\Delta_k + \alpha_k\right) < 0 \tag{21}$$

and firms should follow a decreasing order of the Φ_k^{CS}. The q_k value is irrelevant whereas, ceteris paribus, higher Δ_k and γ_k should go later in the order.

The qualitative implications regarding the ranking of products according to how popular they are (γ_k) and how appealing they are for search (Δ_k) is quite opposite to what we obtained for TIP or social welfare maximization. Indeed the consumer surplus maximization objectives are the reverse of what we found for total profit: consumers retain more surplus for products placed early in the search order because of pricing and also because they expand less search costs to get to them. Hence, the maximization of consumer surplus requires that the likelihood that the early products is bought is as large as possible, which is ensured by having low γ_k products first, and the surplus extracted from consumers is as low as possible, which means that products with a large Δ_k should come later.

By explicitly deriving endogenous prices and considering *ex ante* heterogeneous products, we highlight a conflict between the order that is desirable for firms and that which consumers prefer, which has not been identified in previous literature. For instance, in Athey and Ellison (2012), the preferred order for both consumers and firms is that the most popular products are first. This is because for them the order matters only because they assume that search costs are heterogenous and some consumers stop searching whereas they could have purchased a product that they like (this does not happen in our setting). If we introduce heterogeneous search costs in our search environment, predictions about the optimal order for TIP maximization become ambiguous but the prediction for consumer surplus would be unchanged.[9]

4 Allocation rules

In this section we explore how an allocation rule that relies on the firms' private incentives can implement a "desirable" outcome. Our characterization of optimal rankings suggests the search order that maximizes total industry profit is not necessarily the most attractive for consumers. Our goal here is to investigate how a particular type of auction might lead to joint profit maximization or rather to the maximization of consumer surplus.

[9]In Chen and He (2011), all consumers have the same search cost and the order of products is irrelevant for TIP or CS maximization in the early slots for which they assume alow search cost. However, it is preferable that these early slots are occupied by the most popular products.

In view of our equilibrium characterization, it is unclear whether it is more profitable for a firm to be searched earlier. Although a firm that is earlier in the search order gets to sell more, it charges a lower price in equilibrium. A simple condition that insures that earlier firms earn more profit is that base qualities q_i are large enough so that the percentage drop in price needed to prevent further search is less than the percentage increase in potential searches afforded by an earlier search slot. We assume this is the case in the analysis below.

4.1 Incremental values

We now wish to characterize each firm's willingness to pay for being searched earlier. More specifically we consider a firm's willingness to pay for being placed one slot ahead of another firm. We therefore consider two consecutive slots. Because of the externalities involved, this incremental value cannot be independent of the identity of the firms holding the other slots, before or after the two slots under consideration. Furthermore, it depends on which two slots are at stake. However, we now show that the ranking of incremental values between any two firms A and B is independent of which two slots they are competing for.

Consider again firms A and B: *which has the highest willingness to pay for being in slot i rather than in slot $i+1$?*. We have

$$\pi_A^i - \pi_A^{i+1} > \pi_B^i - \pi_B^{i+1} \Leftrightarrow \pi_A^i + \pi_B^{i+1} > \pi_B^i + \pi_A^{i+1}.$$

Thus, A's incremental value of being one slot ahead of B is larger than B's incremental value of being ahead of A if and only if $\Phi_A^\pi > \Phi_B^\pi$ and this is true no matter which two slots are considered.

Next we look at how this property can be used to characterize an equilibrium of a generalized second price auction.

4.2 Generalized second price auction: per impression bidding

Following previous literature we consider an allocation of the slots on an internet platform through an auction that assigns positions according to the ranking of bids (where higher bidders get earlier positions) and where a firm which wins a position is charged the next highest bid. Bids are per position, meaning that a firm pays a lump sum amount for a position. We call the latter per impression bidding because in our setting, the number of consumers who see any ad is exogenous and corresponds to the entire consumer population: the number of impressions is therefore 1 for each ad so the bid is a lump sum payment.

