Basic Spatial Equilibrium Model

SALLY ZHANG, STANFORD UNIVERSITY

SALLYTZ@STANFORD.EDU

TA Session Agenda

- Review: Basic spatial equilibrium model
- Derive spatial distribution with extreme value shocks
- Sample Matlab code
- Simulation graphs

Spatial Equilibrium

Location common characteristics

- Amenities, productivity, etc.

Idiosyncratic preference/productivity

- Uniform, Gumbel, Frechet, etc.

Prices

- Rents, wages, land prices, etc.

Common characteristics of a spatial equilibrium model

Locations	Workers	Production	Land market
• Distribution of locations • Travel cost?	• Types? • Utility functional form? • Spillovers?	• Production factors? • How many goods? • Spillovers?	• Fixed land supply? • Commercial vs. residential land? • Min/fractional housing demand?

Set up (from lecture notes, slightly different notation)

- S locations, N workers
- Workers: maximize utility:

$$\max_i \frac{w_i A_i \epsilon_i^\omega}{R_i}$$

  - ϵ_i^ω: worker ω's Frechet preference shock at location i

- Firms: pay workers marginal product X_i

$$w_i = X_i$$

- Housing supply:

$$R_i = z + k_i N_i$$

Deriving equilibrium conditions

1. Labor supply = labor demand (abstracted)
2. Housing supply = housing demand (each worker demands one unit of housing)

Worker's location choice ⟷ Rent

Deriving spatial distribution

- Worker ω choose location d if

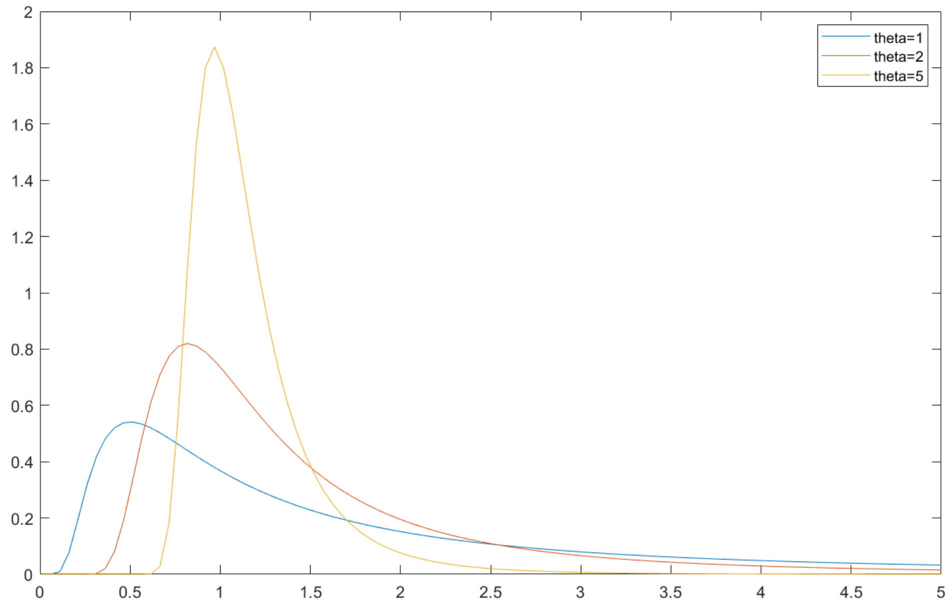
$$\frac{w_i A_i \epsilon_i^{\omega}}{R_i} > \frac{w_j A_j \epsilon_j^{\omega}}{R_j} \qquad \forall j \neq i$$

