Analytical Formulas for the Second Moment in Affine Models with Stochastic Volatility†

Jens H. E. Christensen
Jose A. Lopez
and
Glenn D. Rudebusch

Federal Reserve Bank of San Francisco
101 Market Street, Mailstop 1130
San Francisco, CA 94105

Abstract

In this note, we detail the analytical formula, first derived in Fisher and Gilles (1996), for the second moment in general affine term structure models with stochastic volatility.

†The views in this note are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of San Francisco or the Board of Governors of the Federal Reserve System.

Date: August 24, 2015.


1 Introduction

In this note, we briefly describe the derivation of the analytical formula for the conditional covariance matrix of the state variables in general $A_i(N)$ dynamic term structure models with stochastic volatility first presented in Fisher and Gilles (1996). These formulas are used in Christensen, Lopez, and Rudebusch (2014) in their analysis of arbitrage-free Nelson-Siegel models with stochastic volatility.

2 Affine Term Structure Models

We start from a standard continuous-time affine structure as in Duffie and Kan (1996). To represent an affine diffusion process, define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where the filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions (Williams, 1997). The state variables $X_t$ are assumed to be a Markov process defined on a set $M \subset \mathbb{R}^n$ that solves the following stochastic differential equation (SDE)\(^2\)

$$dX_t = K^P(t)[\theta^P(t) - X_t]dt + \Sigma(t)D(X_t, t)dW_t^P,$$  \hspace{1cm} (1)

where $W^P$ is a standard Brownian motion in $\mathbb{R}^n$, the information of which is contained in the filtration $(\mathcal{F}_t)$. The drift terms $\theta^P : [0, T] \to \mathbb{R}^n$ and $K^P : [0, T] \to \mathbb{R}^{n \times n}$ are bounded, continuous functions.\(^3\) Similarly, the volatility matrix $\Sigma : [0, T] \to \mathbb{R}^{n \times n}$ is assumed to be a bounded, continuous function, while $D : M \times [0, T] \to \mathbb{R}^{n \times n}$ is assumed to have the following diagonal structure

$$(\sqrt{\gamma^1(t)} + \delta^1(t)X_t \quad \ldots \quad 0 \quad \ldots \quad \sqrt{\gamma^n(t)} + \delta^n(t)X_t),$$

where $\gamma(t) = \begin{pmatrix} \gamma^1(t) \\ \vdots \\ \gamma^n(t) \end{pmatrix}$, $\delta(t) = \begin{pmatrix} \delta^1_1(t) & \ldots & \delta^1_n(t) \\ \vdots & \ddots & \vdots \\ \delta^n_1(t) & \ldots & \delta^n_n(t) \end{pmatrix}$, $\gamma : [0, T] \to \mathbb{R}^n$ and $\delta : [0, T] \to \mathbb{R}^{n \times n}$ are bounded, continuous functions, and $\delta^i(t)$ denotes the $i$th row of the $\delta(t)$-matrix.

\(^1\)Our nomenclature follows Dai and Singleton (2000).

\(^2\)The affine property applies to bond prices; therefore, affine models only impose structure on the factor dynamics under the pricing measure.

\(^3\)Stationarity of the state variables is ensured if all the eigenvalues of $K^P(t)$ are positive (if complex, the real component should be positive), see Ahn et al. (2002). However, stationarity is not a necessary requirement for the process to be well defined.
3 The Conditional Covariance Matrix in Affine Models

The conditional mean for multi-dimensional affine diffusion processes is given by

\[ E^P[X_T|X_t] = (I - \exp(-K^P(T - t)))\theta^P + \exp(-K^P(T - t))X_t, \]

where \( \exp(-K^P(T - t)) \) is a matrix exponential.

In general, the conditional covariance matrix of a multi-dimensional affine diffusion process is given by

\[ V^P[X_T|X_t] = \int_t^T \exp(-K^P(T - s))\Sigma D(E^P[X_s|X_t])D(E^P[X_s|X_t])'\Sigma' \exp(-(K^P)'(T - s))ds. \]

First, we note that

\[ D(E^P[X_s|X_t])D(E^P[X_s|X_t])' = \begin{pmatrix} \gamma^1 + \delta^1 E^P[X_s|X_t] & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \gamma^N + \delta^N E^P[X_s|X_t] \end{pmatrix}, \]

where \( \gamma^i \) and \( \delta^i, i = 1, \ldots, N \), have been defined previously.