We extend the search and competition model to have $n \geq 2$ firms, that bid for $n-1$ positions on a platform. Hence, only the firms with the $n-1$ highest bids get a slot and the remaining firm is assumed to be searched last: to simplify the exposition we however say that it is in slot n. Having only one outside firm avoids having to model search and competition among outside firms.[10] The corresponding complete information auction game typically has multiple equilibria. We follow previous literature and focus primarily on envy-free equilibria (Edelman, *et al.*, 2007) (also called symmetric equilibria in Varian, 2007) to refine the equilibrium concept. In those papers, the price paid by firms is per click.

According to the envy-free condition, the firm in slot i should not wish to be in slot $j \neq i$ while paying b^{j+1}, which denotes the $j+1$th highest bid, which is paid by firm j (where b^{n+1} is set to zero because a firm can get the outside slot n for free). For $j > i$, it is equivalent to Nash equilibrium. For $j < i$ it is stronger: it allows i to pay b^{j+1} to be in slot j whereas it would have to pay at least b^j so as to outbid the firm in slot j and be able to deviate to slot j. Formally, no envy says that if some firm A is in slot i in equilibrium then

$$\pi_A^i - b^{i+1} \geq \pi_A^j - b^{j+1}, \tag{22}$$

for all $j = 1, ..., n$.

Now consider again two firms A and B in consecutive slots i and $i+1$. No envy for firm A in slot i not moving to slot $i+1$ can be written as

$$\pi_A^i - \pi_A^{i+1} \geq b^{i+1} - b^{i+2} \tag{23}$$

For firm B in slot $i+1$, no envy *vis-a-vis* slot i yields

$$\pi_B^{i+1} - \pi_B^i \geq b^{i+2} - b^{i+1}, \tag{24}$$

[10]There is no obvious way of modeling the search behavior of consumers among outside firms. Chen and He (2012) assume that search costs outside the platform are high enough that a consumer only searches one firm picked at random among the outsiders.

or equivalently,

$$b^{i+1} - b^{i+2} \geq \pi_B^i - \pi_B^{i+1}. \tag{25}$$

Hence we must have

$$\pi_A^i - \pi_A^{i+1} \geq \pi_B^i - \pi_B^{i+1}. \tag{26}$$

As before, this ranking of incremental value holds if and only if the ranking of A before B maximizes total industry profit. Hence, if there exists an envy-free equilibrium, then it yields the joint profit maximizing order.

This property underpins the result in Edelmaan *et al.* (2007) that an envy free equilibrium yields the same ranking as the VCG allocation. In contrast with the present analysis, their setting is a pure private value setting, where the surplus of each firm only depends on its own characteristics. We allow for externalities imparted by the search order, where a firm's profit is in general affected by the characteristics of competitors that precede it and follow it. However, in the special case where firms only differ with respect to their base qualities q_i, we have a private value environment. It is straightforward to characterize the transfers that should be used in a VCG mechanism. Such a mechanism implements by nature the total profit maximization ranking. However in this instance it is also the consumers' preferred ranking. Transfers are such that, if a firm is placed one slot earlier, then its payment increases by the drop in profit of the firm it demotes (that is its incremental value for staying in its original position as opposed to being moved back).

We now explore whether similar results can be obtained when some common value dimension is introduced (so VCG is no more relevant). Note that it is typically no more the case that a ranking maximizes both total profit and consumer surplus. We first consider the case where the common value stems solely from the search appeal externality (firms have identical γ_i's but different Δ_i's) and then the case where it results from the business stealing externality (identical Δ_i's and different γ_i's).

