Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment*

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Abstract

This paper develops a theory of how organizations make timing decisions. We consider a problem where an uninformed principal decides when to exercise an option and interacts with an informed but biased agent. This problem is common: examples include headquarters deciding when to close a plant, drill an oil well, or launch a product. Because time is irreversible, the direction of the agent's bias is crucial for communication and allocation of authority. When the agent favors late exercise, centralized decision-making, where the principal retains authority and communicates with the agent, often features full information revelation but inefficient delay. Delegation is never optimal in this case. In contrast, when the agent favors early exercise, communication under centralized decision-making is partial, while option exercise is unbiased or delayed. Delegation is optimal if the bias is small or delegation can be timed. Thus, delegating decisions such as plant closures is never optimal, while delegating decisions such as product launches may be optimal.

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1 Introduction

Many decisions in organizations deal with the optimal timing of taking a certain action. Because information in organizations is dispersed, the decision-maker needs to rely on the information of his better-informed subordinates who, however, may have conflicting preferences. Consider the following two examples of such settings. 1) In a typical hierarchical firm, top executives may be less informed than the product manager about the optimal timing of the launch of a new product. It would not be surprising for an empire-building product manager to be biased in favor of an earlier launch. 2) The CEO of a multinational corporation is contemplating when to shut down a plant in a struggling economic region. While the local plant manager is better informed about the prospects of the plant, he may be biased towards a later shutdown due to personal costs of relocation.

These examples share a common theme. An uninformed principal faces an optimal stopping-time problem (when to exercise a real option). An agent is better informed than the principal but is biased towards earlier or later option exercise. In this paper, we study how organizations make timing decisions in such a setting. We first examine the effectiveness of centralized decision-making, where the principal retains formal authority over the decision and gets information via communication with the agent ("cheap talk"). We next compare this with decentralized decision-making, where the principal delegates the decision to the agent, and study the optimal allocation of authority. Since most decisions that organizations make can be delayed and thus have option-like features, our analysis of pure timing decisions is relevant for organizational design more generally.

We show that the economics underlying this problem are quite different from those when the decision is static rather than dynamic, and the decision variable is scale of the action rather than a stopping time. In particular, there is a large asymmetry in the equilibrium properties of communication and decision-making and the optimal allocation of authority depending on the direction of the agent's bias. In the first example above, the agent is biased towards early exercise, while in the second example above, the agent is biased towards late exercise. Unlike in the static problem (e.g., Crawford and Sobel, 1982, and Dessein, 2002), the results for these two cases are not mirror images of each other. For example, within our framework, there is no benefit from delegating decisions for which the agent favors late exercise, such as plant closures, as opposed to decisions for which the agent favors early exercise, such as product launches.

Our setting combines the framework of real option exercise problems with the framework of cheap talk communication between an agent and a principal. The principal must decide when to exercise an option whose payoff depends on an unknown parameter. The agent knows the parameter, but the agent's payoff from exercise differs from the principal's due to a bias. If the

principal retains formal authority over the decision, he relies on communication with the agent: At any point in time, the agent sends a message to the principal about whether or not to exercise the option. Conditional on the received message and the history of the game, the principal chooses whether to exercise or wait. Importantly, not exercising today provides an option to get advice in the future. In equilibrium, the agent's communication strategy and the principal's exercise decisions are mutually optimal, and the principal rationally updates his beliefs about the agent's private information. In most of the paper, we look for stationary equilibria in this setting.

We show that when the agent is biased towards late exercise and the bias is not too high, there is often an equilibrium with full revelation of information. However, the equilibrium timing of the decision always involves delay relative to the principal's preferences. This is different from the static cheap talk setting of Crawford and Sobel (1982), where information is only partially revealed but the decision is conditionally optimal from the principal's standpoint. In contrast, when the agent is biased towards early exercise, all equilibria have a partition structure and thus feature incomplete revelation of information. Conditional on this incomplete information, the equilibrium exercise times are either unbiased or delayed from the principal's standpoint, despite the agent's bias towards early exercise.

The intuition for these strikingly different results for the two directions of the agent's bias lies in the nature of time as a decision variable. While the principal always has the choice to exercise at a point later than the present, he cannot do the reverse, i.e., exercise at a point earlier than the present. If the agent is biased towards late exercise, she can withhold information and reveal it later, exactly at the point where she finds it optimal to exercise the option. When the agent with a late exercise bias recommends exercise, the principal learns that it is too late to do so and is tempted to go back in time and exercise the option in the past. This, however, is not feasible, and hence the principal finds it optimal to follow the agent's recommendation. Knowing that, the agent communicates honestly, but communication occurs with delay. When the principal chooses whether to wait for the agent's recommendation to exercise, he trades off the value of information against the cost of delay. In our stationary setting, the principal always finds it optimal to wait for the agent's recommendation provided that the bias is not too large, and hence full revelation of information occurs. When we consider a non-stationary setting, the principal waits for the agent's recommendation up to a certain cutoff, and hence full revelation of information occurs up to a cutoff. Conversely, if the agent is biased towards early exercise, she does not benefit from withholding information, but when she discloses it, the principal can always postpone exercise if it is not in his best interest. Thus, only partial information revelation is possible.

These results have implications for the informativeness and timeliness of option exercise decisions in organizations where the principal has formal authority. First, other things equal, the

agent's information is likely to explain more variation in the timing of option exercise for decisions with a late exercise bias (e.g., shutting down a plant) than for decisions with an early exercise bias (e.g., launching a new product or making an acquisition). Second, decisions with a late exercise bias are always delayed relative to the optimal exercise time from the principal's perspective. In contrast, the timing of decisions with an early exercise bias is on average unbiased or delayed.

The asymmetric nature of time also has important implications for the optimal allocation of authority in organizations. In particular, we examine the principal's choice between delegating decision-making rights to the agent and retaining authority and communicating with the agent – the problem studied by Dessein (2002) in the context of static decisions. We show that if the agent favors late exercise, as in the case of a plant closure, the principal is always weakly better off keeping authority and communicating with the agent, rather than delegating the decision to the agent. This preference is strict in our non-stationary setting. This result is different from the result for static decisions, where delegation is optimal if the agent's bias is sufficiently small (Dessein, 2002). Intuitively, the inability to go back in time and act on the information before it is received allows the principal to commit to follow the agent's recommendation, i.e., to exercise exactly when the agent recommends to exercise. This commitment ability makes communication sufficiently effective, so that delegation has no further benefit. In fact, we show that the communication equilibrium in this case coincides with the solution under the optimal contract with commitment, and hence the ability to commit to any decision rule does not improve the principal's payoff.

In contrast, if the agent favors early exercise, as in the case of a product launch, delegation is optimal if the agent's bias is not too high. Intuitively, if the agent recommends exercise at her most preferred time, the principal is tempted to delay the decision. Unlike changing past decisions, changing future decisions is possible, and hence time does not have valuable built-in commitment. Thus, communication is not as efficient as in the case of a late exercise bias. As a consequence, delegation can now be optimal because it allows for more effective use of the agent's information. The trade-off between information and bias suggests that delegation is superior when the agent's bias is sufficiently small, similar to the argument for static decisions (Dessein, 2002).

We next allow the principal to time the delegation decision strategically, i.e., to choose the optimal timing of delegating authority to the agent. When the agent favors late exercise, the principal finds it optimal to retain authority forever: His built-in commitment power due to the inability to go back in time makes communication effective and eliminates the need for delegation. In contrast, when the agent favors early exercise, the principal finds it optimal to delegate authority to the agent at some point in time, and delegation occurs later when the agent's bias is higher. In fact, delegating authority at the right time implements the second-best, i.e., there is no mechanism that improves the principal's expected payoff over what he can achieve by simply delegating

authority at the right time. This result further emphasizes that the direction of the agent's bias is the main driver of the allocation of authority for timing decisions. This is different from static decisions, like choosing the scale of the project, where the key drivers of the allocation of authority are the magnitude of the agent's bias and the importance of her private information.

We also study the comparative statics of the communication equilibrium with respect to the parameters of the stochastic environment. We show that when the agent is biased towards early exercise, an increase in volatility or in the growth rate of the option payoff, as well as a decrease in the discount rate, lead to less information being revealed in equilibrium. Intuitively, these changes increase the value of the option to delay exercise and thereby effectively increase the conflict of interest between the principal and the agent with an early exercise bias. Finally, we show that given the same absolute bias, the principal is better off with an agent who favors late exercise.

The paper proceeds as follows. The remainder of this section discusses the related literature. Section 2 describes the setup and solves for the benchmark case of full information. Section 3 provides the analysis of the main model of communication under asymmetric information. Section 4 examines delegation. Section 5 considers comparative statics and other implications. Section 6 shows the robustness of the results to several versions of the model. Finally, Section 7 concludes.

Related literature

Our paper is related to the literature that analyzes decision-making in the presence of an informed but biased expert. The seminal paper in this literature is Crawford and Sobel (1982), who consider a cheap talk setting, where the expert sends a message to the decision-maker and the decision-maker cannot commit to the way he reacts to the message. Our paper differs from Crawford and Sobel (1982) in that communication between the expert and the decision-maker is dynamic and concerns the timing of option exercise, rather than a static decision such as choosing the scale of a project. To our knowledge, ours is the first paper that studies the problem of optimal timing in a cheap talk setting. Surprisingly, even though there is no flow of additional private information to the agent, equilibria differ substantially from those in Crawford and Sobel (1982).

By studying the choice between communication and delegation, our paper contributes to the literature on authority in organizations (e.g., Holmstrom, 1984; Aghion and Tirole, 1997; Dessein, 2002; Alonso and Matouschek, 2008). Gibbons, Matouschek, and Roberts (2013), Bolton and Dewatripont (2013), and Garicano and Rayo (2014) provide comprehensive reviews of this literature. Unlike Crawford and Sobel (1982), where the principal has no commitment power, the papers in this literature allow the principal to have some degree of commitment, although most of them rule out contingent transfers to the agent. Our paper is most closely related to Dessein

(2002), who assumes that the principal can commit to delegate full decision-making authority to the agent. Dessein (2002) studies the principal's choice between delegating the decision and communicating with the agent via cheap talk to make the decision himself, and shows that delegation dominates communication if the agent's bias is not too large. Relatedly, Harris and Raviv (2005, 2008) and Chakraborty and Yilmaz (2013) analyze the optimality of delegation in settings with two-sided private information. Alonso, Dessein, and Matouschek (2008, 2014) and Rantakari (2008) compare centralized and decentralized decision-making in a multidivisional organization that faces a trade-off between adapting divisions' decisions to local conditions and coordinating decisions across divisions.[1] Our paper contributes to this literature by studying delegation of timing decisions and showing that unlike in static settings, the optimality of delegation crucially depends on the direction of the agent's bias. In particular, unlike in the static problem, it is never optimal to delegate decisions where the agent has a delay bias. In contrast, delegating the decision at the right time implements the second-best if the agent has an early exercise bias.