$$U_i \epsilon_i^{\omega} > U_j \epsilon_i^{\omega} \qquad \forall j \neq i$$

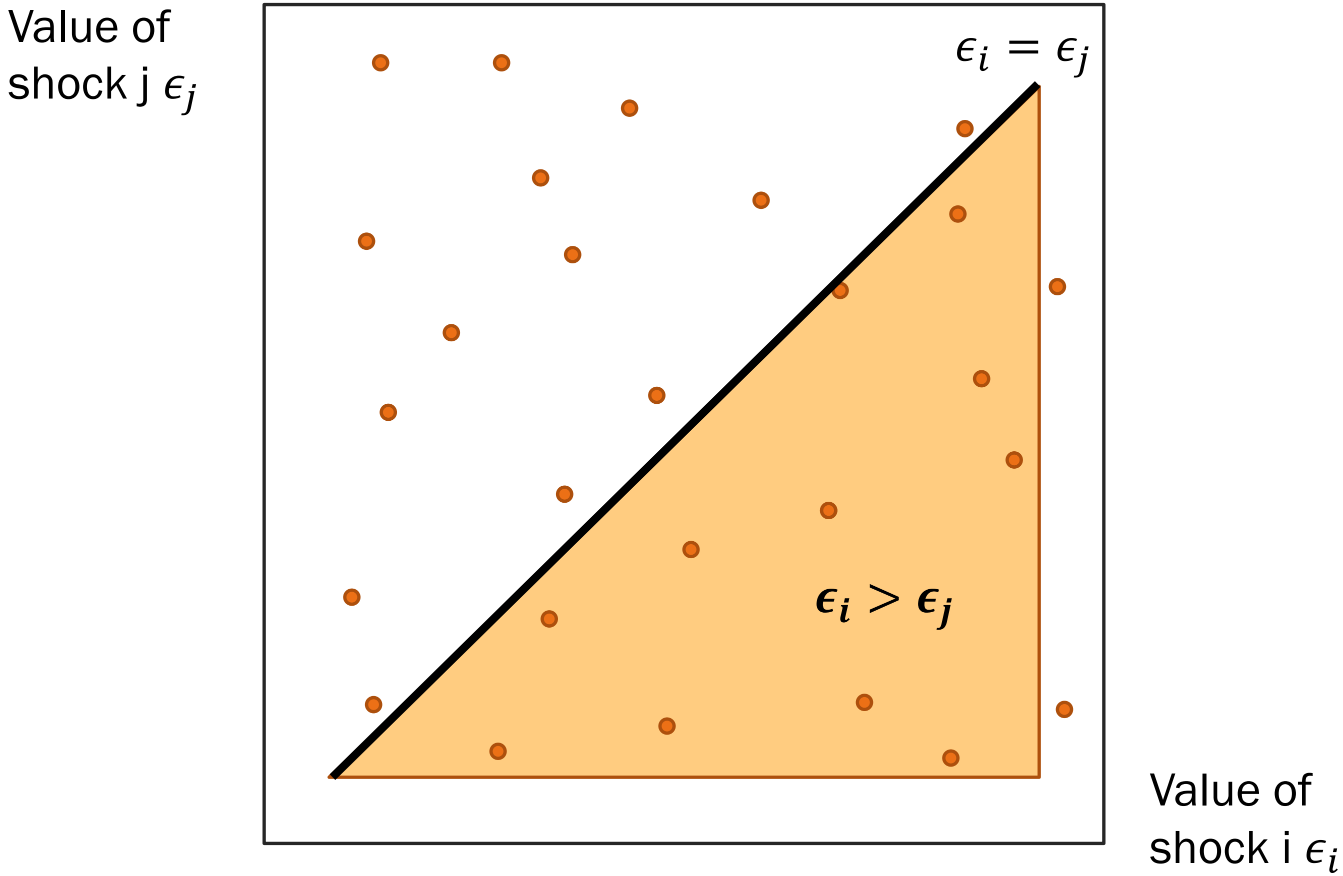
- ϵ is i.i.d. Frechet preference shock

Extreme Value Distributions

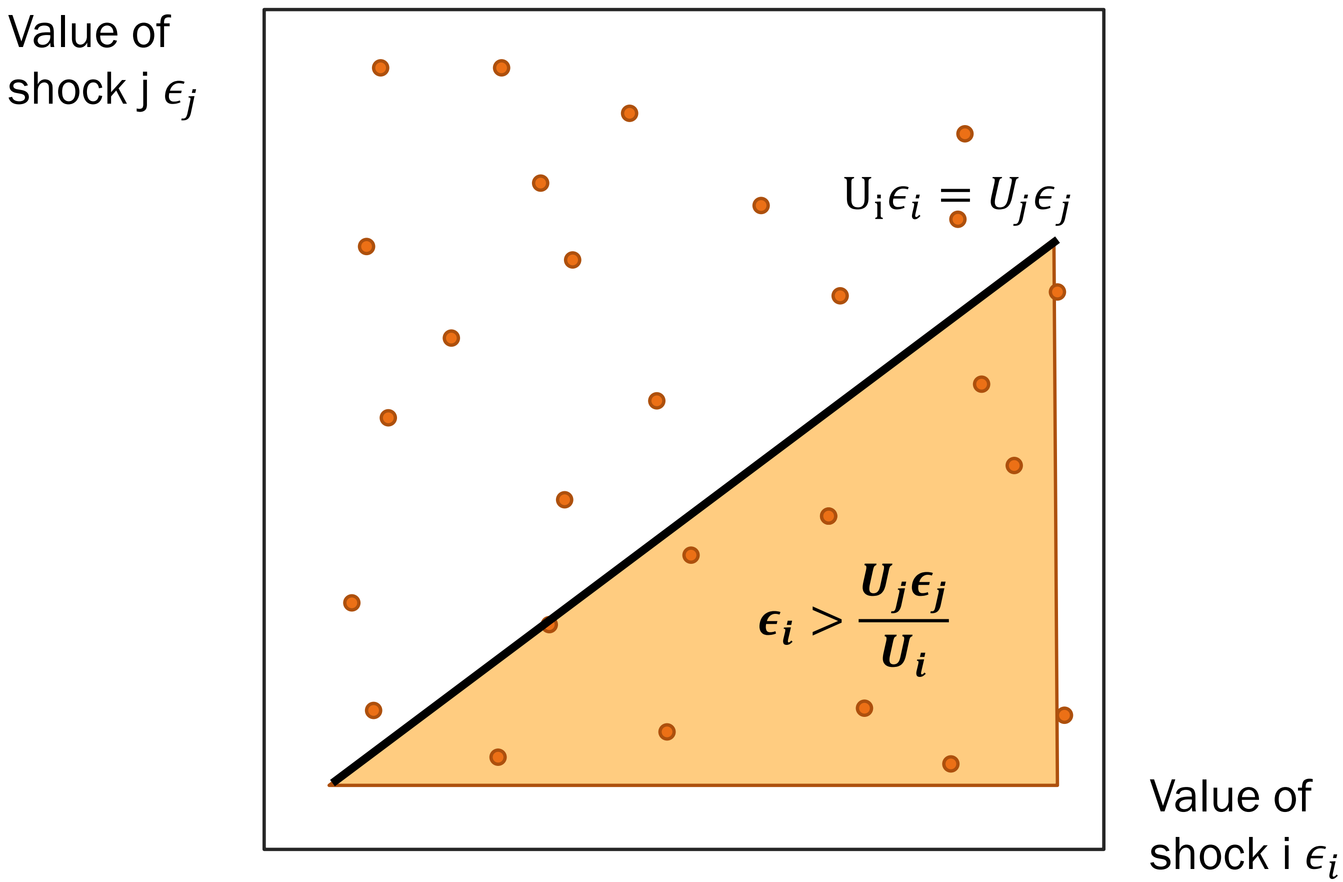
- Maximum of extremely value distributed random variables is itself extremely value distributed
- Type I: Gumbel – Additive, Type II: Frechet – Multiplicative
- Frechet cdf: $F(\epsilon_i) = \exp(-\epsilon_i^{-\theta})$
- Shape parameter determines variance