Second, we assume that \( K^P \) is diagonalizable

\[ K^P = Q\Phi Q^{-1}, \]

where \( \Phi \) is a diagonal matrix containing the eigenvalues of \( K^P \), while \( Q \) is the matrix of eigenvectors of \( K^P \). This implies that

\[ e^{-K^P(s-t)} = Qe^{-\Phi(s-t)}Q^{-1}, \]

whereby we can define and re-write the following terms

\[ D^i(t, s) \equiv \gamma^i + \delta^i E^P[X_s|X_t] \]
\[ = \gamma^i + \delta^i[\theta^P + Qe^{-\Phi(s-t)}Q^{-1}(X_t - \theta^P)] \]
\[ = \gamma^i + \delta^i\theta^P + \delta^i Qe^{-\Phi(s-t)}Q^{-1}(X_t - \theta^P), \quad i = 1, \ldots, N. \]

Third, we let \( \overline{D}(t, s) \) denote the following diagonal matrix\(^4\)

\[ \overline{D}(t, s) \equiv \begin{pmatrix} D^1(t, s) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & D^N(t, s) \end{pmatrix}. \]

\(^4\)That is, \( \overline{D}(t, s) = D(E^P[X_s|X_t])D(E^P[X_s|X_t])' \).
To simplify the notation of the elements in $\overline{D}(t,s)$, we introduce two intermediate terms

\[
\begin{align*}
\overline{Q}^i &= \delta^i Q, \\
\overline{Q}^X &= Q^{-1}(X_t - \theta^P),
\end{align*}
\]

where $\overline{Q}^i$ is a row vector, while $\overline{Q}^X$ is a state-dependent column vector.

Furthermore, we introduce the following term

\[
\overline{S} = Q^{-1}\Sigma.
\]

Given this notation the conditional covariance matrix can be written as

\[
V^P[X_T|X_t] = \int_t^T Q e^{-\Phi(T-s)}\overline{SD}(t,s)\overline{S}' e^{-\Phi(T-s)}Q' ds.
\]

If we exploit the diagonal property of $\overline{D}(t,s)$, we see that the columns of $\overline{D}(t,s)\overline{S}'$ are:

\[
[D(t,s)\overline{S}']_{i,j} = \begin{pmatrix} 
D^1(t,s)\overline{S}'_{1,j} \\
\vdots \\
D^N(t,s)\overline{S}'_{N,j}
\end{pmatrix}.
\]

By implication, element $(i,j)$ in the matrix $\overline{SD}(t,s)\overline{S}'$ is given by

\[
[\overline{SD}(t,s)\overline{S}']_{i,j} = \sum_{k=1}^N \overline{S}_{i,k} \overline{S}'_{k,j} D^k(t,s).
\]

Now, we can insert the terms of $D^k(t,s)$ and separate into constant terms and state-dependent terms:

\[
[\overline{SD}(t,s)\overline{S}']_{i,j} = \sum_{k=1}^N \overline{S}_{i,k} \overline{S}'_{k,j} [\gamma^k + \delta^k \theta^P + \overline{Q}^k e^{-\Phi(s-t)}\overline{Q}^X] = \sum_{k=1}^N \overline{S}_{i,k} \overline{S}'_{k,j} [\gamma^k + \delta^k \theta^P] + \sum_{k=1}^N \overline{S}_{i,k} \overline{S}'_{k,j} \overline{Q}^k e^{-\Phi(s-t)}\overline{Q}^X.
\]

Here, $\overline{Q}^k e^{-\Phi(s-t)}\overline{Q}^X$ is simply a number given by the following sum

\[
\overline{Q}^k e^{-\Phi(s-t)}\overline{Q}^X = \sum_{l=1}^N \overline{Q}^k_{i,l} \overline{Q}^X_{l} e^{-\phi_l(s-t)}.
\]
Since we note that the diagonal property of \(e\) gives us that element \((i, j)\) in the matrix \(e^{-\Phi(T-s)} SD(t, s) e^{-\Phi(T-s)}\) is given by

\[
[S D(t, s)]_{i,j} = \sum_{k=1}^{N} S_{i,k} S_{k,j} (\gamma^k + \delta^k \theta^P) + \sum_{l=1}^{N} \sum_{k=1}^{N} Q_{l}^k Q_{l}^X e^{-\phi_l(s-t)}.
\]

Expand on this equation:

\[
[S D(t, s)]_{i,j} = \sum_{k=1}^{N} S_{i,k} S_{k,j} (\gamma^k + \delta^k \theta^P)
+ \sum_{l=1}^{N} \sum_{k=1}^{N} Q_{l}^1 Q_{1}^X e^{-\phi_1(s-t)} + \ldots + \sum_{l=1}^{N} \sum_{k=1}^{N} Q_{l}^N Q_{N}^X e^{-\phi_N(s-t)}
\]