Other papers in this literature assume that the principal can commit to a decision rule and thus focus on a partial form of delegation: the principal offers the agent a set of decisions from which the agent can choose her preferred one. These papers include Holmstrom (1984), Melumad and Shibano (1991), Alonso and Matouschek (2008), and Goltsman et al. (2009), among others. In Baker, Gibbons, and Murphy (1999) and Alonso and Matouschek (2007), the principal's commitment power arises endogenously through relational contracts. Guo (2014) studies the optimal mechanism without transfers in an experimentation setting where the agent prefers to experiment longer than the principal.[2] The optimal contract in her paper is time-consistent but becomes time-inconsistent if the agent prefers to experiment less than the principal, which is related to the asymmetry of our results in the direction of the agent's bias. Our paper differs from this literature because it focuses on the principal's choice between simple delegation and keeping the control rights and communicating with the agent. We derive the optimal mechanism under commitment as an intermediate result to study the role of delegation in organizations.

Several papers analyze dynamic extensions of Crawford and Sobel (1982). In Sobel (1985), Benabou and Laroque (1992), and Morris (2001), the advisor's preferences are unknown and her messages in prior periods affect her reputation with the decision-maker.[3] Aumann and Hart (2003), Krishna and Morgan (2004), Goltsman et al. (2009), and Golosov et al. (2014) consider settings

[1] See also Dessein, Garicano, and Gertner (2010) and Friebel and Raith (2010). Dessein and Santos (2006) study the benefits of specialization in the context of a similar trade-off, but do not analyze strategic communication.

[2] Halac, Kartik, and Liu (2013) also analyze optimal dynamic contracts in an experimentation problem, but in a different setting and allowing for transfers.

[3] Ottaviani and Sorensen (2006a,b) study a single-period reputational cheap talk setting, where the expert is concerned about appearing well-informed. Boot, Milbourn, and Thakor (2005) compare delegation and centralization when the agent's reputational concerns can distort her recommendations on whether to accept the project.

with persistent private information where the principal actively participates in communication by either sending messages himself or taking an action following each message of the advisor.[4] Our paper differs from this literature because of the dynamic nature of the decision problem: the decision variable is the timing of option exercise, rather than a static variable. The inability to go back in time creates an implicit commitment device for the principal to follow the advisor's recommendations and thereby improves communication, a feature not present in prior literature.

Finally, our paper is related to the literature on option exercise in the presence of agency problems. Grenadier and Wang (2005), Gryglewicz and Hartman-Glaser (2013), and Kruse and Strack (2015) study such settings but assume that the principal can commit to contracts and make contingent transfers to the agent, which makes the problem conceptually different from ours. Several papers study signaling through option exercise.[5] They assume that the decision-maker is informed, while in our setting the decision-maker is uninformed.

2 Model setup

A firm (or an organization, more generally) has a project and needs to decide on the optimal time to implement it. There are two players, the uninformed party (principal, P) and the informed party (agent, A). Both parties are risk-neutral and have the same discount rate $r > 0$. Time is continuous and indexed by $t \in [0, \infty)$. The persistent type θ is drawn and learned by the agent at the initial date $t = 0$. The principal does not know θ. It is common knowledge that θ is a random draw from the uniform distribution over $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, where $0 \leq \underline{\theta} < \bar{\theta}$. Without loss of generality, we normalize $\bar{\theta} = 1$. For much of the paper, we also assume $\underline{\theta} = 0$.

We focus on the case of a call option. We will refer to it as the option to invest, but it can capture any perpetual American call option, such as the option to do an IPO or to launch a new product. We also extend the analysis to a put option (e.g., if the decision is about shutting down a plant) and show that the main results continue to hold (see Section 6.3).

Specifically, the exercise at time t generates the payoff to the principal of $\theta X(t) - I$, where $I > 0$ is the exercise price (the investment cost), and $X(t)$ follows geometric Brownian motion with drift μ and volatility σ:

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dB(t),$$

[4] Ely (2015) analyzes a setting with stochastically changing private information, where the informed party can commit to an information policy that shapes the beliefs of the uninformed party.

[5] Grenadier and Malenko (2011), Morellec and Schuerhoff (2011), Bustamante (2012), Grenadier, Malenko, and Strebulaev (2013).

where $\sigma > 0$, $r > \mu$, and $dB(t)$ is the increment of a standard Wiener process. The starting point $X(0)$ is low enough, so that immediate exercise does not happen. Process $X(t)$, $t \geq 0$ is observable by both the principal and the agent. As an example, consider an oil-producing firm that owns an oil well and needs to choose the optimal time to begin drilling. The publicly observable oil price process is represented by $X(t)$. The top management of the firm has authority over the decision to drill. The regional manager has private information about how much oil the well contains (θ), which stems from her local knowledge and prior experience with neighboring wells.

While the agent knows θ, she is biased. Specifically, upon exercise, the agent receives the payoff of $\theta X(t) - I + b$, where $b \neq 0$ is her commonly known bias. Positive bias $b > 0$ means that the agent is biased towards early exercise: her personal exercise price $(I - b)$ is lower than the principal's (I), so her most preferred timing of exercise is earlier than the principal's for any θ. Similarly, negative bias $b < 0$ means that the agent favors late exercise. These preferences can be viewed as reduced-form implications of an existing revenue-sharing agreement.[6] An alternative way to model the conflict of interest is to assume that $b = 0$ but the players discount the future using different discount rates. An early exercise bias corresponds to the agent being more impatient than the principal, $r_A > r_P$, and vice versa. We have analyzed the setting with different discount rates and shown that the results are identical to those in the bias setting (see Section 6.1).

The principal has formal authority over when to exercise the option. Furthermore, the organization is assumed to have a resource, controlled by the principal, which is critical for the implementation of the project. This resource is the reason why the agent cannot implement the project without the principal's approval. For most of the paper, we adopt an incomplete contracting approach. First, we consider the advising setting, where the principal has no commitment power and can only rely on informal "cheap talk" communication with the agent. This problem is the option exercise analogue of Crawford and Sobel's (1982) cheap talk model. Then, in Section 4, we relax this assumption by allowing the principal to grant the agent authority over the exercise of the option. This problem is the option exercise analogue of Dessein's (2002) analysis on authority and communication. Finally, as an intermediate result, in Section 4.1, we derive the optimal mechanism if the principal could commit to any decision rule.

Following most of the literature on delegation (e.g., Holmstrom, 1984; Aghion and Tirole, 1997; Dessein, 2002; Alonso and Matouschek, 2008), we do not allow the principal to make contingent transfers to the agent. In practice, decision-making inside firms mostly occurs via the allocation of control rights and informal communication, and hence it is important to study

[6] For example, suppose that the principal supplies financial capital $\hat{I}$, the agent supplies human capital ("effort") valued at $\hat{e}$, and the principal and the agent hold fractions α_P and α_A of equity of the realized value from the project. Then, at exercise, the principal's (agent's) expected payoff is $\alpha_P \theta X(t) - \hat{I}$ ($\alpha_A \theta X(t) - \hat{e}$). This is analogous to the specification in the model with $I = \frac{\hat{I}}{\alpha_P}$ and $b = \frac{\hat{I}}{\alpha_P} - \frac{\hat{e}}{\alpha_A}$.

such settings. A plausible rationale for this is that the reallocation of control rights is a simple solution to the problem of complexity of contracts with contingent transfers. Indeed, agents in organizations usually make many decisions, and writing complex contracts that specify transfers for all decisions and all possible outcomes of each decision is prohibitively costly.[7] Furthermore, in some organizational settings, such as in government, transfers are explicitly ruled out by law.

We start by analyzing the advising setting, where authority is not contractible. The timing is as follows. At each time t, knowing the type $\theta \in \Theta$ and the history of the game $\mathcal{H}_t$, the agent decides on a message $m(t) \in M$ to send to the principal, where M is a set of messages. At each t, the principal decides whether to exercise the option or not, given $\mathcal{H}_t$ and the current message $m(t)$. If the principal exercises the option, the game ends. If the principal does not exercise the option, the game continues. Because the game ends when the principal exercises the option, we can only consider histories such that the option has not yet been exercised. Then, the history of the game at time t has two components: the sample path of the public state $X(t)$ and the history of messages of the agent. Formally, it is represented by $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \{X(s), s \leq t; m(s), s < t\}$. Thus, the strategy m of the agent is a family of functions $(m_t)_{t\geq 0}$ such that for any t function m_t maps the agent's information set at time t into the message she sends to the principal: $m_t : \Theta \times \mathcal{H}_t \rightarrow M$. The strategy e of the principal is a family of functions $(e_t)_{t\geq 0}$ such that for any t function e_t maps the principal's information set at time t into the binary exercise decision: $e_t : \mathcal{H}_t \times M \rightarrow \{0, 1\}$. Here, $e_t = 1$ stands for "exercise" and $e_t = 0$ stands for "wait." Let $\tau(e) \equiv \inf\{t : e_t = 1\}$ denote the stopping time implied by strategy e of the principal. Finally, let $\mu(\theta|\mathcal{H}_t)$ and $\mu(\theta|\mathcal{H}_t, m(t))$ denote the updated probability that the principal assigns to the type of the agent being θ given the history $\mathcal{H}_t$ before and after getting message $m(t)$, respectively.