Deriving spatial distribution f

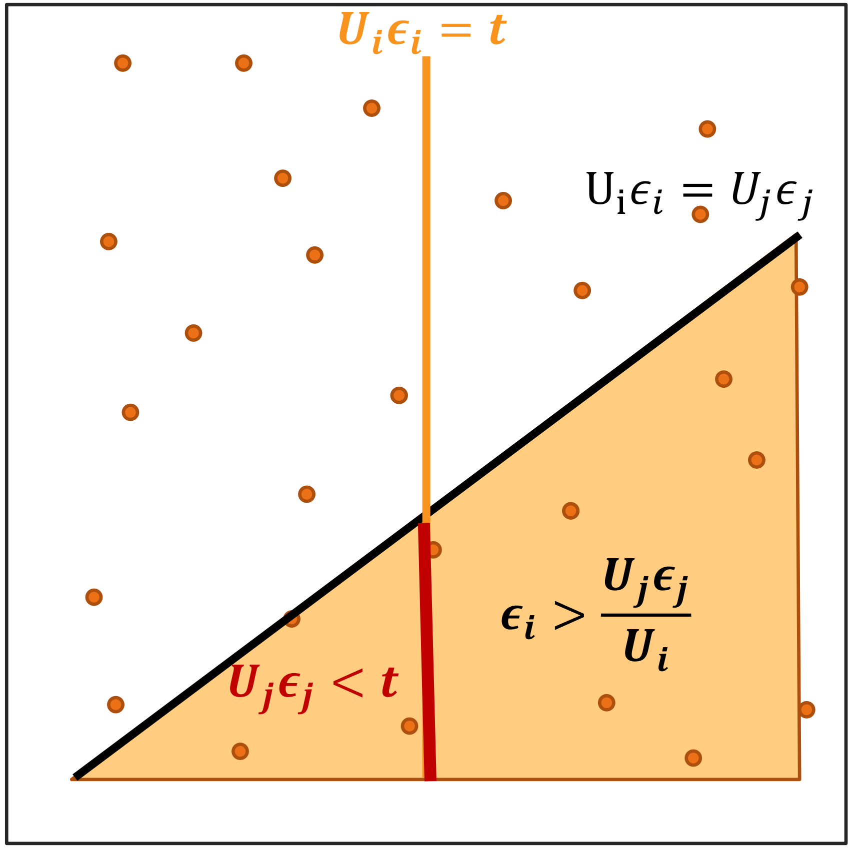


Deriving spatial distribution f



Deriving spatial distribution f

Value of shock j ϵ_j

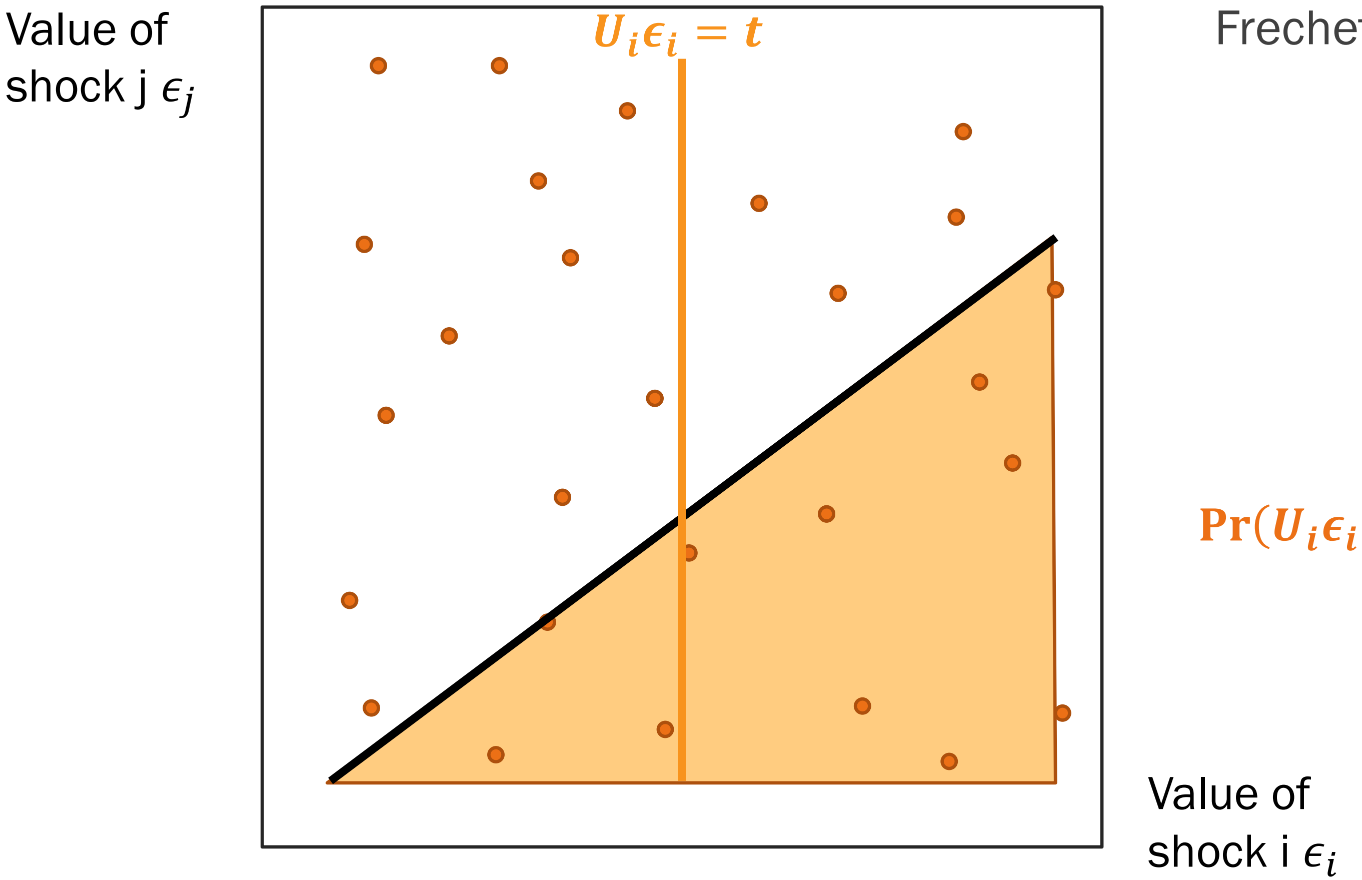


Value of shock i ϵ_i

$$\pi_i = \Pr\left(U_i\epsilon_i > U_j\epsilon_j \; \forall i \neq j\right)$$

$$= \int \boldsymbol{Pr\left(U_j\epsilon_j < t \; \forall i \neq j\right)} \, \Pr(\boldsymbol{U_i\epsilon_i = t}) \, f(t)dt$$