Now, it follows that the last \(N\) terms can be rearranged to yield

\[
[S D(t, s)]_{i,j} = \sum_{k=1}^{N} S_{i,k} S_{k,j} (\gamma^k + \delta^k \theta^P) + \sum_{l=1}^{N} e^{-\phi_l(s-t)} \sum_{k=1}^{N} S_{i,k} S_{k,j} Q_{l}^k Q_{l}^X.
\]

If we return to the expression for the conditional covariance matrix

\[
V^P[X_T|X_t] = Q \int_{t}^{T} e^{-\Phi(T-s)} SD(t, s) SD' e^{-\Phi(T-s)} ds Q',
\]

we note that the diagonal property of \(e^{-\Phi(T-s)}\) gives us that element \((i, j)\) in the matrix \(e^{-\Phi(T-s)} SD(t, s) e^{-\Phi(T-s)}\) is given by

\[
[S D(t, s)]_{i,j} = \sum_{k=1}^{N} S_{i,k} S_{k,j} (\gamma^k + \delta^k \theta^P)
+ \sum_{l=1}^{N} e^{-\phi_l(s-t)} \sum_{k=1}^{N} S_{i,k} S_{k,j} Q_{l}^k Q_{l}^X e^{-\phi_l(s-t)}
\]

Since

\[
e^{-\phi_l(s-t)} e^{-\phi_l(s-t)} = e^{\phi_l (T-t)} e^{-\phi_l(s-t)} e^{-\phi_l(s-t)}
\]

Thus,

\[
e^{-\phi_l(s-t)} e^{-\phi_l(s-t)} = e^{-\phi_l(T-t)} e^{-\phi_l(s-t)}.
\]
it follows that element \((i, j)\) in the matrix \(e^{-\Phi(T-s)SD(t,s)}S e^{-\Phi(T-s)}\) can be written as

\[
[e^{-\Phi(T-s)SD(t,s)}S e^{-\Phi(T-s)}]_{i,j} = \left( \sum_{k=1}^{N} S_{i,k}S_{k,j}^T(\gamma^k + \delta^k \theta^P) \right) e^{-(\phi_i + \phi_j)(T-s)} + \left( \sum_{l=1}^{N} e^{-(\phi_i + \phi_j - \phi_l)(T-s)} e^{-\phi_l(T-t)} \sum_{k=1}^{N} S_{i,k}S_{k,j}^T Q^k_i Q^X_l \right).
\]

When integrating each element of \(e^{-\Phi(T-s)SD(t,s)}S e^{-\Phi(T-s)}\) w.r.t. \(s\), we end up having to solve two types of integrals.

The first type of integral is given by

\[
\mathcal{E}_{i,j}^1(T-t) = \int_t^T e^{-(\phi_i + \phi_j)(T-s)} ds = \left\{ \begin{array}{ll}
\left[ \frac{1}{\phi_i + \phi_j} e^{-(\phi_i + \phi_j)(T-s)} \right]_t^T & \text{if } \phi_i + \phi_j \neq 0 \\
T - t & \text{if } \phi_i + \phi_j = 0.
\end{array} \right.
\]

The second type of integral is given by

\[
\mathcal{E}_{i,j,l}^2(T-t) = e^{-\phi_l(T-t)} \int_t^T e^{-(\phi_i + \phi_j - \phi_l)(T-s)} ds = \left\{ \begin{array}{ll}
e^{-\phi_l(T-t)} \left[ \frac{1}{\phi_i + \phi_j - \phi_l} e^{-(\phi_i + \phi_j - \phi_l)(T-s)} \right]_t^T & \text{if } \phi_i + \phi_j - \phi_l \neq 0 \\
e^{-\phi_l(T-t)}(T - t) & \text{if } \phi_i + \phi_j - \phi_l = 0.
\end{array} \right.
\]

With these terms defined, the integral of element \((i, j)\) in the matrix \(e^{-\Phi(T-s)SD(t,s)}S e^{-\Phi(T-s)}\) equals

\[
\int_t^T [e^{-\Phi(T-s)SD(t,s)}S e^{-\Phi(T-s)}]_{i,j} ds = \left( \sum_{k=1}^{N} S_{i,k}S_{k,j}^T(\gamma^k + \delta^k \theta^P) \right) \mathcal{E}_{i,j}^1(T-t) + \sum_{l=1}^{N} \mathcal{E}_{i,j,l}^2(T-t) \sum_{k=1}^{N} S_{i,k}S_{k,j}^T Q^k_i Q^X_l.
\]

