Term Structure Modeling and the Lower Bound Problem

Day 2: The Lower Bound Problem

Lecture II.1

European University Institute
Florence, September 8, 2015

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Day 2: The Lower Bound Problem

1. Lecture:
   - Estimating Shadow-Rate Term Structure Models with Near-Zero Yields

2. Lecture:
   - Modeling Yields at the Zero Lower Bound: Are Shadow Rates the Solution?

3. Lecture:
   - How Efficient is the Extended Kalman Filter at Estimating Shadow-Rate Models?

4. Lecture:
   - Staying at Zero with Affine Processes: An Application to Term Structure Modeling

5. Lecture:
   - Linear-Rational Term Structure Models
In recent years, a number of the world’s most prominent central banks have held their conventional policy rates at or near their effective lower bound. This makes central bank response functions asymmetric. Thanks to investors’ forward-looking behavior, this gets reflected in bond yields and returns. As a consequence, bond yields exhibit asymmetric behavior near the lower bound. This is a problem for standard Gaussian affine models. The term structure literature offers a few solutions:

- Shadow-rate models
- Square-root models
- Stay-at-zero affine models
- Linear-rational models

Today, we will study the lower bound problem and the models above.
U.S. and Japanese yields respect the zero lower bound.

Characteristics:
- Systematic upward sloping yield curve near ZLB.
- Asymmetric behavior of yield shocks near ZLB.
- Clear compression in yield volatility near ZLB.

Today’s models should be suitable for this type of data.
For U.K. gilt yields the evidence is mixed:

- Is there compression in yield volatility?
- Is there a lower bound?
- Yield curve is not always upward sloping near the lower bound.
For German bund yields and Swiss Confederation bond yields, markets are truly testing whether there is a lower bound.

Also, these yield curves are not always upward sloping.

In such cases, Christensen and Krogstrup (2015) argue that a Gaussian AFNS model appears to be warranted.
Shown are differences between largest and smallest observed two-year yields during 91-day windows.

U.S., U.K., and Japanese series are shown to the left, while German and Swiss series are shown to the right.

Note that German and Swiss yields have periodic bursts of high volatility since 2009. That’s a challenge!
Problem: Standard Gaussian term structure models do not restrict yields to respect a lower bound!

Existing models that do respect a lower bound:

- Shadow-rate models
- Stochastic-volatility models with square-root processes
- Gaussian quadratic models
- Stay-at-zero affine models
- Linear-rational models

Ultimate empirical question: Which model class performs well during normal times and lower bound episodes?

In this paper, we focus on shadow-rate models. Away from the lower bound, they reduce to standard Gaussian affine models. Thus, the huge existing empirical term structure literature remains relevant with a shadow-rate tweak in the current low-yield environment.
Black (1995) shadow-rate term structure models ensure nonnegative yields but have been hard to estimate.

We introduce a tractable shadow-rate arbitrage-free Nelson-Siegel model—the B-AFNS model—using an option-based approach from Krippner (2013).

We estimate the B-AFNS model on Japanese yields.

Our option-based implementation of the B-AFNS model provides a very close approximation to Black arbitrage-free bond pricing.

The B-AFNS model outperforms the standard Gaussian model.

The shadow short rate, which has been used to measure the stance of monetary policy, is sensitive to model specification.
Shortly before his death in 1995, Fisher Black wrote a paper on the impossibility of negative nominal interest rates in a world with currency.

To handle this constraint, Black proposed using standard tools to model a shadow rate, $s_t$, that may be negative, while the observed short rate would be its truncated version due to the option to hold currency at no cost:

$$ r_t = \max\{s_t, 0\}. $$

Numerically, this nonnegativity is straightforward to handle—but computation can be prohibitively burdensome.


Main problem: None of these studies are easily extended to three-factor settings due to computational challenges.
Krippner (2013) provides an option-based approximation to Black’s shadow-rate concept.  

**Key assumptions:**
- There exists a shadow short-rate process, $s_t$, that we only observe when it is nonnegative: $r_t = \max\{s_t, 0\}$.  
- There exists a continuum of *unobserved* shadow discount bonds, $P(t, T + \delta)$, that earn a return equal to $s_t$.  
- Currency exists and can be held at no cost.  

**Key insight:** The existence of currency implies an observed bond price that is equivalent to a portfolio of two assets
- A long position in the corresponding shadow discount bond;  
- A short position in an American call option with strike 1 and the shadow discount bond as the underlying asset.
Shadow-rate zero-coupon bond price, $P(t, T)$, may trade above par.