Heuristically, the timing of events over an infinitesimal time interval $[t, t+dt]$ prior to option exercise can be described as follows: (1) The nature determines the realization of X_t. (2) The agent sends message $m(t) \in M$ to the principal. (3) The principal decides whether to exercise the option or not. If the option is exercised, the principal obtains the payoff of $\theta X_t - I$, the agent obtains the payoff of $\theta X_t - I + b$, and the game ends. Otherwise, the game continues, and the nature draws $X_{t+dt} = X_t + dX_t$.

This is a dynamic game with observed actions (messages and the exercise decision) and incomplete information (type θ of the agent). We focus on equilibria in pure strategies. The equilibrium concept is Perfect Bayesian Equilibrium in Markov strategies, defined as:

Definition 1. *Strategies $m^* = \{m_t^*, t \geq 0\}$ and $e^* = \{e_t^*, t \geq 0\}$, beliefs μ^*, and a message space*

[7] In Section 6.2, we allow the principal to offer simple compensation contracts and show that the setting and implications of our paper are robust.

*M constitute a **Perfect Bayesian equilibrium in Markov strategies (PBEM)** if:*

1. For every t, H_t, $\theta \in \Theta$, and strategy m,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I+b\right)|\mathcal{H}_t,\theta,\mu^*\left(\cdot|\mathcal{H}_t\right),m^*,e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I+b\right)|\mathcal{H}_t,\theta,\mu^*\left(\cdot|\mathcal{H}_t\right),m,e^*\right]. \end{aligned} \tag{1}$$

2. For every t, H_t, $m(t) \in M$, and strategy e,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right)-I\right)|\mathcal{H}_t,\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right),m^*,e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e)}\left(\theta X\left(\tau\left(e\right)\right)-I\right)|\mathcal{H}_t,\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right),m^*,e\right]. \end{aligned} \tag{2}$$

3. Bayes' rule is used to update beliefs $\mu^*\left(\theta|\mathcal{H}_t\right)$ to $\mu^*\left(\theta|\mathcal{H}_t,m\left(t\right)\right)$ whenever possible: For every H_t and $m(t) \in M$, if there exists θ such that $m_t^*\left(\theta,\mathcal{H}_t\right)=m\left(t\right)$, then for all θ

$$\mu^*\left(\theta|\mathcal{H}_t,m\left(t\right)\right)=\frac{\mu^*(\theta|\mathcal{H}_t)\mathbf{1}\{m_t^*(\theta,\mathcal{H}_t)=m\left(t\right)\}}{\int_{\underline{\theta}}^1\mu^*(\tilde{\theta}|\mathcal{H}_t)\mathbf{1}\{m_t^*(\tilde{\theta},\mathcal{H}_t)=m\left(t\right)\}d\tilde{\theta}}, \tag{3}$$

where $\mu^*\left(\theta|\mathcal{H}_0\right)=\frac{1}{1-\underline{\theta}}$ for $\theta\in\Theta$ and $\mu^*\left(\theta|\mathcal{H}_0\right)=0$ for $\theta\notin\Theta$.

4. For every t, H_t, $\theta \in \Theta$, and $m(t) \in M$,

$$m_t^*\left(\theta,\mathcal{H}_t\right) = m^*\left(\theta,X\left(t\right),\mu^*\left(\cdot|\mathcal{H}_t\right)\right); \tag{4}$$

$$e_t^*\left(\mathcal{H}_t,m\left(t\right)\right) = e^*\left(X\left(t\right),\mu^*\left(\cdot|\mathcal{H}_t,m\left(t\right)\right)\right). \tag{5}$$

The first three conditions, given by (1)–(3), are requirements of the Perfect Bayesian equilibrium. Inequalities (1) require the equilibrium strategy m^* to be sequentially optimal for the agent for any possible history $\mathcal{H}_t$ and type realization θ. Similarly, inequalities (2) require equilibrium strategy e^* to be sequentially optimal for the principal. Equation (3) requires beliefs to be updated according to Bayes' rule. Finally, conditions (4)–(5) are requirements that the equilibrium strategies and the message space are Markov.

Bayes' rule does not apply to messages that should not be sent by any type in equilibrium. To restrict beliefs following such off-equilibrium messages, we make the following assumption:

Assumption 1. *If at any t, the principal's belief $\mu\left(\cdot|\mathcal{H}_t\right)$ and the observed message $m(t)$ are such that no type that could exist (according to the belief $\mu\left(\cdot|\mathcal{H}_t\right)$) could send $m(t)$, then the belief is unchanged: If $\{\theta : m_t^*(\theta,\mathcal{H}_t)=m\left(t\right),\mu^*\left(\theta|\mathcal{H}_t\right)>0\}=\emptyset$, then $\mu^*\left(\theta|\mathcal{H}_t,m\left(t\right)\right)=\mu^*\left(\theta|\mathcal{H}_t\right)$.*

This assumption is related to a frequently imposed restriction in models with two types that if, at any point, the posterior assigns probability one to a given type, then this belief persists no matter what happens (e.g., Rubinstein, 1985; Halac, 2012). Because our model features a continuum of types, an action that no one was supposed to take may occur off equilibrium even if the belief is not degenerate. As a consequence, we impose a stronger restriction.

Let stopping time $\tau^*(\theta)$ denote the equilibrium exercise time of the option if the type is θ. In almost all standard option exercise models, the optimal exercise strategy for a perpetual American call option is a threshold: It is optimal to exercise the option at the first instant the state process $X(t)$ exceeds some critical level, which depends on the parameters of the environment. It is thus natural to look for equilibria that exhibit a similar property, formally defined as:

Definition 2. *An equilibrium is a **threshold-exercise** PBEM if for all $\theta \in \Theta$, $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some $\bar{X}(\theta)$ (possibly infinite).*

For any threshold-exercise equilibrium, let $\mathcal{X}$ denote the set of equilibrium exercise thresholds: $\mathcal{X} \equiv \left\{X : \exists\theta \in \Theta \text{ such that } \bar{X}(\theta) = X\right\}$. We next prove two useful auxiliary results that hold in any threshold-exercise PBEM. The first result shows that in any threshold-exercise PBEM, the option is exercised weakly later if the agent has less favorable information:

Lemma 1. *$\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$.*

Intuitively, because talk is "cheap," the agent of type θ_1 can adopt the message strategy of the agent with type $\theta_2 > \theta_1$ (and vice versa). Thus, when choosing between communication strategies that induce exercise at thresholds $\bar{X}(\theta_1)$ and $\bar{X}(\theta_2)$, type θ_1 must prefer the former, and type θ_2 must prefer the latter. This is simultaneously possible only if $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$.

The second auxiliary result is that it is without loss of generality to reduce the message space significantly. Specifically, for any threshold-exercise equilibrium, there exists an equilibrium with a binary message space $M = \{0, 1\}$ and simple equilibrium strategies that implements the same exercise times and hence features the same payoffs of both players:

Lemma 2. *If there exists a threshold-exercise PBEM with thresholds $\bar{X}(\theta)$, then there exists an equivalent threshold-exercise PBEM with the binary message space $M = \{0, 1\}$ and the following strategies of the agent and the principal and beliefs of the principal:*

1. The agent with type θ sends message $m(t) = 1$ if and only if $X(t) \geq \bar{X}(\theta)$:

$$\bar{m}_t(\theta, X(t), \bar{\mu}(\cdot|\mathcal{H}_t)) = \begin{cases} 1, & \text{if } X(t) \geq \bar{X}(\theta), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

2. The posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal).

3. The exercise strategy of the principal as a function of the state process and his beliefs is

$$\bar{e}_t(X(t), \check{\theta}_t, \hat{\theta}_t) = \begin{cases} 1, & \text{if } X(t) \geq \check{X}(\check{\theta}_t, \hat{\theta}_t), \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

for some threshold $\check{X}(\check{\theta}_t, \hat{\theta}_t)$. Function $\check{X}(\check{\theta}_t, \hat{\theta}_t)$ is such that on equilibrium path the option is exercised at the first instant when the agent sends message $m(t) = 1$, i.e., when $X(t)$ hits threshold $\bar{X}(\theta)$ for the first time.

Lemma 2 implies that it is without loss of generality to focus on equilibria of the following simple form. At any time t, the agent can send one of two messages, 1 or 0. Message $m = 1$ can be interpreted as a recommendation of exercise, and message $m = 0$ as a recommendation of waiting. The agent plays a threshold strategy, recommending exercise if and only if $X(t)$ is above threshold $\bar{X}(\theta)$, which depends on her private information θ. The principal also plays a threshold strategy: if he believes that $\theta \in [\check{\theta}_t, \hat{\theta}_t]$, he exercises the option if and only if $X(t)$ exceeds some threshold $\check{X}(\check{\theta}_t, \hat{\theta}_t)$. As a consequence of the agent's strategy, there is a set $\mathcal{T}$ of "informative" times, when the agent's message has information content, i.e., it affects the belief of the principal and, in turn, her exercise decision. These are instances when $X(t)$ first passes a new threshold from the set of possible exercise thresholds $\mathcal{X}$. At all other times, the agent's message has no information content and does not lead the principal to update his belief. In equilibrium, each type θ recommends exercise (sends $m = 1$) at the first time when $X(t)$ passes the threshold $\bar{X}(\theta)$ for the first time, and the principal responds by exercising the option immediately.

The intuition behind Lemma 2 is that at each time the principal faces a binary decision: to exercise or to wait. Because the agent's information is important only for the timing of the exercise, one can achieve the same efficiency by choosing the timing of communicating a binary message as through the richness of the message space. Therefore, message spaces that are richer than binary cannot improve the efficiency of decision making.