If we let \(\mathbf{V}(t, T)\) denote the matrix of these individual terms, the conditional covariance matrix in equation (2) is obtained as

\[
V^F[X_T|X_t] = Q\mathbf{V}(t, T)Q'.
\]
3.1 Steps in Calculating $V(t, T)$

(i) Define the model structure: $K^P$, $\theta^P$, $\Sigma$, $\gamma$, and $\delta$.

(ii) Make the eigenvalue decomposition of $K^P$, which delivers $\phi_1, \ldots, \phi_N$ and $Q$.

(iii) Calculate $E^{1}_{i,j,l}(T-t)$ for all $i, j$ and $E^{2}_{i,j,l}(T-t)$ for all $i, j, l$. Normally, there is only one time horizon to consider. For example with weekly data, $T-t$ is typically fixed at $\frac{1}{52}$ throughout the estimation. Similarly, monthly data is handled by fixing $T-t = \frac{1}{12}$ throughout.

(iv) Calculate $Q_i$ for all $i$ and calculate $S, S'$, and $Q_X$.

(v) Now, make loop to calculate the individual elements of the $V(t,T)$-matrix.

(vi) Finally, this is converted into $V^P[X_T|X_t]$ using equation (3).

Note, that the matrix $\left( \sum_{k=1}^{N} S_{i,k} S'_{k,j}(\gamma^k + \delta^k \theta^P) \right) E^{1}_{i,j}(T-t)$ only has to be calculated once upfront. Thus, denote this

$$V^c(t, T)_{i,j} = \left( \sum_{k=1}^{N} S_{i,k} S'_{k,j}(\gamma^k + \delta^k \theta^P) \right) E^{1}_{i,j}(T-t).$$

Furthermore, to make the calculation of $V^P[X_T|X_t]$ as easy as possible as $X_t$ varies through the sample, it is useful to calculate the following matrices for each state variable $l = 1, \ldots, N$ upfront

$$V^{1,l}(t, T)_{i,j} = \sum_{k=1}^{N} S_{i,k} S'_{k,j} Q_l$$

and

$$V^{2,l}(t, T)_{i,j} = E^{2}_{i,j,l}(T-t).$$

Now, the individual elements in $V(t,T)$ are given by

$$V(t, T)_{i,j} = V^c(t, T)_{i,j} + \sum_{l=1}^{N} V^{2,l}(t, T)_{i,j} V^{1,l}(t, T)_{i,j} Q_l^X.$$  

In the three-factor AFNS models with stochastic volatility, there will be a total of $1 + 3 + 3 = 7$ simple $3 \times 3$ matrices to calculate up front and pass through to the function that calculates the conditional covariance matrix at each observation date.

4 The Unconditional Covariance Matrix in Affine Models

For the unconditional second moments to exist, we need stationarity. Under that assumption, we can let $T-t \to \infty$ in the calculations above. This implies that $E^{2}_{i,j,l}(\infty) = 0$ for all $i, j, l$.  

$^5$Superscript $c$ is used to indicate that this term remains constant during the Kalman filter algorithm.
Furthermore, it follows that
\[ E_{i,j}(\infty) = \frac{1}{\phi_i + \phi_j}. \]
Thus,
\[ \nabla(\infty)_{i,j} = \nabla'(\infty)_{i,j} = \left( \sum_{k=1}^{N} S_{i,k} S'_{k,j}(\gamma_k + \delta_k \theta) \right) E_{i,j}(\infty). \]
Hence, the unconditional covariance matrix is given by
\[ V[X_t] = Q\nabla(\infty)Q'. \]

5 The Conditional Covariance Matrix in Affine Gaussian Models

In this section, we derive the analytical formula for the conditional covariance matrix in Gaussian $A_0(N)$ models. This is a specialization of the general results presented above. The result nests all feasible specifications of the Gaussian AFNS model class introduced in Christensen, Diebold, and Rudebusch (2011).

To begin the analysis, let the state variables $X_t = (X^1_t, \ldots, X^N_t)$ have dynamics described by the following affine stochastic differential equation
\[ dX_t = K\theta - X_t dt + \Sigma dW_t^P. \]
Here, the volatility matrix $\Sigma$ is constant which indicates that we are in the Gaussian $A_0(N)$ class of models.