An obvious candidate for equilibrium has each firm bid its incremental value for moving up one slot, on top of what the next firm down in the order is bidding. That is, if we consider again two consecutive firms, A in slot i and B in slot $i+1$, then B's bid is given by $b^{i+1} = b^{i+2} + \pi_B^i - \pi_B^{i+1}$, where b^{n+1} is taken to be zero, π_B^{i+1} is B's equilibrium profit gross of the bid it pays and π_B^i is B's gross profit if it switches position with firm A. Note that if such an equilibrium exists, firms are necessarily ranked according to the decreasing order of TIP maximization scores Φ_i^π and hence, total profit is maximal. If this were not the case, then there would be two consecutive slots, i and $i+1$, such that the firm in slot i would have a lower incremental value for being in slot i than does the firm in slot $i+1$. Then the firm in slot i would be better off dropping its bid slightly below b^{i+2} in order to be in slot $i+1$, rather than having to pay b^{i+1} and be in slot i: this is because the bid difference reflects the incremental value of the firm in slot $i+1$ for being in slot i, which exceeds that of the firm in slot i. Also note that envy free necessarily holds for any two consecutive firms. Indeed, the additional amount of money the firm in slot i must pay in order to be in slot i rather than in slot $i+1$, exactly reflects the incremental value of the firm in slot $i+1$, so that the latter firm does not wish to be in slot i while having to pay the amount charged for that slot. It is *a priori* less clear whether envy free holds for any two positions, or indeed, whether such incremental value bidding yields an equilibrium. The following result shows that, if all products' consumer bases have the same size, then such an equilibrium exists and is envy-free, provided that base qualities q_i, follow the same order as the scores characterizing the joint profit maximizing order.

Proposition 5 *Suppose* $\gamma_i = \gamma$ *for all* $i = 1, ..., n$, $\Phi_i^\pi > \Phi_{i+1}^\pi$ *and* $q_i > q_{i+1}$ *for* $i = 1, ..., n-1$. *Then there exists an envy-free equilibrium where firm* i *is in slot* i, $i = 1, ..., n$ *and bids are such that:*

- $b^i = b^{i+1} + (1-\gamma)\gamma^{i-2} \left((1-\gamma) \left(q_i - \sum_{j=i+1}^{n} (\Delta_j) \right) - \Delta_{i-1} \right)$, *for* $i = 2, ...n$, *taking* $b^{n+1} = 0$.

Proof. First consider $n = 2$. The two firms are just bidding for one slot and we have a Vickrey auction with only two bidders. Then, despite the externality, it is a dominant strategy for each firm to bid its incremental value for being searched first (this would not be true if there were more than two firms because the incremental value for one firm would depend on the identity of the firm that gets the unique slot if the firm loses). Firm i's incremental value for being searched before firm j, $i, j = 1, 2$, $j \neq i$ is $(1-\gamma)\left((1-\gamma)q_i - (\Delta_j)\right)$, which is larger for firm 1, because $\Phi_1^\pi \geq \Phi_2^\pi$.

Now assume that $n > 2$. The bid structure is such that no firm would wish to deviate one slot up or one slot down. Indeed, the difference between a firm's bid and what it ends up paying is exactly equal to the firm's incremental value for being one slot up. In order to move up one slot, it would have to be paying the equilibrium bid of the firm it would be demoting which would involve an increase in it payment which strictly exceeds its incremental value. Now if a firm chooses to move down by one slot, it decreases its payment by the incremental value of the next firm down for being in its slot. However, because that firm has a lower TIP maximization score, its incremental value is also lower. Hence, the firm that deviates downward by 1 saves less than its own incremental value for staying where it is and this is not profitable.

Next consider a firm in slot $i < n-1$ deviating downward by t slots, $t > 1$. Instead of paying b^{i+1}, it pays b^{i+t+1}. It therefore saves

$$(1-\gamma)\gamma^{i-2}\sum_{k=1}^{t}\gamma^k\left((1-\gamma)\left(q_{i+k} - \sum_{j=i+k+1}^{n}(\Delta_j)\right) - (r_{i+k-1} - q_{i+k-1})\right). \tag{27}$$

Because $\Phi_{i+k-1}^\pi > \Phi_{i+k}^\pi$, we have $(1-\gamma)q_{i+k-1} - (r_{i+k} - q_{i+k}) > (1-\gamma)q_{i+k} - (r_{i+k-1} - q_{i+k-1})$. Using $q_i \geq q_{i+k-1}$, it follows that firm i's saving when it deviates is bounded above by

$$(1-\gamma)\gamma^{i-2}\sum_{k=1}^{t}\gamma^k\left((1-\gamma)\left(q_i - \sum_{j=i+k+1}^{n}(\Delta_j)\right) - (r_{i+k} - q_{i+k})\right). \tag{28}$$