In what follows, we focus on threshold-exercise PBEM of the form in Lemma 2 and refer to them as simply "equilibria." When $\underline{\theta} = 0$, the problem exhibits stationarity in the following

sense. Because the prior distribution of types is uniform over $[0,1]$ and the payoff structure is multiplicative, a time-t sub-game in which the posterior belief of the principal is uniform over $[0,\hat{\theta}]$ is equivalent to the game where the belief is that θ is uniform over $[0,1]$, the true type is $\frac{\theta}{\hat{\theta}}$, and the modified state process is $\tilde{X}(t)=\hat{\theta}X(t)$. Because of this scalability of the game, it is natural to restrict attention to stationary equilibria, which are formally defined as follows:

Definition 3. *Suppose $\underline{\theta}=0$. A threshold-exercise PBEM (m^*,e^*,μ^*,M) is **stationary** if whenever posterior belief $\mu^*(\cdot|\mathcal{H}_t)$ is uniform over $[0,\hat{\theta}]$ for some $\hat{\theta}\in(0,1)$, then for all $\theta\in[0,\hat{\theta}]$:*

$$\begin{aligned} m^*\left(\theta, X(t), \mu^*(\cdot|\mathcal{H}_t)\right) &= m^*\left(\frac{\theta}{\hat{\theta}}, \hat{\theta}X(t), \mu^*(\cdot|\mathcal{H}_0)\right), && (8)\\ e^*\left(X(t), \mu^*(\cdot|\mathcal{H}_t, m(t))\right) &= e^*\left(\hat{\theta}X(t), \mu^*(\cdot|\mathcal{H}_0, m(t))\right), && (9)\end{aligned}$$

Condition (8) means that every type $\theta\in[0,\hat{\theta}]$ sends the same message when the public state is $X(t)$ and the posterior is uniform over $[0,\hat{\theta}]$ as type $\frac{\theta}{\hat{\theta}}$ when the public state is $\hat{\theta}X(t)$ and the posterior is uniform over $[0,1]$. Condition (9) means that the exercise strategy of the principal is the same when the public state is $X(t)$ and his belief is that θ is uniform over $[0,\hat{\theta}]$ as when the public state is $\hat{\theta}X(t)$ and his belief is that θ is uniform over $[0,1]$.

From now on, if $\underline{\theta}=0$, we focus on threshold-exercise PBEM in the form stated in Lemma 2 that are stationary. We refer to these equilibria as *stationary equilibria.*

2.1 Benchmark cases

As benchmarks, we consider two simple settings: one in which the principal knows θ and the other in which the agent has formal authority to exercise the option.

Optimal exercise for the principal. Suppose that the principal knows θ, so communication with the agent is irrelevant. Let $V_P^*(X,\theta)$ denote the value of the option to the principal in this case if the current value of $X(t)$ is X. In the Appendix, we show that following the standard arguments (e.g., Dixit and Pindyck, 1994), in the range prior to exercise, $V_P^*(X,\theta)$ solves

$$rV_P^*(X,\theta)=\mu X\frac{\partial V_P^*(X,\theta)}{\partial X}+\frac{1}{2}\sigma^2X^2\frac{\partial^2V_P^*(X,\theta)}{\partial X^2}. \tag{10}$$

Suppose that type θ exercises the option when $X(t)$ reaches threshold $X_P^*(\theta)$. Then,

$$V_P^*(X_P^*(\theta),\theta)=\theta X_P^*(\theta)-I. \tag{11}$$

Solving (10) subject to the boundary condition (11) and condition $V_P^*(0,\theta)=0$,[8] we obtain

$$V_P^*(X,\theta)=\begin{cases}\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I), & \text{if } X\leq X_P^*(\theta)\\ \theta X-I, & \text{if } X>X_P^*(\theta),\end{cases}\tag{12}$$

where

$$\beta=\frac{1}{\sigma^2}\left[-\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right]>1\tag{13}$$

is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r=0$.

The optimal exercise trigger $X_P^*(\theta)$ maximizes the value of the option (12) and is given by

$$X_P^*(\theta)=\frac{\beta}{\beta-1}\frac{I}{\theta}.\tag{14}$$

Optimal exercise for the agent. Suppose that the agent has formal authority over when to exercise the option. If $b<I$, then substituting $I-b$ for I in (10)–(14), the agent's optimal exercise strategy is to exercise the option at the first moment when $X(t)$ exceeds the threshold

$$X_A^*(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}.\tag{15}$$

If $b\geq I$, the optimal exercise strategy for the agent is to exercise the option immediately.

3 Communication game

By Lemmas 1 and 2, the history of the game at time t on the equilibrium path can be summarized by two cutoffs, $\check{\theta}_t$ and $\hat{\theta}_t$. Moreover, before the agent recommends exercise, $\check{\theta}_t=\underline{\theta}$, and the history of the game can be summarized by a single cutoff $\hat{\theta}_t$, where $\hat{\theta}_t\equiv\sup\left\{\theta:\bar{X}(\theta)>\max_{s\leq t}X(s)\right\}$. Indeed, on the equilibrium path, the principal exercises the option at the first time t with $X(t)\in\mathcal{X}$ at which the agent sends $m(t)=1$. If the agent has not recommended exercise by time t, the principal infers that θ does not exceed $\hat{\theta}_t$. Thus, process $\hat{\theta}_t$ summarizes the principal's belief at time t, provided that he has not deviated from his equilibrium strategy of exercising at the first instant when $X(s)\in\mathcal{X}$ and the agent recommends exercise.

Consider the stationary case $\underline{\theta}=0$. If $b\geq I$, the agent prefers immediate exercise regardless

[8] $V_P^*(0,\theta)=0$ because $X=0$ is an absorbing barrier: if the value of $X(t)$ is zero, it will remain zero forever.

of her type, and hence the principal must exercise the option at his optimal uninformed threshold

$$\bar{X}_u = \frac{\beta}{\beta - 1} 2I. \tag{16}$$

Hence, we focus on $b < I$. Using Lemma 1 and stationarity, we conclude that any stationary equilibrium must either have partitioned exercise or continuous exercise, as explained below.

First, if the equilibrium has a partition structure, i.e., the set of types is partitioned into intervals with each interval inducing exercise at a given threshold, then stationarity implies that the set of partitions must be infinite and take the form $[\omega, 1]$, $[\omega^2, \omega]$, ..., $[\omega^n, \omega^{n-1}]$, ..., $n \in \mathbb{N}$, for some $\omega \in [0, 1)$, where $\mathbb{N}$ is the set of natural numbers. This implies that the set of exercise thresholds $\mathcal{X}$ is given by $\left\{\bar{X}, \frac{\bar{X}}{\omega}, \frac{\bar{X}}{\omega^2}, ..., \frac{\bar{X}}{\omega^n}, ...\right\}$, $n \in \mathbb{N}$, for some $\bar{X} > 0$, such that if $\theta \in (\omega^n, \omega^{n-1})$, the option is exercised at threshold $\frac{\bar{X}}{\omega^{n-1}}$. We refer to an equilibrium of this form as a ω-*equilibrium.*

For ω and $\bar{X}$ to constitute an equilibrium, the incentive compatibility (IC) conditions for the principal and the agent must hold. Because the problem is stationary, it is sufficient to only consider the IC conditions for the game up to reaching the first threshold $\bar{X}$. First, consider the agent's problem. Pair $\left(\omega, \bar{X}\right)$ satisfies the agent's IC condition if and only if types above ω have incentives to recommend exercise ($m = 1$) at threshold $\bar{X}$ rather than to wait, whereas types below ω have incentives to recommend delay ($m = 0$). From the agent's point of view, the set of possible exercise thresholds is given by $\mathcal{X}$: The agent can induce exercise at any threshold in $\mathcal{X}$ by sending $m = 1$ at the first instant when $X(t)$ reaches a desired point in $\mathcal{X}$, but cannot induce exercise at any point not in $\mathcal{X}$. This implies that the agent's IC condition holds if and only if type ω is exactly indifferent between exercising the option at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$:

$$\left(\frac{X(t)}{\bar{X}}\right)^{\beta} \left(\omega \bar{X} + b - I\right) = \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta} \left(\omega \frac{\bar{X}}{\omega} + b - I\right). \tag{17}$$

which simplifies to $\omega \bar{X} + b - I = \omega^{\beta} \left(\bar{X} + b - I\right)$. Indeed, if (17) holds, then $\left(\frac{X(t)}{\bar{X}}\right)^{\beta} \left(\theta \bar{X} + b - I\right) \gtreqless \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta} \left(\theta \frac{\bar{X}}{\omega} + b - I\right)$ if $\theta \gtreqless \omega$. Hence, if type ω is indifferent between exercise at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$, then any higher type strictly prefers recommending exercise at $\bar{X}$, while any lower type strictly prefers recommending delay at $\bar{X}$. By stationarity, if (17) holds, then type ω^2 is indifferent between recommending exercise and recommending delay at threshold $\frac{\bar{X}}{\omega}$, so types in (ω^2, ω) strictly prefer recommending exercise at threshold $\frac{\bar{X}}{\omega}$, and so on. Thus, (17) is necessary and sufficient for the agent's IC condition to hold. Equation (17) is equivalent to the following

relation between the first possible exercise threshold $\bar{X}$ and ω:

$$\bar{X} = Y(\omega) \equiv \frac{\left(1-\omega^{\beta}\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)}. \tag{18}$$

The partitions in a ω-equilibrium are illustrated in Figure 1.

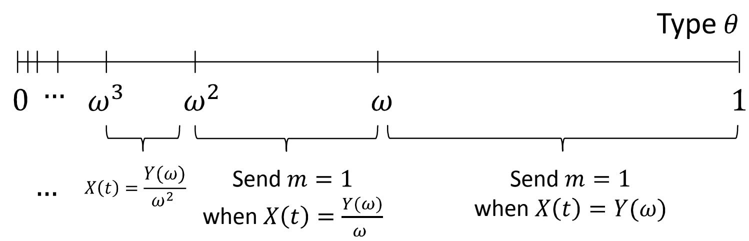

Figure 1. Partitions in a ω-equilibrium.