For $X_t$ to be stationary, the real component of the eigenvalues of $K\theta$ must be positive. This is imposed in all estimations in Christensen at al. (2011).

It is a well-established fact that the conditional mean for multi-dimensional affine diffusion processes is given by
\[ E^P[X_T | X_t] = (I - \exp(-K\theta(T-t)))\theta + \exp(-K\theta(T-t))X_t, \]
where $\exp(-K\theta(T-t))$ is a matrix exponential. Since $\exp(-K\theta(T-t))$ is easily calculated numerically and only has to be calculated once upfront, there is no need for an analytical formula for this expression. Furthermore, due to the imposed stationarity of $X_t$, the unconditional mean is simply given by
\[ E[X_t] = \theta. \]

Similarly, it is well-established that the conditional covariance matrix in affine Gaussian models is given by
\[ V^P[X_T | X_t] = \int_0^{T-t} e^{-K\theta \Sigma \Sigma'} e^{-(K\theta')'s} ds. \]
To get to analytical formulas for this integral, start by assuming that $K^P$ is diagonalizable

$$\exp(K^P) = Q\Phi Q^{-1},$$

where $\Phi$ is a diagonal matrix containing the eigenvalues of $K^P$, while $Q$ is the matrix of eigenvectors of $K^P$.

Since $e^{-K^P s}$ equals $Qe^{-\Phi s}Q^{-1}$, we can replace $e^{-K^P s}$ in the integral for the conditional covariance matrix to obtain

$$V^P[X_T|X_t] = \int_0^{T-t} Qe^{-\Phi s}Q^{-1}\Sigma\Sigma'[Qe^{-\Phi s}Q^{-1}]' ds = Q \int_0^{T-t} e^{-\Phi s}Q^{-1}\Sigma\Sigma'(Q^{-1})'e^{-\Phi s} ds Q'.$$

Now, define the following $N \times N$ matrix

$$\overline{S} = Q^{-1}\Sigma\Sigma'(Q^{-1})'$$

and exploit that

$$e^{-\Phi s} = \begin{pmatrix} e^{-\phi_1 s} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{-\phi_N s} \end{pmatrix}$$

to see that element $(i, j)$ in $e^{-\Phi s}Q^{-1}\Sigma\Sigma'(Q^{-1})'e^{-\Phi s}$ is given by

$$[e^{-\Phi s}\overline{S}e^{-\Phi s}]_{i,j} = \overline{S}_{i,j} e^{-(\phi_i + \phi_j)s}.$$

The integral of this term is

$$\int_0^{T-t} [e^{-\Phi s}\overline{S}e^{-\Phi s}]_{i,j} ds = \frac{\overline{S}_{i,j}}{\phi_i + \phi_j} [1 - e^{-(\phi_i + \phi_j)(T-t)}].$$

Now, combine the analytical result for the individual integrals into the matrix $\overline{V}(t, T)$

$$\overline{V}(t, T) = \begin{pmatrix} \frac{\overline{S}_{1,1}}{2\phi_1}[1 - e^{-2\phi_1(T-t)}] & \ldots & \frac{\overline{S}_{1,N}}{\phi_1 + \phi_N}[1 - e^{-(\phi_1 + \phi_N)(T-t)}] \\ \vdots & \ddots & \vdots \\ \frac{\overline{S}_{N,1}}{\phi_1 + \phi_N}[1 - e^{-(\phi_1 + \phi_N)(T-t)}] & \ldots & \frac{\overline{S}_{N,N}}{2\phi_N}[1 - e^{-2\phi_N(T-t)}] \end{pmatrix}.$$

Finally, the conditional covariance matrix itself can be calculated as

$$V^P[X_T|X_t] = Q\overline{V}(t, T)Q'.$$

Due to the imposed stationarity of $X_t$, the limit of $\overline{V}(t, T)$ for $T - t \to \infty$ exists and is given
by

$$\mathbf{V}(\infty) = \begin{pmatrix}
\frac{\bar{S}_{1,1}}{2\phi_1} & \ldots & \frac{\bar{S}_{1,N}}{\phi_1 + \phi_N} \\
\vdots & \ddots & \vdots \\
\frac{\bar{S}_{N,1}}{\phi_1 + \phi_N} & \ldots & \frac{\bar{S}_{N,N}}{2\phi_N}
\end{pmatrix}.$$ 

Hence, the unconditional covariance matrix can be obtained as

$$V^p[X_t] = Q\mathbf{V}(\infty)Q'.$$

References