Deriving spatial distribution



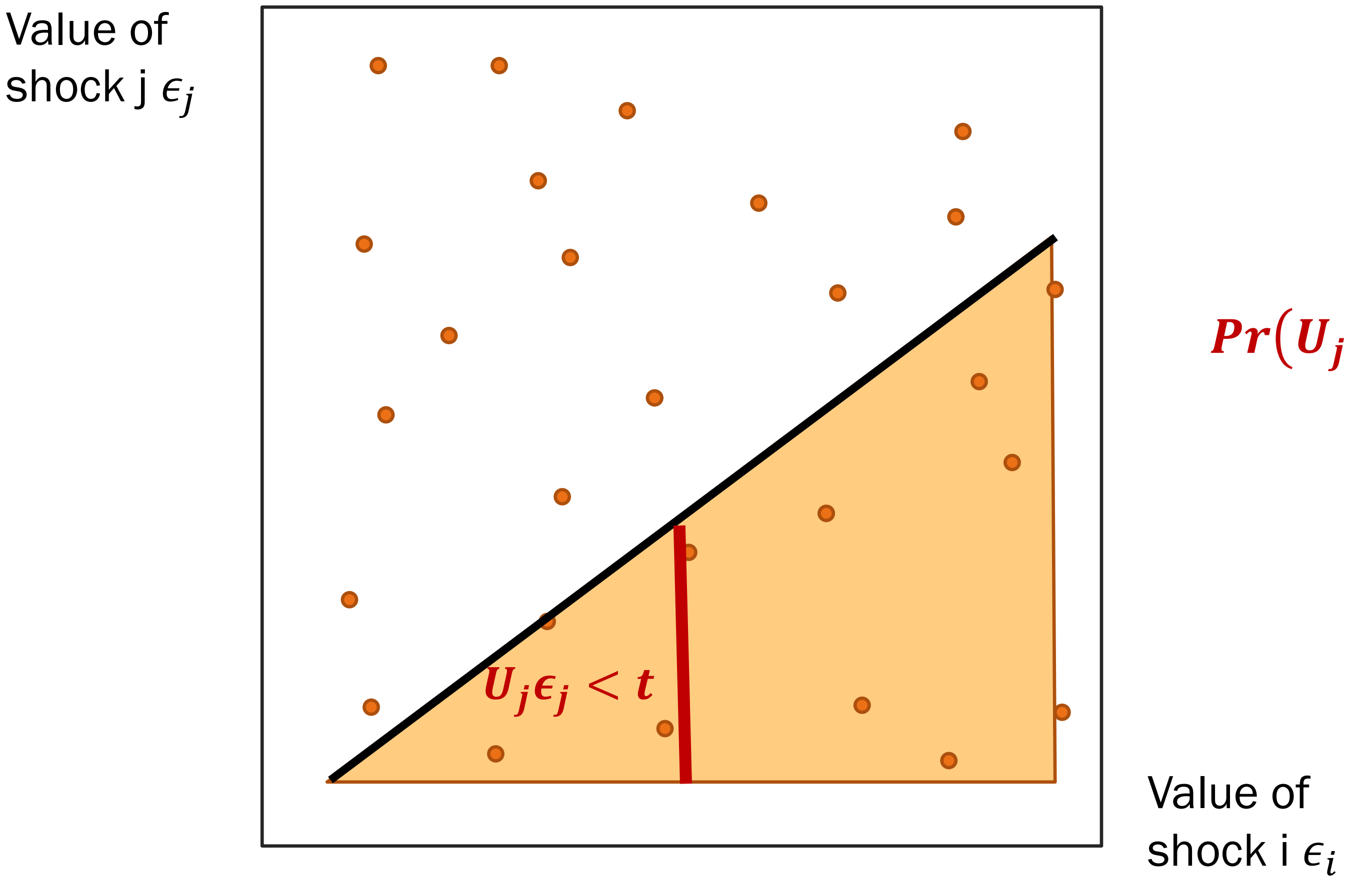
Frechet cdf:

$$F(\epsilon_i) = \exp(-\epsilon_i^{-\theta})$$

$$F\left(\frac{t}{U_i}\right) = \exp\left(-\frac{t}{U_i}^{-\theta}\right)$$

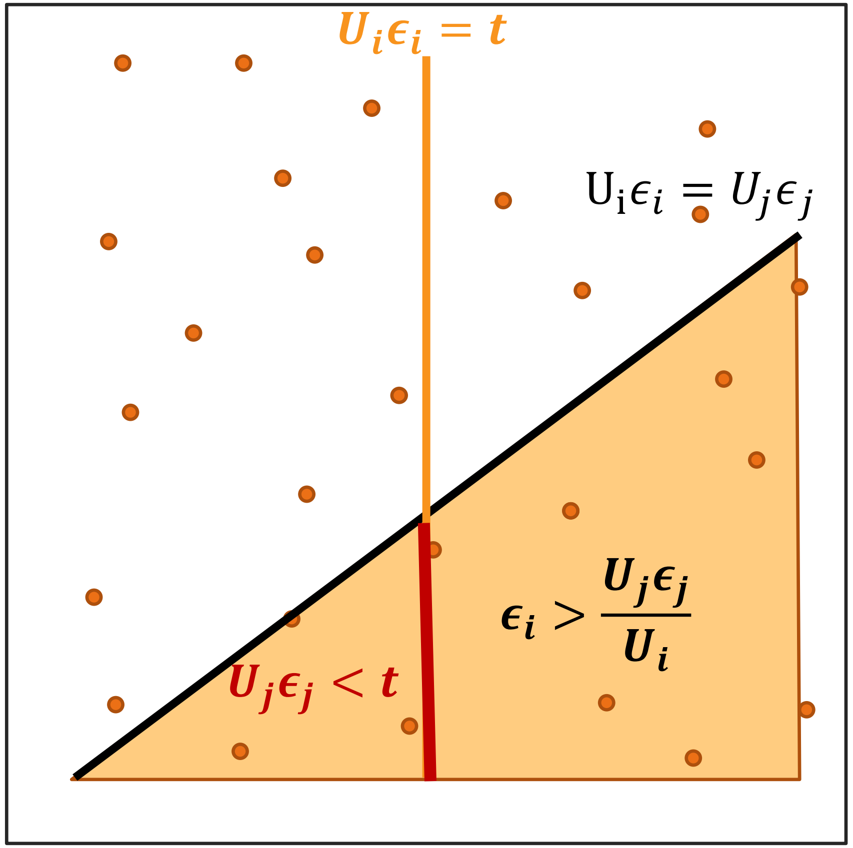
$$\mathbf{Pr}(U_i\epsilon_i = t) = f\left(\frac{t}{U_i}\right) = \exp\left(-\frac{t}{U_i}^{-\theta}\right)\theta U_i^{\theta} t^{-\theta-1}$$