Observed zero-coupon bond price, $\underline{P}(t, T)$, will not rise above par, so nonnegative yields will never be observed.

Relationship between the two bond prices:

$$\underline{P}(t, T) = P(t, T) - C^A(t, T, T; 1),$$

where $C^A(t, t, T; 1)$ is the value of an American call option at time $t$ with maturity $T$ and strike price 1 written on the shadow bond maturing at $T$.

In a world with currency, bond investors effectively sell off the possible gain from above-par bond prices.
Value of American option and constrained bond as a function of the shadow bond price.

**Problem:** Early exercise premium is difficult to value.

**Solution:** The last incremental forward rate of any bond is nonnegative due to the future existence of currency and can be isolated ...
Krippner introduces an auxiliary bond price equation

\[ P^{aux.}(t, T + \delta) = P(t, T + \delta) - C^E(t, T, T + \delta; 1), \]

where \( C^E(t, T, T + \delta; 1) \) is the value of a European call option at time \( t \) with maturity \( T \) and strike price 1 written on the shadow discount bond maturing at \( T + \delta \).

By letting \( \delta \to 0 \), the last incremental nonnegative forward rate, denoted \( f(t, T) \), is identified as

\[ f(t, T) = \lim_{\delta \to 0} \left[ -\frac{d}{d\delta} P^{aux.}(t, T + \delta) \right]. \]

Krippner uses this to define observed bond prices that are approximately arbitrage-free:

\[ P^{app.}(t, T) = e^{-\int_t^T f(t,s)ds}. \]

NB: This price is only approximately arbitrage-free because the auxiliary bond price, \( P^{aux.}(t, T + \delta) \), is not identical to the true observed bond price, \( P(t, T) \), discussed on the previous slides.
Krippner shows that the nonnegative instantaneous forward rate of the observed bond prices is

$$f(t, T) = f(t, T) + z(t, T),$$

where $f(t, T)$ is the instantaneous forward rate on the shadow bond, that may go negative, while $z(t, T)$ is given by

$$z(t, T) = \lim_{\delta \to 0} \left[ \frac{d}{d\delta} \left\{ \frac{C^E(t, T, T + \delta; 1)}{P(t, T + \delta)} \right\} \right],$$

where

- $P(t, T + \delta)$ is the price of the shadow discount bond maturing at $T + \delta$.
- $C^E(t, T, T + \delta; 1)$ is the value of the European call option maturing at $T$ with $P(t, T + \delta)$ as the underlying price.

**Note:** These formulas have general validity independent of model-dimension and assumed factor dynamics.
In the corporate credit literature, researchers typically care about the value process $V_t$ of a firm and its volatility $\sigma_V$. However, we never observe this value process.

Instead, what we actually observe are the prices of the firm’s debt and equity, which are derivative contracts with $V_t$ as the underlying asset process, see Merton (1974).

Researchers use those derivative prices combined with Merton-like pricing formulas to draw inference about the value and parameters of the $V_t$ process.

**Bottom line:** Provided people have few reservations about this type of financial analysis, there is no obvious reason not to embrace shadow-rate models for similarly useful monetary analysis when yields are near zero.

Now, let’s make this operational ...
For Gaussian factor dynamics, Krippner derives the nonnegative instantaneous forward rate:

\[
  f(t, T) = f(t, T)\Phi\left(\frac{f(t, T)}{\omega(t, T)}\right) + \omega(t, T)\frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} \left[ \frac{f(t, T)}{\omega(t, T)} \right]^2 \right),
\]

where

\[
  \omega(t, T)^2 = \frac{1}{2} \lim_{\delta \to 0} \frac{\partial^2 \nu(t, T, T + \delta)}{\partial \delta^2}
\]

and \(\nu(t, T, T + \delta)\) is the conditional variance in the shadow bond option price formula.

To make this operational, we need:

- A Gaussian model;
- The analytical formula for bond option prices within the model;
- The limit of the double derivative of the conditional variance in the bond option formula w.r.t. difference btw. option and bond maturity.