Now consider firm i's drop in profit gross of the bid payment when going down one slot at a time, with all the other firms remaining in the equilibrium order. When it goes from slot $i+k-1$ to slot $i+k$, it loses a fraction γ of its demand (those who stop at firm $i+k$ which is now in slot $i+k-1$) but it increases its price by Δ_{i+k} because firm $i+k$ is now ahead of firm i. Hence its change in profit is $(1-\gamma)\gamma^{i+k-2}\left(\Delta_{i+k} - (1-\gamma)\left(q_i - \sum_{j=i+k+1}^{n}(\Delta_j)\right)\right)$. Taking the sum over $k = 1, ..., t$ yields minus the expression in (28) which is the total loss in gross profit from moving from slot i to slot $i+t$. The saving in bidding payments is therefore lower than the drop in gross profit so the deviation is not profitable.

Consider now some firm $i = 3, ..., n$ that deviates from slot i to slot $i - t$, $t \geq 2$. In order to encompass the no envy condition, which is stricter than the equilibrium condition in that case, assume it can do so while having to pay only b^{i-t+1}, the amount that firm $i - t$ is paying in equilibrium. The additional cost for firm i is then $b^{i-t+1} - b^{i+1}$, that is

$$(1-\gamma)\gamma^{i-t-2}\sum_{k=1}^{t}\gamma^k\left((1-\gamma)\left(q_{i-t+k} - \sum_{j=i-t+k+1}^{n}(\Delta_j)\right) - \Delta_{i-t+k-1}\right). \tag{29}$$

Because $\Phi^{\pi}_{i-t+k} > \Phi^{\pi}_i$, we have $(1-\gamma)q_{i-t+k} - \Delta_i > (1-\gamma)q_i - \Delta_{i-t+k}$. Now using $q_{i-t+k} \geq q_i$, we have $q_{i-t+k} - \Delta_i > q_i - \Delta_{i-t+k}$ so that a lower bound for $b^{i-t+1} - b^{i+1}$ is

$$(1-\gamma)\gamma^{i-t-2}\sum_{k=1}^{t}\gamma^k\left((1-\gamma)\left(q_i - \sum_{j=i-t+k}^{n}(\Delta_j) + \Delta_i\right) - \Delta_{i-t+k-1}\right). \tag{30}$$

Now $(1-\gamma)\gamma^{i-t+k-2}\left((1-\gamma)\left(q_i - \sum_{j=i-t+k}^{n}(\Delta_j) + \Delta_i\right) - \Delta_{i-t+k+1}\right)$ is the incremental profit firm i earns by moving from slot $i-t+k$ to $i-t+k-1$ while all other firms are ordered as in the candidate equilibrium, so that the above lower bound is exactly the increase in gross profit for firm i if it moves up from slot i to slot $i-t$. Hence firm i would not gain from such a deviation, even if it had to pay only b^{i-t+1} for being in slot $i-t$, so that no envy is satisfied. This in turn implies that we have an equilibrium. ■

The above proposition shows that, when the common value dimension is induced by the search appeal externality, then an envy free equilibrium exists under fairly mild conditions and thus the auction maximizes total profit. In general the resulting ranking of firms is not the most favorable to consumers: in order for producers' and consumers' interests to coincide, the ranking of the TIP indices $|Phi^{\pi}_i$ should be mostly driven by the differences in base qualities q_i.

Now consider a situation where firm heterogeneity arises only because they have different probabilities of selling a product that fits a consumer's need, $\gamma_1 > \gamma_2 >, ..., > \gamma_n$. To illustrate, suppose there are 3 firms with $\gamma_1 > \gamma_2 > \gamma_3$. Joint profit maximization requires that firm 1 be ranked before firm 2, which in turn should precede firm 3. In a candidate equilibrium that implements this order with incremental bidding, bids would satisfy:

$$b^3 = \gamma_1(1-\gamma_3)((1-\gamma_2)q - (\Delta)) \tag{31}$$

$$b^2 = b^3 + (1-\gamma_2)((1-\gamma_1)(q-(\Delta)) - (\Delta)). \tag{32}$$

Firm 1 can bid any value that exceeds b^2. Because of incremental bidding, no firm wishes to deviate by only one slot. We now show, however, that this cannot be an envy-free equilibrium, because firm 3 is always better off in slot 1 while paying b^2 than staying in slot 3.