Deriving spatial distribution



$$Pr\left(U_j \epsilon_j < t\right) = \Pr\left(\epsilon_j < \frac{t}{U_j}\right) = \exp\left(-t^{-\theta} U_j^{\theta}\right)$$

Deriving spatial distribution



$$\pi_i = \Pr\left(U_i\epsilon_i > U_j\epsilon_j \; \forall i \neq j\right)$$

$$= \int \boldsymbol{Pr}\left(\boldsymbol{U_j\epsilon_j < t \; \forall i \neq j}\right) \Pr\left(U_i\epsilon_i = t\right) f(t)dt$$

$$= \int_0^\infty \prod_{j \neq i} \exp\left(-\frac{t}{U_j}^{-\theta}\right) \exp\left(-\frac{t}{U_i}^{-\theta}\right) \theta U_i^\theta t^{-\theta-1} dt$$

$$= \frac{U_i^\theta}{\sum_i U_i^\theta}$$

Deriving equilibrium conditions

1. Labor supply = labor demand (abstracted)
2. Housing supply = housing demand

Housing supply: $R_i = z + k_i N_i$

Housing demand: $N_i = \frac{U_i^\theta}{\sum_i U_i^\theta} = \frac{\left(\frac{w_i A_i}{R_i}\right)^\theta}{\sum_i \left(\frac{w_i A_i}{R_i}\right)^\theta}$

Conditional expected utility does not depend on location

$E(U_i\epsilon_i | choose\ i) = E(t | choose\ i)$

$= \frac{1}{\pi_i}\int_0^\infty f(t)tdt$

$= \frac{1}{\pi_i}\int_0^\infty \Pr(U_i\epsilon_i = t)\Pr(U_j\epsilon_j < t\ \forall j \neq i)\ tdt$

$= \left(\sum_i {U_i}^\theta\right)^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta})$

And equals unconditional expected utility

$E(U_i \epsilon_i) = \sum_i \pi_i \, E(U_i \epsilon_i | choose\ i)$

$= \sum_i \pi_i \left(\sum_i {U_i}^{\theta}\right)^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta})$

$= \left(\sum_i {U_i}^{\theta}\right)^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta})$

Matlab algorithm

1. Guess spatial distribution
2. Calculate rent using housing supply equation
3. Calculate spatial distribution using housing demand equation
4. Update guess, until convergence

Matlab Code

```
%% Loop
% initial guess of spatial distribution is that all workers are equally
% distributed across space
pi_0 = ones(1,S)./S;
% set up difference
diff = 1;
```

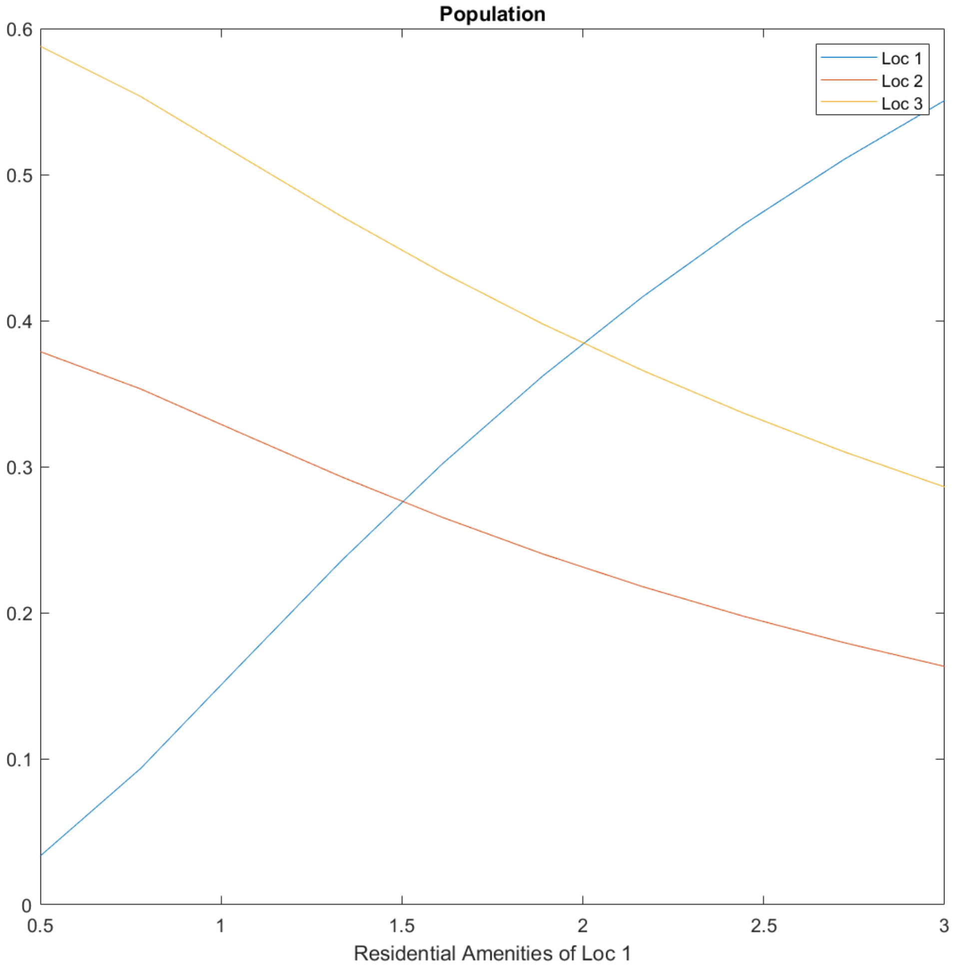
```matlab
% loop until convergence

while diff>error

    % Compute spatial distribution of population

    N = pi_0.*H;

    % Calculate rent

    R = z + k.*N;

    % Calculate spatial distribution given rent

    pi_1 = (w.*A./R).^theta;

    pi_1 = pi_1./sum(pi_1);

    % Check convergence

    diff = max(abs(pi_0-pi_1));

    % Update rent

    pi_0 = 0.7.*pi_0 + 0.3.*pi_1;

end
```