Enter the AFNS model class!
Proposition: If the risk-free rate is defined by

\[ r_t = L_t + S_t \]

and the \( Q \)-dynamics of \( X_t = (L_t, S_t, C_t) \) are given by

\[
\begin{pmatrix}
\frac{dL_t}{dt} \\
\frac{dS_t}{dt} \\
\frac{dC_t}{dt}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & \lambda & -\lambda \\
0 & 0 & \lambda
\end{pmatrix}
\begin{pmatrix}
\theta_1^Q \\
\theta_2^Q \\
\theta_3^Q
\end{pmatrix} -
\begin{pmatrix}
L_t \\
S_t \\
C_t
\end{pmatrix}
\]

\[ dt + \Sigma dW_t^Q, \]

where \( \Sigma \) is a constant matrix, then zero-coupon yields have the Nelson-Siegel factor structure:

\[
y_t(\tau) = L_t + \left(1 - e^{-\lambda \tau}\right) S_t + \left(1 - e^{-\lambda \tau}\right) - e^{-\lambda \tau} C_t - \frac{A(\tau)}{\tau}.
\]

- This defines the AFNS model class.
- The constant yield-adjustment term, \( A(\tau)/\tau \), ensures absence of arbitrage.
- Nice analytical pricing formulas to work with.
In the shadow-rate AFNS class of models, the shadow rate is
\[ s_t = L_t + S_t, \]
while the \( Q \)-dynamics are as in the original AFNS model.

The instantaneous shadow forward rates are given by
\[ f(t, T) = L_t + e^{-\lambda(T-t)}S_t + \lambda(T - t)e^{-\lambda(T-t)}C_t + A^f(t, T), \]
where \( A^f(t, T) \) is an analytical function of model parameters.

The European call option with maturity \( T \) and strike price \( K \) written on the zero-coupon shadow bond maturing at \( T + \delta \) is given by
\[
C(t, T, T + \delta; K) = E_t^Q \left[ e^{-\int_t^T s_u du} \max\{P(T, T + \delta) - K, 0\} \right] = P(t, T + \delta)\Phi(d_1) - KP(t, T)\Phi(d_2),
\]
\[
d_1 = \ln \left( \frac{P(t, T + \delta)}{P(t, T)K} \right) + \frac{1}{2}\nu(t, T, T + \delta), \quad d_2 = d_1 - \sqrt{\nu(t, T, T + \delta)},
\]
and \( \nu(t, T, T + \delta) \) is the conditional variance mentioned earlier.
Recall that the nonnegative instantaneous forward rate in any Gaussian model within the option-based shadow-rate framework is
\[ f(t, T) = f(t, T)\Phi\left(\frac{f(t, T)}{\omega(t, T)}\right) + \omega(t, T)\frac{1}{\sqrt{2\pi}} \exp\left( - \frac{1}{2} \left[ \frac{f(t, T)}{\omega(t, T)} \right]^2 \right). \]

Within the shadow-rate AFNS model, we obtain
\[
\omega(t, T)^2 = \frac{1}{2} \lim_{\delta \to 0} \frac{\partial^2 v(t, T, T + \delta)}{\partial \delta^2} = \sigma_{11}^2 (T - t) + (\sigma_{21}^2 + \sigma_{22}^2) \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} + (\sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2) \times \left[ \frac{1 - e^{-2\lambda(T-t)}}{4\lambda} - \frac{1}{2}(T - t)e^{-2\lambda(T-t)} - \frac{1}{2}\lambda(T - t)^2e^{-2\lambda(T-t)} \right] + 2\sigma_{11}\sigma_{21} \frac{1 - e^{-\lambda(T-t)}}{\lambda} + 2\sigma_{11}\sigma_{31} \left[ - (T - t)e^{-\lambda(T-t)} + \frac{1 - e^{-\lambda(T-t)}}{\lambda} \right] + (\sigma_{21}\sigma_{31} + \sigma_{22}\sigma_{32}) \left[ - (T - t)e^{-2\lambda(T-t)} + \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} \right].
\]

To be consistent with the notation in Kim and Singleton (2012), we refer to this as the B-AFNS(3) model class.
The yield-to-maturity is defined the usual way as

\[ y_t(\tau) = \int_t^{t+\tau} f_t(s) ds \]

\[ = \frac{1}{\tau} \int_t^{t+\tau} \left[ f_t(s) \Phi \left( \frac{f_t(s)}{\omega(s)} \right) + \omega(s) \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{1}{2} \left( \frac{f_t(s)}{\omega(s)} \right)^2 \right) \right] ds. \]