Along the lines of the method used in the proof of Proposition 5, we proceed by writing the incremental profit for firm 3 if it moves from slot 3 to slot 2, $\gamma_1(1-\gamma_3)((1-\gamma_2)q-(\Delta))$, and then the incremental profit from moving from slot 2 to slot 1 $(1-\gamma_3)((1-\gamma_1)(q-(\Delta))-(\Delta))$. Note that the former term is merely b^3. Hence in order for firm 3 not to be willing to pay b^2 to be in slot 1, the latter term must be less than $b^2 - b^3$. Comparing the two expressions it is readily seen that firm 3's incremental profit exceeds the bid difference, so firm 3 is better off in slot 1 paying b^2. Hence, no envy free equilibrium can be sustained in this manner.

In this 3-firm example, it is always possible to make firm 3's deviation unprofitable by picking b^1 to be large enough. Still, in order to have an equilibrium, it is necessary to check that 1 does not want to deviate to slot 3. It is straightforward to construct a counterexample where such a deviation is profitable with incremental bidding. Consider a case where both γ_3 and γ_2 are small as compared to γ_1, meaning that firms 2 and 3 have significantly larger consumer bases then firm 1. Then firm 3's incremental value for toping firm 2 is quite large because it has potentially a lot of customers and could loose a large share of them to firm 2 if the latter is searched first. However, because firm 1 can only hope to sell to a few people, its incremental value for being anywhere is small and it will prefer to drop down to third place rather than paying firm 2's bid, which embodies the large incremental value of firm 3. This intuition generalizes even if bids do not follow the incremental value pattern. This is because, if both firms 2 and 3 are large, firm 2's bid should be large enough to deter firm 3 from outbidding it to get in second place: again, this is more than firm 1 is willing to pay to be in first place rather than in third place, if firm 1's consumer base is narrow.

Let us now consider some alternative equilibrium patterns that might arise systematically and lead to an outcome that is not joint profit maximizing. It is instructive to start by considering a situation where $\Delta_i = 0$ for all $i = 1, ..., n$ and firms share the same base quality q. Then they all identically price at q in equilibrium and we are in a setting that is analogous to that considered in Chen and He (2011) (note however some differences: in our setting this means that all firms are equally attractive to search whereas in Chen and He a lower γ_i makes a firm more attractive to search). Then, our analysis of optimal rankings shows that all orders yield the same total industry profit. This also means that, if we consider two firms in two consecutive slots, then they both have the same incremental value for being in the top slot. However, if there are more than 2 firms, then it is readily seen that incremental bidding is not consistent with ordering firms from the largest to the smallest value of γ_i. It is, on the contrary, consistent with ordering firms from the smallest to the largest value of γ_i. This can easily be illustrated with the three firms example.

As before, in order to check whether we have an equilibrium, we only need to check whether firm 1 might want to deviate to third place and pay nothing. Because, for $\Delta = 0$, the bid difference $b^2 - b^3$ exactly reflects its profit increase from being in first place in front of firm 2 rather than in second place behind it, we merely need to compare b^3 with firm 1's profit gain from being in front of firm 3 rather than behind while firm 2 is in first place. We have $b^3 = \gamma_1(1 - \gamma_3)(1 - \gamma_2)q$ whereas firm 1's profit gain between slots 3 and 2 is $\gamma_2(1 - \gamma_1)(1 - \gamma_3)q$. The difference is $(\gamma_1 - \gamma_2)(1 - \gamma_3)q$ and it should be negative in order to have an equilibrium. Hence, firms are ranked with firm 1 in front and firm 3 in last place in an incremental value bidding equilibrium if and only if $\gamma_1 \leq \gamma_2$.