Next, consider the principal's problem. For ω and $\bar{X}$ to constitute an equilibrium, the principal must have incentives: (1) to exercise the option immediately when the agent sends message $m=1$ at a threshold in $\mathcal{X}$; and (2) not to exercise the option before getting message $m=1$. We refer to the former (latter) IC condition as the *ex-post (ex-ante) IC constraint*. Suppose that $X(t)$ reaches threshold $\bar{X}$ for the first time, and the principal receives recommendation $m=1$ at that instant. By Bayes' rule, the principal updates his beliefs to θ being uniform on $[\omega, 1]$. If the principal exercises immediately, his expected payoff is $\frac{\omega+1}{2}\bar{X}-I$. If the principal delays, he expects that there will be no further informative communication in the continuation game. Thus, upon receiving message $m=1$ at threshold $\bar{X}$, the principal faces the standard perpetual call option exercise problem (e.g., Dixit and Pindyck, 1994) as if the type of the project were $\frac{\omega+1}{2}$. Immediate exercise is optimal if and only if exercising at threshold $\bar{X}$ dominates waiting until $X(t)$ reaches a higher threshold $\hat{X}$ and exercising the option then for any possible $\hat{X}>\bar{X}$:

$$\bar{X} \in \arg\max_{\hat{X}\geq\bar{X}} \left(\frac{\bar{X}}{\hat{X}}\right)^{\beta}\left(\frac{\omega+1}{2}\hat{X}-I\right). \tag{19}$$

Using $\bar{X}=Y(\omega)$ and the fact that the right-hand side is an inverted U-shaped function of $\hat{X}$ with a maximum at $\hat{X}^*=\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, the ex-post IC condition for the principal is equivalent to

$$Y(\omega) \geq \frac{\beta}{\beta-1}\frac{2I}{\omega+1}. \tag{20}$$

This condition has a clear intuition. It means that at the moment when the agent recommends to exercise the option, it must be "too late" for the principal to delay exercise. If (20) is violated,

the principal delays exercise, so the recommendation loses its responsiveness as the principal does not follow it. In contrast, if (20) holds, the principal's optimal response to getting message $m = 1$ is to exercise immediately. As with the IC condition of the agent, stationarity implies that if (20) holds, then a similar condition holds for all higher thresholds in $\mathcal{X}$. The fact that constraint (20) is an inequality rather than an equality highlights the built-in asymmetric nature of time: When the agent recommends exercise, the principal can either exercise immediately or can delay, but cannot go back in time and exercise in the past, even if it is tempting to do so.

Let $V_P(X(t), \hat{\theta}_t; \omega)$ denote the expected value to the principal in the ω-equilibrium, given that the public state is $X(t)$ and the principal's belief is that θ is uniform over $[0, \hat{\theta}_t]$. In the appendix, we solve for the principal's value in closed form and show that if $\hat{\theta}_t = 1$,

$$V_P(X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X}{Y(\omega)} \right)^{\beta} \left(\frac{1}{2}(1+\omega) Y(\omega) - I \right) \tag{21}$$

for any $X \leq Y(\omega)$. By stationarity, (21) can be generalized to any $\hat{\theta}$:

$$V_P(X, \hat{\theta}; \omega) = V_P(\hat{\theta} X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X\hat{\theta}}{Y(\omega)} \right)^{\beta} \left(\frac{1}{2}(1+\omega) Y(\omega) - I \right). \tag{22}$$

The principal's ex-ante IC constraint requires that the principal is better off waiting, rather than exercising immediately, at any time prior to receiving message $m = 1$ at $X(t) \in \mathcal{X}$:

$$V_P(X(t), \hat{\theta}_t; \omega) \geq \frac{\hat{\theta}_t}{2} X(t) - I \tag{23}$$

for any $X(t)$ and $\hat{\theta}_t = \sup\{\theta : \bar{X}(\theta) > \max_{s \leq t} X(s)\}$. By stationarity, it is sufficient to verify the ex-ante IC constraint for $X(t) \leq \bar{X}(1) = Y(\omega)$ and beliefs equal to the prior:

$$V_P(X, 1; \omega) \geq \frac{1}{2} X - I \quad \forall X \leq Y(\omega). \tag{24}$$

This inequality states that at any point up to threshold $Y(\omega)$, the principal is better off waiting than exercising the option. If (24) does not hold for some $X \leq Y(\omega)$, then the principal is better off exercising the option when $X(t)$ reaches X, rather than waiting for informative recommendations from the agent. If (24) holds, then the principal does not exercise the option prior to reaching threshold $Y(\omega)$. By stationarity, if (24) holds, then a similar condition holds for the n^{th} partition for any $n \in \mathbb{N}$, which implies that (24) and (23) are equivalent. To summarize, a ω-equilibrium exists if and only if conditions (18), (20), and (24) are satisfied.

So far, we have considered partition equilibria, which satisfy $\bar{X}(\theta) = \bar{X}(1)$ for any $\theta \in (\omega, 1]$.

In addition, there may be equilibria with $\bar{X}(\theta) \neq \bar{X}(1)$ for all $\theta < 1$. We refer to such equilibria, if they exist, as *equilibria with continuous exercise*, and analyze them below.

3.1 Preference for late exercise

Suppose that the agent favors late exercise, $b < 0$. We start with the stationary case $\underline{\theta} = 0$. First, consider equilibria with continuous exercise. By stationarity, $\mathcal{X} = \{X : X \geq \underline{X}\}$ for some $\underline{X}$. The IC condition of the agent of type θ requires that the equilibrium exercise threshold $\bar{X}(\theta)$ satisfies

$$\bar{X}(\theta) \in \arg\max_{\hat{X} \geq \underline{X}} \left(\frac{X(t)}{\hat{X}}\right)^{\beta} \left(\theta \hat{X} - I + b\right).$$

It implies that exercise occurs at the agent's most preferred threshold as long as it is above $\underline{X}$:

$$\bar{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta - 1} \frac{I - b}{\theta}. \tag{25}$$

Stationarity requires that separation must hold for all types, including $\theta = 1$, which implies that (25) holds for any $\theta \in \Theta$. Hence, $\mathcal{X} = \{X : X \geq X_A^*(1)\}$. This exercise schedule satisfies the ex-post IC condition of the principal. Indeed, because the agent is biased towards delay and recommends exercise at her most preferred threshold, it follows that when the agent recommends exercise, the principal infers that it is already too late and thus does not benefit from delaying exercise even further. Formally, $X_P^*(\theta) < X_A^*(\theta)$.

Consider the ex-ante IC condition of the principal. Let $V_P^c(X, \hat{\theta})$ denote the expected value to the principal in the equilibrium with continuous exercise, given that the public state is X and the principal's belief is that θ is uniform over $[0, \hat{\theta}]$. If the agent's type is θ, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. Hence,

$$V_P^c(X, \hat{\theta}) = \int_0^{\hat{\theta}} \frac{1}{\hat{\theta}} X^{\beta} \left(\frac{\beta}{\beta - 1} \frac{I - b}{\theta}\right)^{-\beta} \frac{I - \beta b}{\beta - 1} d\theta = \frac{(X\hat{\theta})^{\beta}}{\beta + 1} \left(\frac{\beta}{\beta - 1}(I - b)\right)^{-\beta} \frac{I - \beta b}{\beta - 1}. \tag{26}$$

By stationarity, it is sufficient to verify the principal's ex-ante IC constraint for $\hat{\theta} = 1$, which yields

$$V_P^c(X, 1) \geq \frac{1}{2} X - I \quad \forall X \leq X_A^*(1). \tag{27}$$

The proof of Proposition 1 shows that this constraint holds if and only if $b \geq -I$.

Next, consider equilibria with partitioned exercise, characterized by ω and illustrated in Figure 1. To be an equilibrium, the implied exercise thresholds must satisfy the IC conditions of the principal (20) and (24). As the proof of Proposition 1 demonstrates, the principal's ex-post IC condition is satisfied for any $\omega \in (0, 1)$. Intuitively, this is because the agent is biased towards

late exercise, and hence the principal does not benefit from further delay. The principal's ex-ante IC condition is satisfied if communication is informative enough, which puts a lower bound on ω, denoted $\underline{\omega} > 0$. The set of these equilibria is illustrated in Figure 4(a) below.

The following proposition summarizes the set of all stationary equilibria:[9]

Proposition 1. *If $b \in (-I, 0)$, the set of non-babbling stationary equilibria is given by:*

(1) Equilibrium with continuous exercise. The principal exercises at the first time t at which the agent sends $m = 1$, provided that $X(t) \geq X_A^*(1)$ and $X(t) = \max_{s \leq t} X(s)$. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses her most-preferred threshold $X_A^*(\theta)$.

(2) Equilibria with partitioned exercise (ω-equilibria), indexed by $\omega \in [\underline{\omega}, 1)$, where $0 < \underline{\omega} < 1$, and $\underline{\omega}$ is the unique solution to $V_P(X, 1; \underline{\omega}) = \left(\frac{X}{\bar{X}_u}\right)^{\beta} \left(\frac{1}{2}\bar{X}_u - I\right)$, where $\bar{X}_u$ is given by (16). The principal exercises at time t at which $X(t)$ crosses threshold $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, ... for the first time, provided that the agent sends message $m = 1$ at that point, where $Y(\omega)$ is given by (18). The principal does not exercise the option at any other time. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses threshold $Y(\omega)\frac{1}{\omega^n}$, where $n \geq 0$ is such that $\theta \in \left(\omega^{n+1}, \omega^n\right)$. There exists a unique equilibrium for each $\omega \in [\underline{\omega}, 1)$.

If $b = -I$, the unique non-babbling stationary equilibrium is the equilibrium with continuous exercise. If $b < -I$, the principal exercises the option at his optimal uninformed threshold $\frac{\beta}{\beta-1}2I$.

Thus, for $b > -I$, there exist an infinite number of stationary equilibria: one equilibrium with continuous exercise and infinitely many equilibria with partitioned exercise. All these equilibria feature delay relative to the principal's optimal timing given the information available to him at the time of exercise.[10]

Clearly, not all of these equilibria are equally reasonable. It is common in cheap talk games to focus on the equilibrium with the most information revelation, which here corresponds to the equilibrium with continuous exercise.[11] It turns out that the equilibrium with continuous exercise dominates all other possible equilibria in the Pareto sense: it leads to a weakly higher expected

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[9] As always in cheap talk games, there exists a "babbling" equilibrium in which the agent's recommendations are uninformative, and the principal exercises at his optimal uninformed threshold, $\frac{\beta}{\beta-1}2I$. We do not consider this equilibrium unless it is the unique equilibrium of the game.

[10] The equilibrium delay in option exercise is consistent with Atkin et al. (2014), who implement an experiment that shows a strikingly slow adoption of a new technology among soccer-ball producers. This delay comes from the misalignment of incentives between agents and principals and from agents withholding information from the principals about the value of the technology.