This is the measurement equation in the Kalman filter.
We complete the model by specifying the link between the risk-neutral and real-world yield dynamics using the essentially affine risk premiums introduced by Duffee (2002):

\[
\begin{pmatrix}
    dL_t \\
    dS_t \\
    dC_t
\end{pmatrix} =
\begin{pmatrix}
    \kappa_{11}^P & \kappa_{12}^P & \kappa_{13}^P \\
    \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P \\
    \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P
\end{pmatrix} \begin{pmatrix}
    \theta_1^P \\
    \theta_2^P \\
    \theta_3^P
\end{pmatrix} - \begin{pmatrix}
    L_t \\
    S_t \\
    C_t
\end{pmatrix} dt + \Sigma \begin{pmatrix}
    dW_t^{P,L} \\
    dW_t^{P,S} \\
    dW_t^{P,C}
\end{pmatrix},
\]

where \( \Sigma \) is a lower triangular matrix, i.e. the maximally flexible AFNS specification with 19 model parameters.

- This is the transition equation in the Kalman filter.
- Due to the nonlinearity of the measurement equation, we use the extended Kalman filter.
- Importantly, the ease and robustness of the regular AFNS models carries over to its shadow-rate equivalent ⇒ The B-AFNS(3) model class is highly empirically tractable.
The Extended Kalman Filter Estimation (1)

- In shadow-rate models, zero-coupon bond yields are not affine functions of the state variables.
- Instead, the measurement equation takes the general form
  \[ y_t = z(X_t; \psi) + \varepsilon_t. \]
- In the extended Kalman filter, this equation is linearized through a first-order Taylor expansion around the best guess of \( X_t \) in the prediction step of the Kalman filter algorithm, \( X_{t|t-1} \).
- Thus, the approximation is given by
  \[ z(X_t; \psi) \approx z(X_{t|t-1}; \psi) + \left. \frac{\partial z(X_t; \psi)}{\partial X_t} \right|_{X_t=X_{t|t-1}} (X_t - X_{t|t-1}). \]
- Now, define
  \[ A_t(\psi) \equiv z(X_{t|t-1}; \psi) - \left. \frac{\partial z(X_t; \psi)}{\partial X_t} \right|_{X_t=X_{t|t-1}} X_{t|t-1}, \quad B_t(\psi) \equiv \left. \frac{\partial z(X_t; \psi)}{\partial X_t} \right|_{X_t=X_{t|t-1}}. \]
For the purposes of the Kalman filter, the measurement equation can be thought of as having an affine form:

\[ y_t = A_t(\psi) + B_t(\psi)X_t + \varepsilon_t. \]

Thus, \( B_t(\psi) \) as defined on the previous slide will be used in the Kalman filter algorithm.

Importantly, the error term in the Kalman filter is calculated directly (and not from the Taylor approximation)

\[ \nu_t = y_t - E[y_t|Y_{t-1}] \]
\[ = y_t - z(X_{t|t-1}; \psi). \]

**Note:** \( A_t(\psi) \) is not actually calculated or used.

Beyond these two adjustments, the steps in the algorithm proceed as previously described.
We extend the Kim-Singleton weekly sample (1995-2008) with data from Bloomberg until May 3, 2013, for 6 maturities.

We estimate one-, two-, and three-factor Gaussian models with and without the shadow-rate interpretation on this data.

Long sample of near-zero yields, ideal for our purposes!
We estimated all shadow-rate models with the extended and unscented Kalman filter.

In terms of RMSEs, differences are barely noticeable.

We conclude that the extended Kalman filter is efficient at estimating shadow-rate models.

This is also consistent with the findings in Christoffersen et al. (2014) regarding nonlinear filtering.

Further evidence from a simulation study will be provided later.
We assess the approximation provided by the option-based B-AFNS(3) model via Monte Carlo simulations.

**Input:** Estimated $\hat{K}^P$, $\hat{\Sigma}$, $\hat{\theta}^P$, $\hat{\lambda}$, and filtered $\hat{X}_t$.

The continuous-time $P$-dynamics are given by

$$dX_t = \hat{K}^P(\hat{\theta}^P - X_t)dt + \hat{\Sigma}dW_t^P.$$ 

These are approximated with the Euler approximation:

$$X_t^i = X_{t-1}^i + \hat{\kappa}^P_{ii}(\hat{\theta}_i^P - X_{t-1}^i)\Delta t + \hat{\kappa}^P_{ij}(\hat{\theta}_j^P - X_{t-1}^j)\Delta t + \hat{\sigma}_{ii}\sqrt{\Delta t}z_{it}^i + \hat{\sigma}_{ij}\sqrt{\Delta t}z_{jt}^i,$$

where $z_{it}^i, z_{jt}^j \sim N(0, 1)$, $\Delta t = 0.0001$, and $X_0 = \hat{X}_t$.