The above result however is not robust to the introduction of $\Delta > 0$, no matter how small it is. Indeed, from our previous analysis, if $\gamma_1 < \gamma_2$, firm 1 would no more be indifferent between the two slots and would choose to drop to slot 2 and pay b^3 rather than staying in slot 1 and pay b^2. That is, if we require bids to follow the incremental value structure, the equilibrium with more popular products in front does not survive the introduction of even the slightest search appeal. We now show that it is nonetheless possible to specify bids that sustain such a search order. This can easily be illustrated in the three firms example. Assume bids are now given by

$$b^3 = \gamma_1(1-\gamma_2)((1-\gamma_3)q-(\Delta)) \tag{33}$$

$$b^2 = b^3 + (1-\gamma_1)((1-\gamma_2)(q-(\Delta))-(\Delta)). \tag{34}$$

In other words, the bid increment for a firm in slot $i+1$ is the incremental value of the next firm up (the firm in slot i) for being in slot i rather than in slot $i+1$. This means that, taking $i=1$, firm 1 is indifferent between staying in slot 1 or dropping down to slot 2. Furthermore, if it moves from slot 2 to slot 3 while letting firm 3 take over slot 2, firm 1's gross profit decreases by

$$\gamma_2(1-\gamma_1)((1-\gamma_3)q-(\Delta)) \tag{35}$$

and this is strictly above b^3 for $\Delta = 0$ and $\gamma_1 < \gamma_2$. Thus, for Δ close enough to zero, if $\gamma_1 < \gamma_2$ then firm 1 would nor deviate by dropping down to slot 3 with this bid structure. (How small should Δ be? How does the condition compare to what is needed to get strictly positive incremental values?) Because bids are not determined by the incremental values, we also need to check that firm 3 would not be willing to pay b^2 to demote firm 2. Now for $\Delta = 0$, the incremental profit for firm 3 is exactly equal to b^3 so firm 3 would not be willing to pay $b^2 > b^3$. Hence the above bids implement an equilibrium with larger firms searched first as long as Δ is not too large. The following proposition establishes that this result generalizes to n firms and n slots.

Proposition 6 *Assume $q_i = q$ and $\Delta_i = \Delta$ for all i and $\gamma_{i+1} > \gamma_i$ for all $i = 1, ..., n+1$. Then, if Δ is close enough to zero, there exists an equilibrium of the GSP such that firms are ranked with firm i in front of firm $i+1$, $i = 1, ..., n-1$.*

Proof. Assuming that firms are ordered from firm 1 to firm n, consider the following bid structure:

- $b^n = \Pi_{k=1}^{n-2}\gamma_k(1-\gamma_{n-1})\left((1-\gamma_n)q-\Delta\right)$,
- $b^i = b^{i+1} + \Pi_{k=1}^{i-2}\gamma_k(1-\gamma_{i-1})\left((1-\gamma_i)\left(q-(n-i)\Delta\right)-\Delta\right)$, for $i = 2, ..., n-1$,

- $b^1 \geq b^3 + (1-\gamma_2)\left((1-\gamma_1)(q-(n-2))\Delta) - \Delta\right)$.

Note that, because $\gamma_1 < \gamma_2$, $b^1 > b^2$.

Consider now some firm i in slot i moving to some slot $i+t$, $t = 1, ..., n-i$. Its drop in gross profit can be decomposed into t terms: each term is the drop in profit that corresponds to moving to slot j after having already moved to slot $j-1$ while all other firms are ranked according to the equilibrium order. For any $j = i+1, ..., i+t$, the associated drop in gross profit is

$$\Pi_{k=1}^{i-1}\gamma_k \Pi_{k=i+1}^{j-1}\gamma_k (1-\gamma_i)\left((1-\gamma_j)(q-(n-j)\Delta) - \Delta\right), \tag{36}$$

where the above expression reflects that, if firm i starts in slot $j-1$, then firm $j-1$ has been moved up to slot $j-2$. This is to be compared to how much firm i is saving in bids, which is

$$b^j - b^{j+1} = \Pi_{k=1}^{i-1}\gamma_k \Pi_{k=i+1}^{j-2}\gamma_k \gamma_i (1-\gamma_{j-1})\left((1-\gamma_j)(q-(n-j)\Delta) - \Delta\right). \tag{37}$$

for $\Delta = 0$, this saving in bids is less than the drop in profit if and only if

$$\gamma_{j-1}(1-\gamma_i) \geq \gamma_i(1-\gamma_{j-1}). \tag{38}$$

This inequality indeed holds strictly if $\gamma_i < \gamma_{j-1}$. Hence, for $\Delta = 0$, a deviation to slot $i+1$ yeilds the same profit as in equilibrium whereas a deviation to slot $i+t$, $t > 1$ strictly decreases firm i profit. For $\Delta > 0$, a deviation to slot $i+1$ leaves, by construction, firm i's profit unchanged and, for Δ small enough a deviation to some slot $i+2$ or beyond strictly decreases its profit.