[11] In general, equilibrium selection in cheap-talk games is a delicate issue. Unfortunately, most equilibrium refinements that reduce the set of equilibria in costly signaling games do not work well in games of costless signaling (i.e., cheap talk). Some formal approaches to equilibrium selection in cheap-talk games are provided by Farrell (1993) and Chen, Kartik, and Sobel (2008).

payoff for both the principal and all types of the agent. Indeed, in this equilibrium, exercise occurs at the unconstrained optimal time of any type θ of the agent. Therefore, the payoff of any type of the agent is higher in this equilibrium than in any other possible equilibrium. In addition, as Section 4 shows, the exercise times implied by the optimal mechanism if the principal could commit to any mechanism, coincide with the exercise times in the equilibrium with continuous exercise. Thus, the principal's expected payoff in this equilibrium exceeds his expected payoff under the exercise rule implied by any other equilibrium. We conclude:

Proposition 2. *The equilibrium with continuous exercise from Proposition 1 dominates all other possible equilibria in the Pareto sense: both the agent's expected payoff for each realization of θ and the principal's expected payoff are higher in this equilibrium than in any other equilibrium.*

Using Pareto dominance as a selection criterion, we conclude that there is full information revelation if the agent's bias is not very large, $b \geq -I$. However, although information is communicated fully, communication and exercise are inefficiently (from the principal's perspective) delayed. Using the terminology of Aghion and Tirole (1997), the equilibrium features unlimited real authority of the agent, even though the principal has unlimited formal authority. Figure 2 illustrates this equilibrium for parameters $r = 0.15$,[12] $\mu = 0.05$, $\sigma = 0.2$, $I = 1$, and $b = -0.25$.

Next, consider the non-stationary case of $\underline{\theta} > 0$. Instead of continuous exercise, as in the stationary case of $\underline{\theta} = 0$, the equilibrium now features continuous exercise up to a cutoff:

Proposition 3. *Suppose that $\underline{\theta} > 0$. The equilibrium with continuous exercise from Proposition 1 does not exist. However, if $b \in (-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0]$, the equilibrium with continuous exercise up to a cutoff exists. In this equilibrium, there is a cutoff $\hat{X}$ such that the principal's strategy is: (1) to exercise at the first time t at which the agent sends $m = 1$, provided that $X(t) \in [X_A^*(1), \hat{X}]$ and $X(t) = \max_{s \leq t} X(s)$; (2) to exercise at the first time t at which $X(t) \geq \hat{X}$, regardless of the agent's message. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses the minimum between her most-preferred threshold $X_A^*(\theta)$ and $\hat{X}$. Threshold $\hat{X}$ is given by*

$$\hat{X} = \frac{\beta}{\beta - 1} \frac{I + b}{\underline{\theta}} = X_A^*(\hat{\theta}^*),$$

where $\hat{\theta}^* \equiv \frac{I-b}{I+b}\underline{\theta} < 1$. If $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the principal exercises at the uninformed threshold $\frac{\beta}{\beta-1}\frac{2I}{1+\underline{\theta}}$.

[12] The discount rate 0.15 can be interpreted as the sum of the risk-free interest rate 0.05 and the intensity 0.1 with which the investment opportunity disappears.

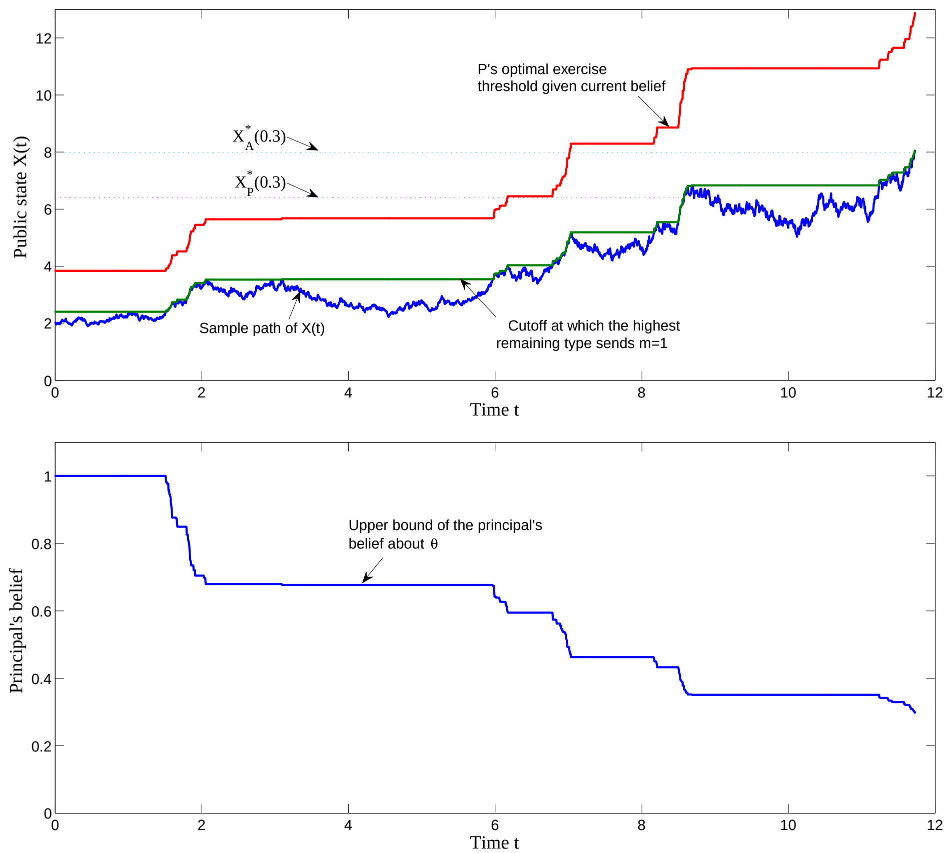


Figure 2. Equilibrium for the case $\underline{\theta} = 0$, $b < 0$. The figure presents the equilibrium with continuous exercise for parameters $\underline{\theta} = 0$, $r = 0.15$, $\mu = 0.05$, $\sigma = 0.2$, $I = 1$, and $b = -0.25$. The blue line in the top panel shows a sample path of the process $X(t)$. The green line represents the running maximum of $X(t)$ starting with the point when $X(t)$ first crosses $X_A^*(1)$ and determines the principal's beliefs: it equals the cutoff at which the highest remaining type, $\hat{\theta}_t$, recommends exercise. The blue line in the lower panel represents the dynamics of $\hat{\theta}_t$, the upper bound of the principal's belief, and the red line in the top panel represents the principal's optimal exercise threshold given these beliefs. The two horizontal lines in the top panel indicate the equilibrium exercise threshold of type $\theta = 0.3$ (top line) and the principal's optimal exercise threshold if he knew that $\theta = 0.3$ in advance (bottom line).

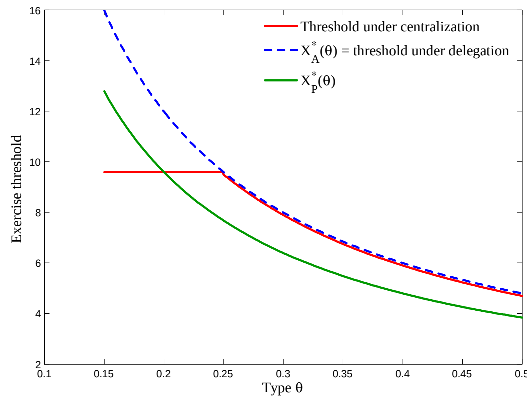


Figure 3. Equilibrium for the case $\underline{\theta} > 0$, $b < 0$. The figure presents the equilibrium with continuous exercise up to a cutoff for parameters $\underline{\theta} = 0.15$, $r = 0.15$, $\mu = 0.05$, $\sigma = 0.2$, $I = 1$, and $b = -0.25$.

The intuition is as follows. At any time, the principal who obtains a recommendation to delay exercise faces the following trade-off. On the one hand, he can wait and see what the agent will recommend in the future. This leads to more informative exercise because the agent communicates her information, but has a drawback in that exercise will be delayed. On the other hand, the principal can disregard the agent's recommendation and exercise immediately. This results in less informative exercise, but not in excessive delay. Thus, the trade-off is between the value of information and the cost of delay. When $\underline{\theta} = 0$, the problem is stationary and the trade-off persists over time even though the principal updates his belief about θ: If the agent's bias is not too high ($b > -I$), waiting for the agent's recommendation is strictly better, while if the agent's bias is too high ($b < -I$), waiting is too costly and communication does not happen. However, when $\underline{\theta} > 0$, the problem is non-stationary, and the trade-off between information and delay changes over time. Specifically, as time goes by and the agent continues recommending against exercise, the principal learns that θ is not too high (below $\hat{\theta}_t$ at time t), and the interval $[\underline{\theta}, \hat{\theta}_t]$ shrinks over time. Because $\underline{\theta} > 0$, the shrinkage of this interval implies that the remaining value of the agent's information declines over time. Once the interval shrinks to $[\underline{\theta}, \hat{\theta}^*]$, which happens at threshold $\hat{X}$, the remaining value of the agent's information becomes sufficiently small to make it optimal for the principal to exercise immediately. The comparative statics of the cutoff type $\hat{\theta}^*$ are intuitive. As b decreases, i.e., the conflict of interest gets bigger, $\hat{\theta}^*$ increases and $\hat{X}$ decreases, implying that the principal waits less for the agent's recommendation. The red line in Figure 3 illustrates the exercise threshold in this equilibrium.

3.2 Preference for early exercise

Suppose that $b > 0$, i.e., the agent is biased in the direction of early exercise. We focus on the stationary case $\underline{\theta} = 0$. Because the principal's optimal exercise time is later than the agent's, there is no equilibrium with continuous exercise. Indeed, if the agent follows the strategy of recommending exercise at her most-preferred threshold $X_A^*(\theta)$, the principal infers the agent's type perfectly and prefers delay over immediate exercise upon getting the recommendation to exercise. Knowing this, the agent is tempted to change her recommendation strategy, mimicking a lower type. Thus, no equilibrium with continuous exercise exists in this case.

