On the Network Topology of Variance Decompositions: Measuring the Connectedness of Financial Firms

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Financial and Economic Connectedness

- Market Risk, Portfolio Concentration Risk
  (return connectedness)
- Credit Risk
  (default connectedness)
- Counterparty Risk, Gridlock Risk
  (bilateral and multilateral contractual connectedness)
- Systemic Risk
  (system-wide connectedness)
- Business Cycle Risk
  (local or global real output connectedness)
Covariance

- So pairwise...
- So linear...
- So Gaussian...
Two Natural Questions

A natural modeling question:
*What fraction of the H-step-ahead prediction-error variance of variable i is due to shocks in variable j, \( \forall i, j \)?*

Variance decomposition: \( d_{ij}^H, \forall i, j \)

A natural financial/economic connectedness question:
*What fraction of the H-step-ahead prediction-error variance of variable i is due to shocks in variable j, \( \forall j \neq i \)?*

Non-own elements of the variance decomposition: \( d_{ij}^H, \forall j \neq i \)
Orthogonal system, correctly-specified, known parameters:

\[ x_t = B(L) \varepsilon_t \]

\[ C(x, H, B(L)) \]
## Variance Decompositions and the Connectedness Table

### $N$-Variable Connectedness Table

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>...</th>
<th>$x_N$</th>
<th>From Others to $i$</th>
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</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$d_{11}^H$</td>
<td>$d_{12}^H$</td>
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<td>$d_{1N}^H$</td>
<td>$\sum_{j=1}^N d_{1j}^H, j \neq 1$</td>
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<td>$x_2$</td>
<td>$d_{21}^H$</td>
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<td>$d_{NN}^H$</td>
<td>$\sum_{j=1}^N d_{Nj}^H, j \neq N$</td>
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</tbody>
</table>

| To Others | $\sum_{i=1}^N d_{i1}^H$ | $\sum_{i=1}^N d_{i2}^H$ | ... | $\sum_{i=1}^N d_{iN}^H$ | $\sum_{i,j=1}^N d_{ij}^H$ |
| From $j$  | $i \neq 1$           | $i \neq 2$          | ... | $i \neq N$          | $i \neq j$          |

Upper-left block is variance decomposition matrix, $D$

Connectedness involves the **non-diagonal** elements of $D$
Connectedness Measures

- **Pairwise Directional:** \( C_{i \leftarrow j}^H = d_{ij} \) ("i’s imports from j")
- **Net:** \( C_{ij}^H = C_{j \leftarrow i}^H - C_{i \leftarrow j}^H \) ("ij bilateral trade balance")

---

- **Total Directional:**
  - From others to i: \( C_{i \leftarrow \bullet}^H = \sum_{j=1}^{N} d_{ij} \) ("i’s total imports")
  - To others from j: \( C_{\bullet \leftarrow j}^H = \sum_{i=1}^{N} d_{ij} \) ("j’s total exports")
  - Net: \( C_{i}^H = C_{\bullet \leftarrow i}^H - C_{i \leftarrow \bullet}^H \) ("i’s multilateral trade balance")

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- **Total:** \( C^H = \frac{1}{N} \sum_{i,j=1 \atop i \neq j}^{N} d_{ij}^H \) ("total world exports")
Networks I: Representation

Adjacency Matrix (Symmetric)

\[ A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix} \]

\[ A_{ij} = 1 \text{ if nodes } i, j \text{ linked} \]

\[ A_{ij} = 0 \text{ otherwise} \]
Networks I: Degree

Degree of node $i$, $d_i$:

$$d_i = \sum_{j=1}^{N} A_{ij}$$

Discrete degree distribution, $P(d)$, on $0, \ldots, N - 1$

Mean degree, $E(d)$, is the key connectedness measure

Beautiful results (e.g., “small world”) involve the mean degree:

$$diameter \approx \frac{\ln N}{\ln E(d)}$$
Networks II: Representation (Weighted, Directed)

\[ A = \begin{pmatrix} 
0 & .5 & .7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .3 & 0 \\
0 & 0 & 0 & .7 & 0 & .3 \\
.3 & .5 & 0 & 0 & 0 & 0 \\
.5 & 0 & 0 & 0 & 0 & .3 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \]

“to \( i \), from \( j \)”
Networks II: Degree (Weighted, Directed)

\( A_{ij} \in [0, 1] \) depending on connection strength

Two degrees:

\[
\begin{align*}
    d_i^{\text{from}} &= \sum_{j=1}^{N} A_{ij} \\
    d_j^{\text{to}} &= \sum_{i=1}^{N} A_{ij}
\end{align*}
\]

Continuous “from” and “to” degree distributions on \([0, N - 1]\)

Mean degree \( E(d) \) remains the key connectedness measure
Central Observation: \( D \) is a Weighted, Directed Network