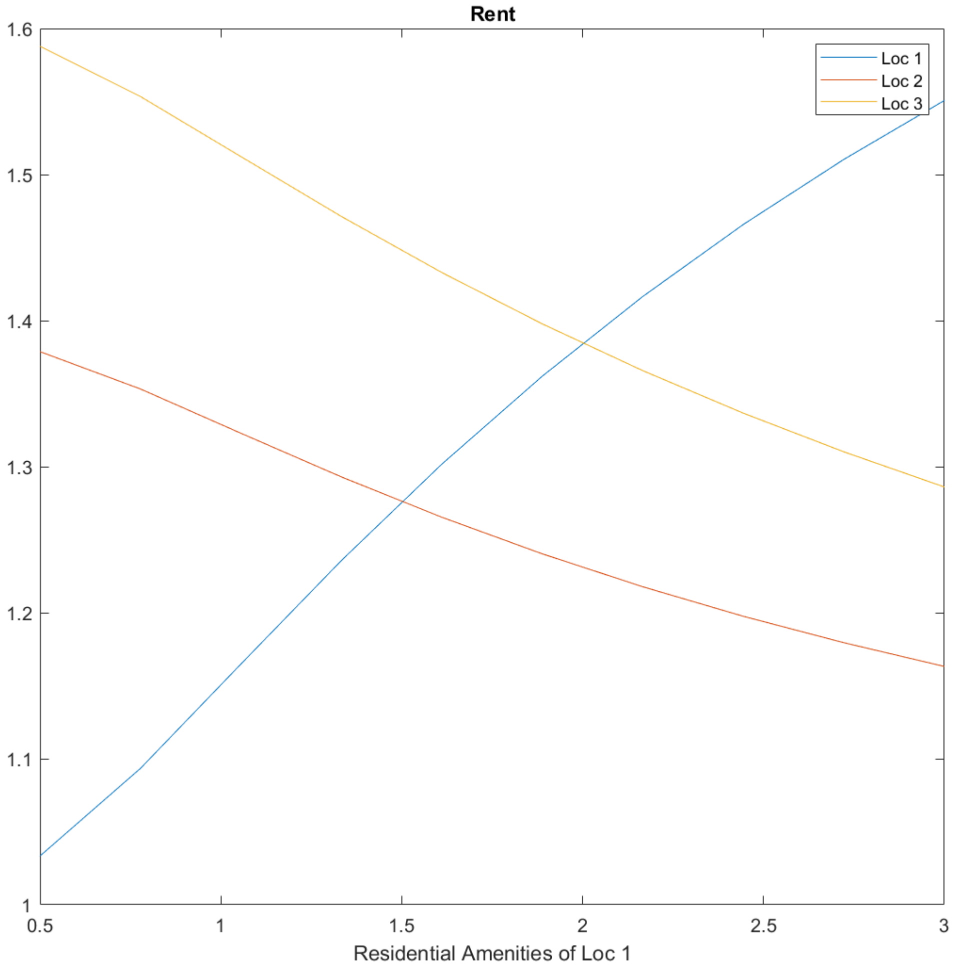
Simulation Set Up

- 3 locations, measure 1 of workers
- Baseline amenities: location 1< location 2 < location 3
- Equal wages, housing elasticity

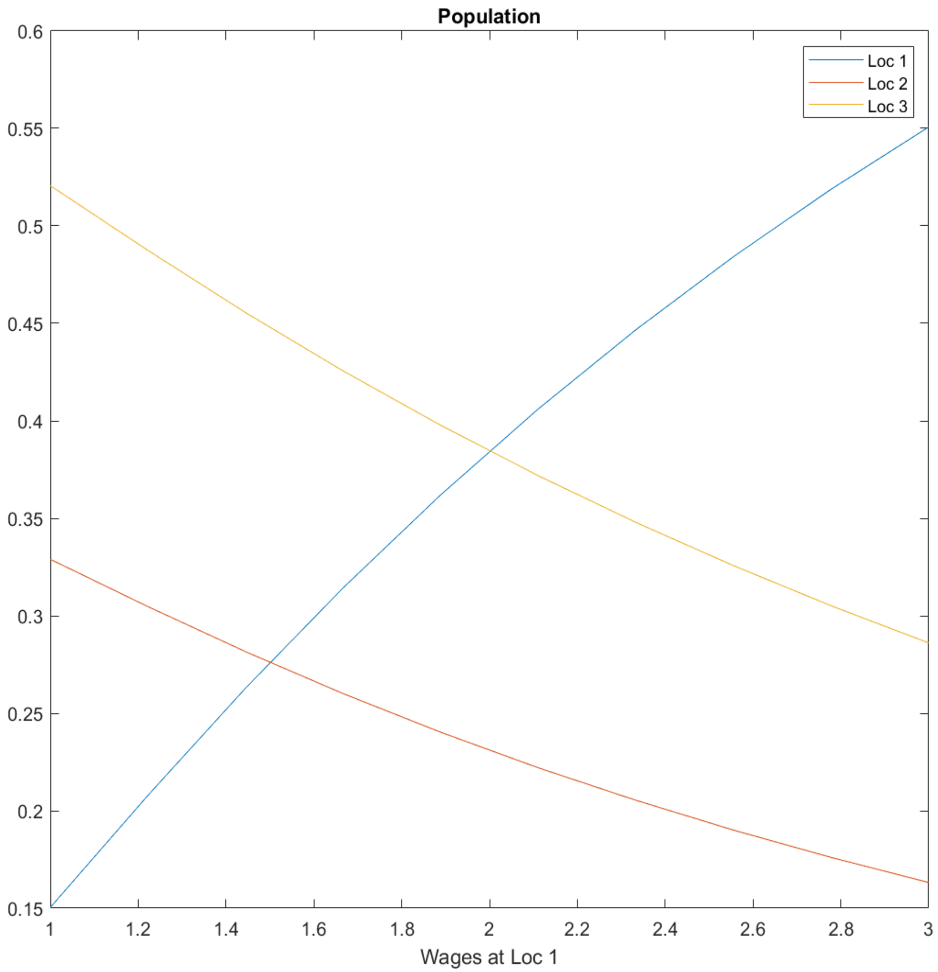
- Vary amenities, wages, housing elasticity of location 1
- Vary Frechet parameter