Consider the ω-equilibria with partitioned exercise, illustrated in Figure 1. For a ω-equilibrium to exist, the expected value $V_P(X, 1; \omega)$ that the principal gets from waiting for recommendations of the agent and the threshold $Y(\omega)$ must satisfy the ex-post and the ex-ante IC conditions (20) and (24). First, consider equilibria where the ex-post IC condition (20) holds as an equality: $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$. Then, using the expression (18) for $Y(\omega)$, we can find ω as the solution to:

$$\omega = \frac{1}{\frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}\frac{2I}{I-b} - 1}. \tag{28}$$

The next lemma shows that when $b \in (0, I)$, equation (28) has a unique solution, denoted ω^*:

Lemma 3. *Suppose that $0 < b < I$. In the range $[0,1]$, equation (28) has a unique solution $\omega^* \in (0,1)$, where ω^* decreases in b, $\lim_{b \to 0} \omega^* = 1$, and $\lim_{b \to I} \omega^* = 0$.*

Second, consider equilibria where the ex-post IC condition (20) holds as a strict inequality: $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. As the proof of Proposition 4 shows, this is equivalent to $\omega < \omega^*$. The intuition why the ex-post IC condition is violated if ω is large is similar to the standard intuition of why sufficiently efficient information revelation is impossible in cheap talk games. Specifically, since the agent has an early exercise bias and the principal can wait and exercise later after getting the agent's message to exercise, the agent's message cannot be too informative about his type.

The ω-equilibrium will then exist as long as $\omega \leq \omega^*$ and the ex-ante IC condition (24) holds as well. The proof of Proposition 4 shows that (24) is satisfied if and only if ω is high enough and is not satisfied for ω close to zero, which puts a lower bound on ω, denoted $\underline{\omega} > 0$. The set of these equilibria is illustrated in Figure 4(b) and is summarized in the following proposition:

Proposition 4. *Suppose that $0 < b < I$. The set of non-babbling equilibria is given by equilibria with partitioned exercise (ω-equilibria), indexed by $\omega \in [\underline{\omega}, \omega^*]$, where $0 < \underline{\omega} < \omega^* < 1$, ω^* is the*

unique solution to (28), and $\underline{\omega}$ is the unique solution to $V_P(X,1;\underline{\omega}) = \left(\frac{X}{\bar{X}_u}\right)^{\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$, where $\bar{X}_u$ is given by (16). The principal exercises at time t at which $X(t)$ crosses threshold $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, ... for the first time, provided that the agent sends message $m = 1$ at that point, where $Y(\omega)$ is given by (18). The principal does not exercise the option at any other time. The agent of type θ sends message $m = 1$ at the first moment when $X(t)$ crosses threshold $\frac{1}{\omega^n}Y(\omega)$, where $n \geq 0$ is such that $\theta \in (\omega^{n+1}, \omega^n)$. There exists a unique equilibrium for each $\omega \in [\underline{\omega}, \omega^*]$.

If $b \geq I$, the principal exercises the option at his optimal uninformed threshold $\frac{\beta}{\beta-1}2I$.

Similar to the case $b < 0$, the equilibria for the case $b > 0$ can also be ranked by informativeness. The most informative equilibrium is the one with the smallest partitions, i.e., ω^*. In this equilibrium, exercise is unbiased: since the principal's ex-post IC condition holds as an equality, the exercise rule maximizes the principal's payoff given that the agent's type lies in a given partition. In all other equilibria, there is both loss of information and delay in option exercise. Interestingly, delay in exercise occurs despite the fact that the agent is biased towards early exercise.

As the next result demonstrates, the expected utility of the principal and the ex-ante expected utility of the agent (before the agent's type is realized) is higher in the ω^*-equilibrium than in any other stationary equilibrium with partitioned exercise. Intuitively, this is because the ω^*-equilibrium is both the most informative and features no delay, which is detrimental for both the principal and the agent with a bias towards early exercise.

Proposition 5. *The ω^*-equilibrium dominates other equilibria with partitioned exercise in the following sense: both the principal's expected payoff and the ex-ante expected payoff of the agent before θ is realized are higher in the ω^*-equilibrium than in the ω-equilibrium for any $\omega < \omega^*$.*

Motivated by this result, we focus on the ω^*-equilibrium in the remainder of the paper.

3.3 The role of dynamic communication

In this section, we highlight the role of dynamic communication by comparing our model to a model where communication is restricted to a one-shot interaction at the beginning of the game. Specifically, we analyze a restricted version of the basic model where instead of communicating with the principal continuously, the agent sends a single message m_0 at time $t = 0$, and there is no subsequent communication. After receiving the message, the principal updates his beliefs about θ and exercises the option at the optimal threshold given these beliefs.

First, note that for any equilibrium of the static communication game, there exists an equi-

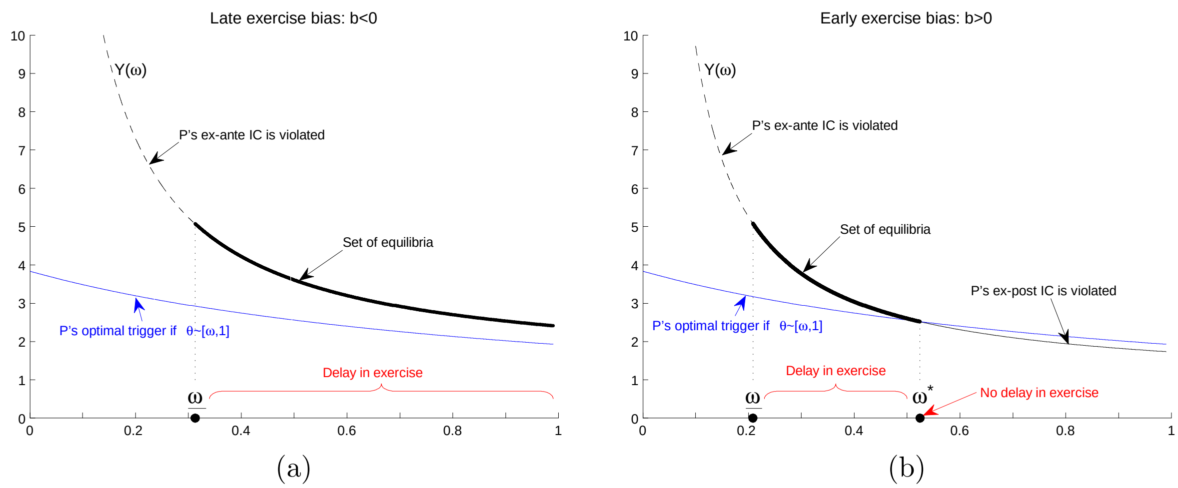


Figure 4. Equilibria with partitioned exercise. The figures present the partition equilibria for $\underline{\theta} = 0$, $r = 0.15$, $\mu = 0.05$, $\sigma = 0.2$, and $I = 1$. The agent's bias is $b = -0.25$ in figure (a) and $b = 0.1$ in figure (b). In both figures, the black line represents the agent's IC condition, i.e., the function $Y(\omega)$, and the blue line represents the function $\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, i.e., the principal's optimal exercise trigger if θ is uniform on $[\omega, 1]$.

valent equilibrium of the dynamic communication game where all communication after $t = 0$ is uninformative (babbling). However, the opposite is not true: many equilibria of the dynamic communication game do not exist in the static communication game. Specifically:

Proposition 6. *Suppose $\underline{\theta} = 0$. If $b < 0$, there is no non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game. If $b > 0$, the only non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game is the ω^*-equilibrium.*

The intuition is as follows. All non-babbling stationary equilibria of the dynamic communication game for $b < 0$ feature delay relative to what the principal's optimal timing of exercise would have been ex ante, given the information he learns in equilibrium. In a dynamic communication game, this delay is feasible because the principal learns information with delay, after his optimal (conditional on this information) exercise time has passed. However, in a static communication game, this delay cannot be sustained: since the principal learns all the information at time zero, his exercise decision is always optimal given the available information.[13] By the same argument, the only sustainable equilibrium of the dynamic communication game for $b > 0$ is the one that features no delay relative to the principal's optimal threshold, i.e., the ω^*-equilibrium.

[13] Similarly, if $\underline{\theta} > 0$, the equilibrium with continuous exercise up to a cutoff, described in Proposition 3, does not exist in the static communication game either.

Thus, even though the agent's information is persistent, the ability to communicate dynamically makes the analysis different from the static problem. When the agent favors late exercise, dynamic communication expands the set of equilibria in a way that improves both players' payoffs. Timing the recommendation strategically helps both players because it ensures that the principal follows the agent's recommendation and thereby makes communication effective.

4 Delegation versus communication

So far, we have assumed that the principal has no commitment power at all. In this section, we relax this assumption by allowing the principal to choose between delegating formal authority to exercise the option to the agent and keeping formal authority but playing the communication game analyzed in the previous section. Formally, we consider the problem studied by Dessein (2002) in the context of static decisions, but focus on stopping time decisions. In an insightful paper, Dessein (2002) shows that delegating the decision to the informed but biased agent dominates keeping the authority and communicating with the agent if the agent's bias is small enough. We show that the choice between delegation and centralization can be quite different for decisions about timing. Because most decisions in organizations can be delayed and thus involve the stopping time component, these results have important implications for organizational design.