### Connectedness Table

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To Others
\[
\sum_{i \neq 1} d_{i1}^H \quad \sum_{i \neq 2} d_{i2}^H \quad \ldots \quad \sum_{i \neq N} d_{iN}^H \quad \sum_{i \neq j} d_{ij}^H
\]

\[
C_{i \leftarrow \bullet}^H = \sum_{j=1}^N d_{ij}^H, \text{ are the “from degrees”}
\]

\[
C_{\bullet \leftarrow j}^H = \sum_{i=1}^N d_{ij}^H, \text{ are the “to degrees”}
\]

\[
C^H = \frac{1}{N} \sum_{i,j=1}^N d_{ij}^H, \text{ is the mean degree (to or from)}
\]
Relationships to “Systemic Risk” Measures

Two prominent market-based measures:

- Marginal expected shortfall
- CoVaR
$MES_{T+1\mid T}^{j\mid mkt} = E_T [r_{j, T+1} \mid \mathbb{C} (r_{mkt, T+1})]$

- Sensitivity of firm $j$’s return to extreme market event $\mathbb{C}$
- Market-based “stress test” of firm $j$’s fragility
- Like “total directional connectedness from” (from degree)

“From others to $j$”
CoVaR

\[ \text{VaR}_{T+1|T}^{p,j} : p = P_T \left( r_{j,T+1} < -\text{VaR}_{T+1|T}^{p,j} \right) \]

\[ \text{CoVaR}_{T+1|T}^{p,j|i} : p = P_T \left( r_{j,T+1} < -\text{CoVaR}_{T+1|T}^{p,j|i} \mid \mathbb{C} (r_{i,T+1}) \right) \]

\[ \text{CoVaR}_{T+1|T}^{p,mkt|i} : p = P_T \left( r_{mkt,T+1} < -\text{CoVaR}_{T+1|T}^{p,mkt|i} \mid \mathbb{C} (r_{i,T+1}) \right) \]

- Measures tail-event linkages
- Leading choice of \( \mathbb{C} (r_{i,T+1}) \) is that firm \( i \) breaches its VaR
- Like “total directional connectedness to” (to degree)
  
  “From \( i \) to others”
Estimating Connectedness

Thus far we’ve worked under correct specification, in population:

\[ C(x, H, B(L)) \]

Now we want:

\[ \hat{C} \left( x, H, B(L), M(L; \hat{\theta}) \right), \]

and similarly for other variants of connectedness.
Many Interesting Issues

- x objects: Returns? **Return volatilities**? Real activities?
- x universe: How many and which ones? 
  (≈ 15 major financial institutions)
- x frequency: **Daily**? Monthly? Quarterly?

- Approximating model $M$: **VAR**? Structural?

- Horizon $H$: **Match VaR horizon**? Holding period?

- Identification of variance decompositions:
  Cholesky? **Generalized**? Structural?

- Estimation: **Classical**? Bayesian?
Connectedness of Major U.S. Financial Institutions

\[ \hat{C} \left( x, H, B(L), M(L; \hat{\theta}) \right) \]

- **x**: Thirteen daily realized stock return volatilities
  - Commercial banks: JP Morgan Chase (JPM), Bank of America (BAC), CitiGroup (C), Wells Fargo (WFC), Bank of New York Mellon (BK), U.S. BankCorp (USB), PNC Bank (PNC)
  - Investment Banks: Goldman Sachs (GS), Morgan Stanley (MS)
  - GSEs: Fannie Mae (FNM), Freddie Mac (FRE)
  - Insurance: AIG (AIG)
  - Specialized: American Express (AXP)

- **H**: 12 days

- **M(L; \theta)**: logarithmic VAR(3), generalized identification, 5/4/1999 - 4/30/2010
## Full-Sample Connectedness Table

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Estimating Time-Varying Connectedness

Before:

\[ C(x, H, B(L), M(L; \theta)) \]
\[ \hat{C}(x, H, B(L), M(L; \hat{\theta})) \]

Now:

\[ C_t(x, H, B_t(L), M(L; \theta_t)) \]
\[ \hat{C}_t(x, H, B_t(L), M(L; \hat{\theta}_t)) \]

- Time-varying parameters: **Rolling estimation**? Smooth TVP model? Regime-switching?

(100-day estimation window)
Rolling Total Connectedness

![Rolling Total Connectedness Graph](image-url)

The graph above represents the rolling total connectedness from the year 2000 to 2009. The x-axis denotes the years, and the y-axis represents the total connectedness, ranging from 0 to 90. The data shows fluctuations over the years, with peaks and troughs indicating varying levels of connectedness.
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Rolling Total Directional Connectedness Distributions

- Total Directional Connectedness "From"
- Total Directional Connectedness "To"
Net Pairwise Directional Connectedness:
The Lehman Bankruptcy, September 17, 2008
Concluding Remarks

- Natural framework with direct motivation
- Firmly grounded in network theory
- Tracks transmissions and receipts, from highly granular to highly aggregated
- *MES* and *CoVaR* perspectives are effectively special cases