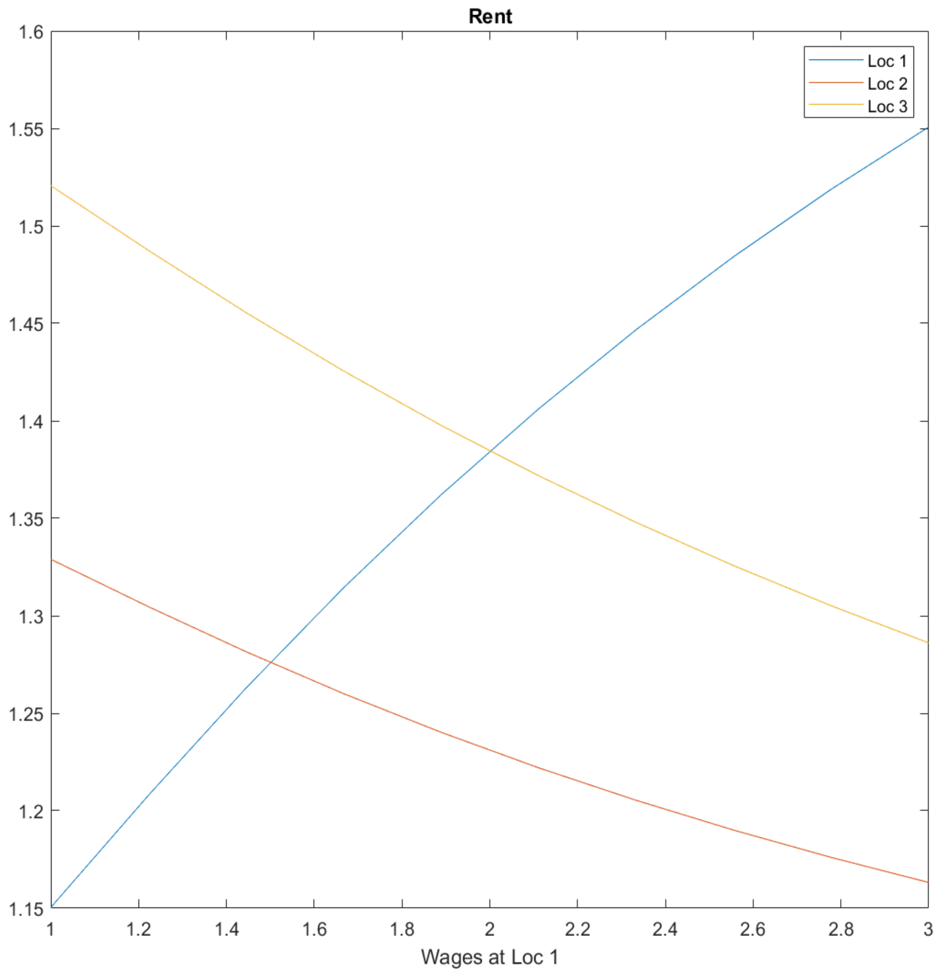
Better amenities: higher rent, higher population