Simulated shadow and Black (1995) bond prices are given by

$$P_t^S(n) = P_{t-1}(n) \exp(- (X_1^t + X_2^t)\Delta t), \quad P_0^S(n) = 1$$

$$P_t^B(n) = P_{t-1}(n) \exp(- \max(X_1^t + X_2^t, 0)\Delta t), \quad P_0^B(n) = 1.$$

The resulting bond prices for any maturity $t > 0$ are:

$$P_t^S = \frac{1}{N} \sum_{n=1}^{N} P_t^S(n) \quad \text{and} \quad P_t^B = \frac{1}{N} \sum_{n=1}^{N} P_t^B(n).$$
We simulate $N = 25,000$ ten-year-long factor paths and calculate shadow and Black zero-coupon bond yields.

The difference between simulated and analytical shadow bond yields provides a measure of the simulation accuracy.

The difference between simulated Black and option-based yields measures the accuracy of the option-based approximation.
Average Absolute Simulation Error in Shadow Yields and Average Absolute Approximation Error in Observed Yields:

Averaged over 19 dates—first observation in each year in sample.

<table>
<thead>
<tr>
<th>Maturity in months</th>
<th>12</th>
<th>36</th>
<th>60</th>
<th>84</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shadow yields</td>
<td>0.14</td>
<td>0.29</td>
<td>0.44</td>
<td>0.55</td>
<td>0.72</td>
</tr>
<tr>
<td>Yields</td>
<td>0.09</td>
<td>0.22</td>
<td>0.43</td>
<td>0.82</td>
<td>2.25</td>
</tr>
</tbody>
</table>

- Differences (in bps) between simulated and option-implied nonnegative yields are only marginally larger than differences between simulated and analytical shadow yields.

- We conclude that the option-based B-AFNS(3) model provides a very close approximation to the Black model for our data.

- **Note:** If greater accuracy is required, Priebsch (2013) provides a second-order approximation, but it is very time consuming.
## Fit of Standard and Shadow-Rate Models (RMSE)

<table>
<thead>
<tr>
<th></th>
<th>Maturity in months</th>
<th>All yields</th>
<th>Max log $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
<td>12</td>
<td>24</td>
</tr>
<tr>
<td>One-factor models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V(1)</td>
<td>5.8</td>
<td>0.0</td>
<td>12.1</td>
</tr>
<tr>
<td>B-V(1)</td>
<td>4.9</td>
<td>0.2</td>
<td>10.9</td>
</tr>
<tr>
<td>Two-factor models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFNS(2)</td>
<td>5.9</td>
<td>0.0</td>
<td>9.0</td>
</tr>
<tr>
<td>B-AFNS(2)</td>
<td>6.6</td>
<td>0.3</td>
<td>8.9</td>
</tr>
<tr>
<td>Three-factor models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFNS(3)</td>
<td>0.0</td>
<td>2.4</td>
<td>0.2</td>
</tr>
<tr>
<td>B-AFNS(3)</td>
<td>0.4</td>
<td>2.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

- Standard one-, two-, and three-factor models, denoted $V(1)$, $AFNS(2)$, and $AFNS(3)$, respectively.
- Equivalent shadow-rate models, denoted $B-V(1)$, $B-AFNS(2)$, and $B-AFNS(3)$, respectively.
- Shadow-rate models have closer fit, higher log $L$. 
Fitted yield curves from standard and shadow-rate two- and three-factor AFNS models on July 1, 2005.

- For two-factor models, there is a clear visible gain from adopting a shadow-rate implementation.
- For three-factor models, the improvement is less clear.
The models tend to agree about the location of the shadow short rate when it is positive.

On the other hand, there is much disagreement about the shadow short rate when it is negative.
In the AFNS models, we assess the probability of negative future short rates at any horizon $t + \tau > t$.