Consider now a possible deviation of firm $i = 3, ..., n$ to slot $i-t$, $t \geq 1$ (note that b^1 is such that firm 2 would never want to outbid firm 1 because $b^1 - b^3$ is larger than its incremental value). Once again, the change in gross profit can be written as the sum of t terms: each corresponds to firm i's gain from moving from slot j to slot $j-1$, $j = i-t+1, ..., i$ while all other firms are ordered according to the equilibrium ranking. Each such term is given by

$$\Pi_{k=1}^{j-2}\gamma_k (1-\gamma_i)\left((1-\gamma_{j-1})(q-(n-j)\Delta) - \Delta\right). \tag{39}$$

When moving from slot i to slot $i-1$, firm i's payment increases by

$$\begin{aligned} b^{i-1} - b^{i+1} = \quad & \Pi_{k=1}^{i-3}\gamma_k \left[\gamma_{i-2}(1-\gamma_{i-1})\left((1-\gamma_i)\left(q-(n-i)\Delta\right)-\Delta\right)\right] \\ & +\Pi_{k=1}^{i-3}\gamma_k \left[(1-\gamma_{i-2})\left((1-\gamma_{i-1})\left(q-(n-i+1)\Delta\right)-\Delta\right)\right] \end{aligned}$$

and when moving from slot j to slot $j-1$ with $j < i$, the corresponding expression is

$$b^{j-1} - b^{j} = \Pi_{k=1}^{j-3}\gamma_k \left[(1-\gamma_{j-2})\left((1-\gamma_{j-1})\left(q-(n-j+1)\Delta\right)-\Delta\right)\right]$$

Consider again $\Delta = 0$. Then the incremental profit associated with moving from slot i to slot $i+1$ is exactly equal to the first term in $b^{i-1} - b^{i+1}$ (which is $b^{i} - b^{i+1}$) and because the second term (which is $b^{i-1} - b^{i}$) is strictly positive, the additional payment strictly exceeds the incremental profit. Now for a move from slot j to slot $j-1$, $j < i$, the additional payment strictly exceeds the incremental profit because $\gamma_{j-2} < 1$ and $\gamma_i > \gamma_{j-2}$. Hence, for Δ close enough to zero no upward deviation is profitable and the equilibrium is sustained. ■

Thus, for Δ small enough, there always exists an equilibrium that puts large firms in front of small firms, so that consumer surplus is maximized but not total profit (and we have provided a counter example that shows that there may not exist an equilibrium inducing the search order that is most favorable to consumers).

4.3 Per click bidding

As mentioned above, previous literature has considered per click bidding. Then the profit of firm A in slot i is given by $\pi_A^i - \lambda_i b^{i+1}$, where λ_i denotes the number of clicks in slot i. It is easy to see that the argument about no envy equilibria maximizing joint profit maximization remains valid in the case where all products have the same consumer base $\gamma_i = \gamma$ for all $i = 1, ..., n$. This is because λ_i and λ_{i+1} are independent of which firm comes first and we have $\lambda_{i+1} = \gamma\lambda_i$.

When products differ in terms of consumer base, it is no more clear whether, with per click bidding, a no envy equilibrium would actually yield total industry profit maximization. The work by Chen and He (2012) and Athey and Ellison (2012) rather suggest the opposite.

They have firms charging essentially exogenous prices and differing only in terms of their consumer base. Both papers characterize an equilibrium where firms with a larger consumer base bid more and are ranked earlier. However, our analysis of joint profit maximization shows that it requires, all other things equal, that firms with a large γ_i (small consumer base) are first.