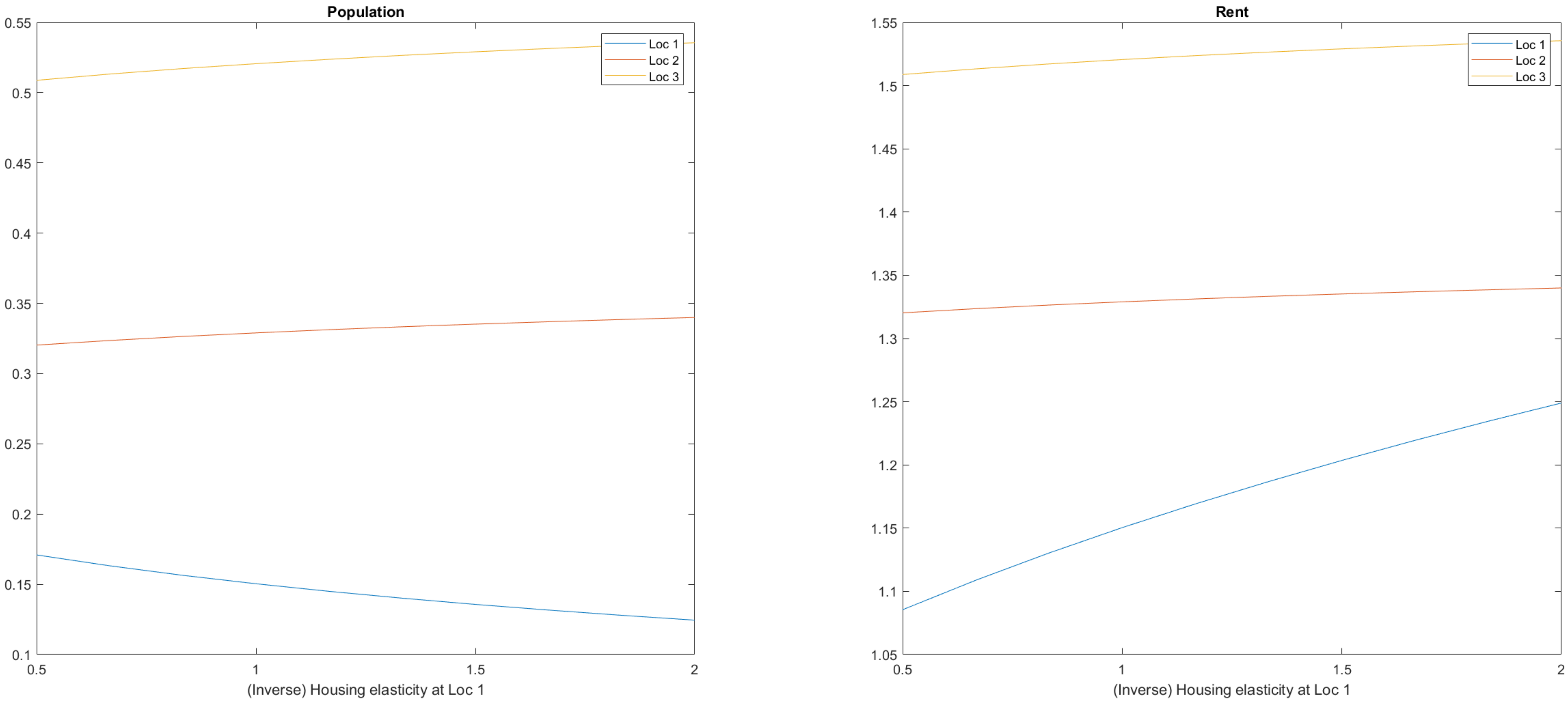


Higher wages: higher rent, higher population

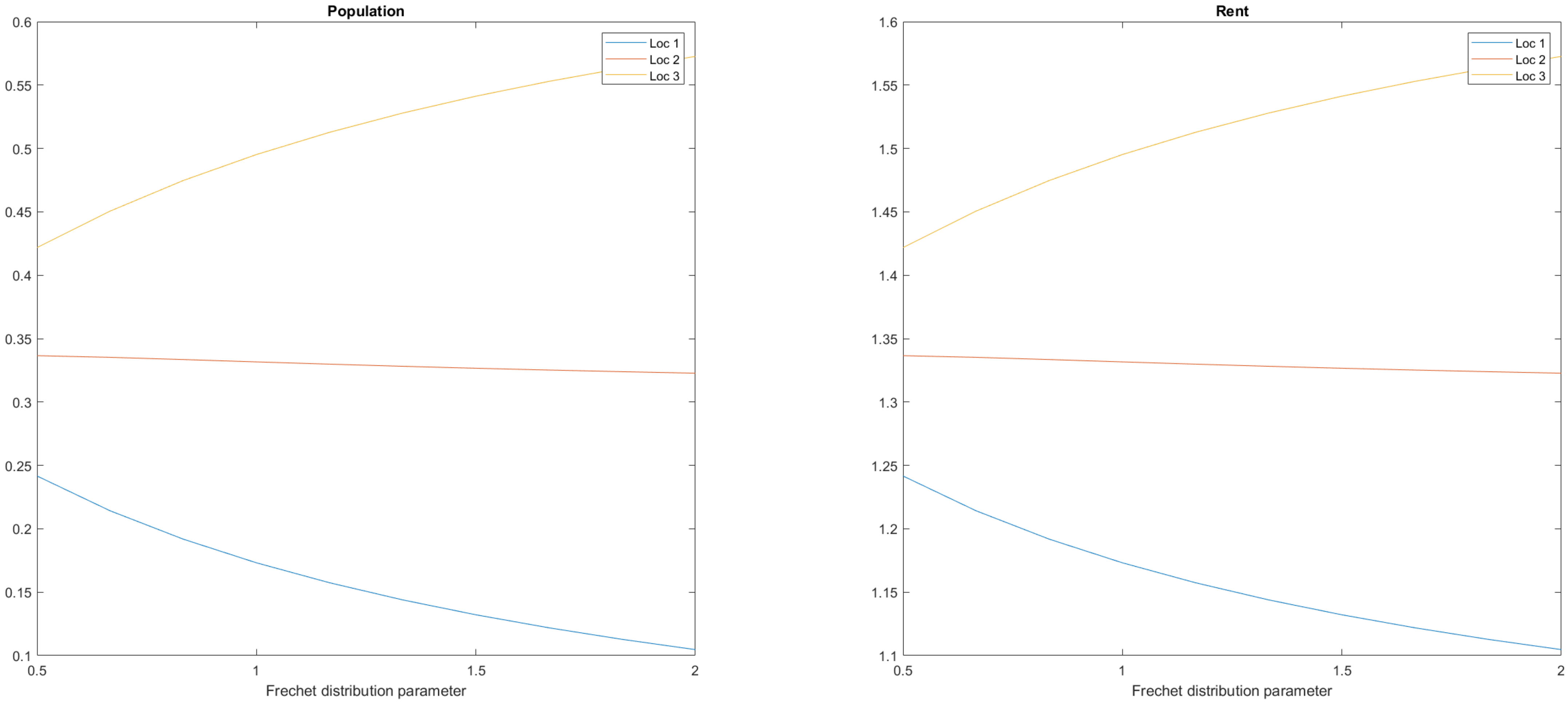




Lower elasticity: higher rent, lower population



Lower shock variance: less heterogeneity



Parameter Estimation (Tsivanidis 2019)

Parameters Calibrated to Literature

- Commercial floorspace share
- Elasticity of substitution

Parameters Estimated from Data

- Housing expenditure share
- Commute costs

Remaining Parameters from GMM

- Solved to match average expenditure share on housing and cars, college wage premium
- Estimated using exogenous variation

Non targeted Moments

- Wages
- Commute flows
- Employment by skill group