- **Input**: Estimated $\hat{K}^P$, $\hat{\Sigma}$, $\hat{\theta}^P$, $\hat{\lambda}$, and filtered $\hat{X}_t$.
- The continuous-time $P$-dynamics are given by
  \[
  dX_t = \hat{K}^P (\hat{\theta}^P - X_t) dt + \hat{\Sigma} dW_t^P.
  \]
- The conditional mean of $X_t$ under the $P$-measure is
  \[
  E_t^P [X_{t+\tau}] = (I - \exp(-\hat{K}^P \tau)) \hat{\theta}^P + \exp(-\hat{K}^P \tau) \hat{X}_t.
  \]
- The conditional covariance matrix of $X_t$ under the $P$-measure is
  \[
  V_t^P [X_{t+\tau}] = \int_0^\tau e^{-\hat{K}^P s} \hat{\Sigma} \hat{\Sigma}' e^{-(\hat{K}^P)' s} ds.
  \]
- Now, the short rate is normally distributed:
  \[
  E_t^P [r_{t+\tau}] = (1 \ 1 \ 0) E_t^P [X_{t+\tau}], \quad V_t^P [r_{t+\tau}] = (1 \ 1 \ 0) V_t^P [X_{t+\tau}] \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
  \]
  \[
  P^P (r_{t+\tau} \leq 0) = \Phi \left( \frac{-E_t^P [r_{t+\tau}]}{\sqrt{V_t^P [r_{t+\tau}]}} \right).
  \]
Illustration of the probability that the short rate will be negative at a three-month forecast horizon according to the estimated AFNS(3) model.
In standard affine models, where zero-coupon yields are affine functions of the state variables, model-implied conditional predicted yield volatilities are given by the square root of

\[ V_t^P[y_T(\tau)] = \frac{1}{\tau^2} B(\tau)' V_t^P[X_T] B(\tau), \]

where
- \( T - t \) is the prediction period;
- \( \tau \) is the yield maturity;
- \( B(\tau) \) contains the yield factor loadings;
- \( V_t^P[X_T] \) is the conditional covariance matrix.

In shadow-rate models, zero-coupon bond yields are nonlinear functions of the state variables. Thus, model-implied conditional yield volatilities have to be generated via Monte Carlo Simulations.
Input: Estimated $\hat{K}_P$, $\hat{\Sigma}$, $\hat{\theta}_P$, $\hat{\lambda}$, and filtered $\hat{X}_t$.

The continuous-time $P$-dynamics are given by

$$dX_t = \hat{K}_P(\hat{\theta}_P - X_t)dt + \hat{\Sigma}dW_t^P.$$ 

The conditional mean of $X_T$ under the $P$-measure:

$$E_t^P[X_T] = (1 - \exp(-\hat{K}_P(T - t)))\hat{\theta}_P + \exp(-\hat{K}_P(T - t))\hat{X}_t.$$ 

The conditional covariance matrix of $X_T$ under the $P$-measure:

$$V_t^P[X_T] = \int_0^{T-t} e^{-\hat{K}_P s} \hat{\Sigma} \hat{\Sigma}' e^{-(\hat{K}_P)'s} ds.$$ 

We simulate $N = 1,000$ draws directly at the desired horizon:

$$X^n_T \sim N\left(E_t^P[X_T], V_t^P[X_T]\right), \quad n = 1, \ldots, 1000.$$ 

The resulting conditional bond volatility is:

$$\sqrt{\frac{1}{N-1} \sum_{n=1}^{N} (y(X^n_T, \tau) - \bar{y}(X_T, \tau))^2}, \quad \bar{y}(X_T, \tau) = \frac{1}{N} \sum_{n=1}^{N} y(X^n_T, \tau).$$
Standard Gaussian models assume constant yield volatility.

However, near the ZLB, there is a clear positive correlation between yield levels and volatilities.

Gaussian shadow-rate models can replicate this phenomenon for short- and medium-term yield maturities.
Conclusion

- We combine Krippner’s general option-based shadow-rate framework with the arbitrage-free Nelson-Siegel model class.
- This delivers highly tractable models that are approximately arbitrage-free and respect the zero lower bound for yields.
- We estimate one-, two-, and three-factor versions of the option-based shadow-rate AFNS model and matching standard models on a long sample of near-zero Japanese yields.
- We find that introducing the shadow-rate interpretation improves model fit and allows the models to capture other important aspects of yields near the ZLB, in particular the positive correlation between yield levels and volatilities.
- We show that the B-AFNS(3) model provides a very close approximation to the corresponding Black shadow-rate model.
- Finally, results for the shadow short rate are sensitive to factor dimension and dynamics. Thus, it is not likely to be a useful tool for assessing the stance of monetary policy near the ZLB.