5 Conclusion

Ordered search seems clearly to characterize the lion's share of the modern online economy, which is only growing in importance. Yet research so far (though see the major advances in Armstrong's 2016 recent survey and extension piece) has been stymied for lack of a tractable set-up, even in the symmetric case, let alone dealing with the full set of product distinguishers that we do here. One main accomplishment of the paper is to deliver a clean analysis for ordered search under asymmetry, which we effectuate by invoking a positive lowest willingness to pay for interested consumers, thus taking off the table the problem of returning consumers.

This device enables us to address 3 forms of asymmetries among firms' products. Two of these constitute position externalities. The first of these is the demand strength of earlier firms in attracting consumers. The second is the search appeal of later firms, which plays out in equilibrium pricing. In the model, the identities of those firms coming before matters for the size of incoming demand, and the identities of those coming after matters to equilibrium prices. However, the *order* in which predecessors of successors are presented has no bearing on a firm's profit at a particular position in the search order. This key property enables us to determine summary statistics for firms, which are firm specific and independent of position. These statistics enable us to determine optimal rankings of firms under different criteria, namely the rankings that maximize total profits, social welfare, and consumer surplus. Comparing these rankings enables us to determine the tensions between the various parties.

We then move to endogenous determination of positions via a second-price auction. One strong result holds when consecutive firms bid their incremental values for positions (that is, each firm is willing to bid up to the extra amount it would gain by being in front of the next one up or down). We show that this is exactly what characterizes total profit maximization, and therefore underpins the efficiency of the equilibrium in this regard: this echoes and extends the case without position externalities, as treated by Varian (2007) and Edelman et al. (2007).

However, while these local equilibrium conditions are readily determined, establishing the existence of an incremental value bidding equilibrium (or indeed, any equilibrium) is substantially more challenging in the presence of position externalities (especially so for asymmetric business stealing externalities – the search appeal externality is less problematic). In this regard, it is interesting to note that despite the billions of dollars spent on position auctions, there is little work beyond the classic 2007 papers (which close down the externalities), with the notable exceptions of Chen and He (2012) and Athey and Ellison (2012), who assume prices are exogenous. Athey and Ellison (2012) conclude that the outcome to the position auction is efficient for both firms and for consumers. However, our set-up reveals tensions: in a two-sided market setting, this substantiates giving a role to relevance scores that can encourage consumer participation by shifting the order of presentation to improve early matches and reduce prices.

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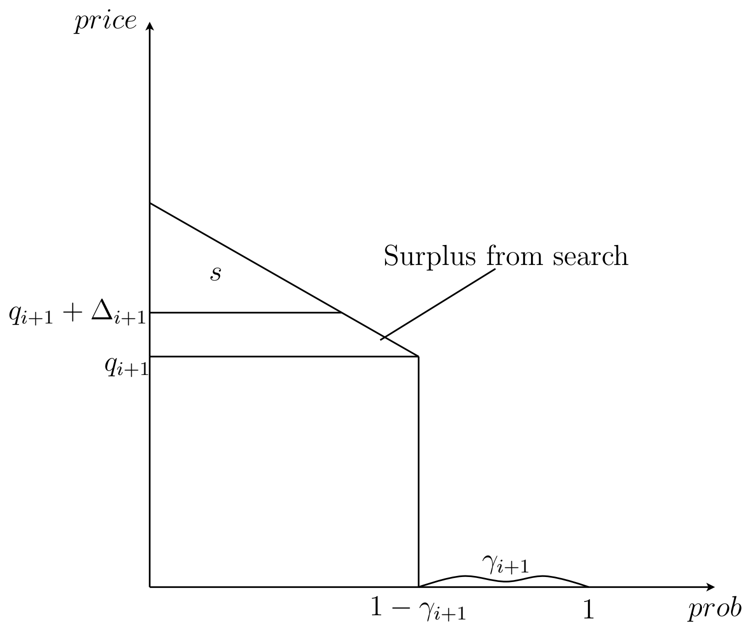

Figure 1: If the price difference reflects the quality difference, $p_i = p_{i+1} + (q_i - q_{i+1})$, a consumer holding q_i at Firm i will buy at Firm $i+1$ if $v_{i+1} \geq q_{i+1}$
and the corresponding gain in surplus is $v_{i+1} - p_{i+1} - (q_i - p_i) = v_{i+1} - q_{i+1}$.