4.1 Optimal mechanism with commitment

To analyze the choice between delegation and communication, it is helpful to derive an auxiliary result: what the optimal mechanism would be if the principal could commit to any mechanism. By the revelation principle, we can restrict attention to direct revelation mechanisms, i.e., those in which the message space is $\Theta = [\underline{\theta}, 1]$ and that provide the agent with incentives to report her type θ truthfully. It is also easy to show that a mechanism in which exercise does not occur at a first passage time cannot be optimal. Hence, we can restrict attention to mechanisms of the form $\{\hat{X}(\theta) \geq X(0), \theta \in \Theta\}$: If the agent reports θ, the principal exercises when $X(t)$ first passes threshold $\hat{X}(\theta)$. Let $\hat{U}_A(\hat{X}, \theta)$ and $\hat{U}_D(\hat{X}, \theta)$ denote the time-zero expected payoffs of the agent and the principal, respectively, when type is θ and the exercise occurs at threshold $\hat{X}$. The optimal mechanism maximizes the principal's expected payoff subject to the agent's IC constraint:

$$\max_{\{\hat{X}(\theta), \theta \in \Theta\}} \int_{\underline{\theta}}^{1} \hat{U}_D(\hat{X}(\theta), \theta) \frac{1}{1-\underline{\theta}} d\theta$$
$$s.t.\ \hat{U}_A(\hat{X}(\theta), \theta) \geq \hat{U}_A(\hat{X}(\hat{\theta}), \theta)\ \forall \theta, \hat{\theta} \in \Theta.$$

The next result characterizes the optimal decision-making rule under commitment:

Lemma 4. *The optimal incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta \in \Theta$, is given by:*

- *If $b \in \left(-\infty, -\frac{1-\underline{\theta}}{1+\underline{\theta}} I\right] \cup \left[\frac{1-\underline{\theta}}{1+\underline{\theta}} I, \infty\right)$, then $\hat{X}(\theta) = \frac{\beta}{\beta-1} \frac{2I}{\underline{\theta}+1}$ for any $\theta \in \Theta$.*
- *If $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}} I, 0\right]$, then* $\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1} \frac{I+b}{\underline{\theta}}, & \text{if } \theta < \left(\frac{I-b}{I+b}\right) \underline{\theta}; \\ \frac{\beta}{\beta-1} \frac{I-b}{\theta}, & \text{if } \theta \geq \left(\frac{I-b}{I+b}\right) \underline{\theta}. \end{cases}$
- *If $b \in \left[0, \frac{1-\underline{\theta}}{1+\underline{\theta}} I\right)$, then* $\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1} \frac{I-b}{\theta}, & \text{if } \theta < \frac{I-b}{I+b}; \\ \frac{\beta}{\beta-1} (I+b), & \text{if } \theta \geq \frac{I-b}{I+b}. \end{cases}$

The reasoning behind this result is similar to the reasoning of why the optimal decision-making rule in the static linear-quadratic model features perfect separation of types up to a cutoff and pooling beyond the cutoff (Melumad and Shibano, 1991; Goltsman et al., 2009). Intuitively, because the agent does not receive additional private information over time and the optimal stopping rule can be summarized by a threshold, the optimal dynamic contract is similar to the optimal contract in a static game with equivalent payoff functions.

By comparing previous results with Lemma 4, it is easy to see that if the agent favors late exercise ($b < 0$), the equilibrium of the communication game and the solution under commitment coincide for any $\underline{\theta} \geq 0$. This result does not hold when the agent favors early exercise ($b > 0$). In this case, the principal would benefit from commitment power. This asymmetry occurs because of the asymmetric nature of time: Even without formal commitment power, as time passes, the principal effectively commits not to exercise earlier because she cannot go back in time. The next proposition summarizes these results:

Proposition 7. *If $b < 0$, the exercise threshold in the most informative equilibrium of the advising game coincides with the optimal exercise threshold under commitment: $\bar{X}(\theta) = \hat{X}(\theta)$ for all $\theta \in \Theta$. In particular, the equilibrium payoffs of both parties in the advising game coincide with their payoffs under the optimal commitment mechanism. If $b > 0$, the payoff of the principal in the advising game is lower than his payoff under the optimal commitment mechanism.*

From the organizational design perspective, this result implies that investing in commitment power is not important for decisions where the agent wants to delay exercise, as in the case of headquarters seeking a local plant manager's advice on closing the plant. In contrast, investing in commitment power is important for decisions where the agent is biased towards early exercise, such as the decision when to drill an oil well, launch a new product, or make an acquisition.

4.2 Delegation when the agent has a preference for late exercise

It follows from Proposition 7 and the asymmetric nature of the equilibrium in the communication game that implications for delegation are different between the "late exercise bias" and the "early exercise bias" cases. First, consider the late exercise bias case, $b < 0$. If the principal does not delegate the decision and instead communicates with the agent, the option is exercised either at the agent's most preferred threshold $X_A^*(\theta)$ (if $\underline{\theta} = 0$) or at the agent's most preferred threshold up to a cutoff (if $\underline{\theta} > 0$). If the principal delegates formal authority to the agent, the agent exercises the option at her most preferred threshold $X_A^*(\theta)$. Clearly, if $\underline{\theta} = 0$, delegation and communication are equivalent. However, if $\underline{\theta} > 0$, they are not equivalent: Not delegating the decision and playing the communication game implements *conditional* delegation (delegation up to a cutoff), while delegation implements *unconditional* delegation. By Proposition 7, the principal is strictly better off with the former rather than the latter. This result is illustrated in Figure 3 and summarized in the following proposition.

Proposition 8. *If $b < 0$, i.e., the agent is biased towards late exercise, the principal always weakly prefers retaining control and getting advice from the agent to delegating the exercise decision. The preference is strict if $\underline{\theta} > 0$. If $\underline{\theta} = 0$, retaining control and delegation are equivalent.*

This result contrasts with the implications for static decisions, such as choosing the scale of the project. Dessein (2002) shows that in the leading quadratic-uniform setting of Crawford and Sobel (1982), regardless of the direction of the agent's bias, delegation always dominates communication as long as the agent's bias is not too high so that at least some informative communication is possible. For general payoff functions, Dessein (2002) shows that delegation is optimal if the agent's bias is sufficiently small. In contrast, we show that if the agent favors late exercise, then regardless of the magnitude of her bias, the principal never wants to delegate decision-making authority to her. Intuitively, the inability to go back in time allows the principal to commit to follow the recommendations of the agent. This built-in commitment role of time ensures that communication is sufficiently effective so that delegation has no further benefit.[14]

4.3 Delegation when the agent has a preference for early exercise

In contrast to the case where the agent is biased towards late exercise, delegation is beneficial if the agent is biased towards early exercise and the bias is low enough. Specifically:

[14] Note also that in our context, exercise occurs with delay even under centralization. This is different from Bolton and Farrell (1990), where centralization helps avoid inefficient delay caused by coordination problems between competing firms.

Proposition 9. *Suppose $b > 0$, i.e., the agent is biased towards early exercise, and consider the most informative equilibrium of the advising game, ω^*. There exist $\underline{b}$ and $\bar{b}$, such that the principal's expected value in the ω^*-equilibrium is lower than his expected value under delegation if $b < \underline{b}$, and is higher than under delegation if $b > \bar{b}$.*

The result that delegation is beneficial when the agent's bias is small enough is similar to the result of Dessein (2002) for static decisions and shows that Dessein's argument extends to stopping time decisions when the agent favors early exercise. Intuitively, the principal faces a trade-off: delegation leads to early exercise due to the agent's bias but uses the agent's information more efficiently. When the agent's bias is small enough, the cost from early exercise is smaller than the cost due to the loss of the agent's information, and hence delegation dominates.

4.4 Optimal timing of delegation

In a dynamic setting, the principal does not need to delegate authority to the agent from the start: he may retain authority for some time and delegate later. In this section, we study whether timing delegation strategically may help the principal. In particular, consider the following game: The principal and the agent play the communication game of Section 3, but at any time, the principal may delegate decision-making authority to the agent. After authority is granted, the agent retains it until the end of the game and thus is free to choose her optimal exercise threshold.

According to Proposition 7, if the agent favors late exercise, the advising equilibrium implements the optimal commitment mechanism. Hence, the principal cannot do better with delegation than with keeping authority forever and communicating with the agent. In contrast, when the agent favors early exercise, simply communicating with the agent brings a lower payoff than under the optimal mechanism. However, the next result shows that for any $\underline{\theta} \geq 0$, the principal can implement the optimal mechanism by delegating the decision at the right time.

Proposition 10. *If $b > 0$, there exists the following equilibrium. The principal delegates authority to the agent at the first moment when $X(t)$ reaches the threshold $X_d \equiv \min(\frac{\beta(I+b)}{\beta-1}, \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1})$ and does not exercise the option before that. For any θ, the agent sends message $m = 0$ at any point before she is granted authority. If $\theta \geq \frac{I-b}{I+b}$, the agent exercises the option immediately after she is given authority, and if $\theta \leq \frac{I-b}{I+b}$, the agent exercises the option when $X(t)$ first reaches her preferred exercise threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. The exercise threshold in this equilibrium coincides with the optimal exercise threshold under commitment.*

Intuitively, timing delegation strategically ensures that the information of low types ($\theta \leq \frac{I-b}{I+b}$) is used efficiently, and that all types above $\frac{I-b}{I+b}$ exercise immediately at the time of delegation, exactly as in the optimal contract. The higher is the agent's bias, the later will delegation occur.

Propositions 7 and 10 imply that the direction of the conflict of interest is the key driver of the allocation of authority for timing decisions. If the agent favors late exercise, the principal should always retain control and rely on communication with the agent. In contrast, if the agent favors early exercise, it is optimal to delegate the decision to the agent at some point in time.

5 Implications

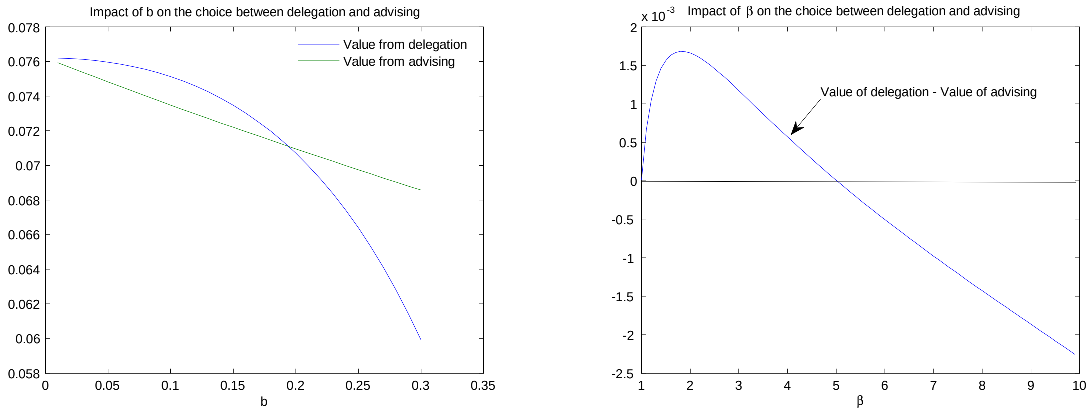
5.1 Comparative statics

The model delivers interesting comparative statics results. We focus on the case $\underline{\theta} = 0$. First, consider the communication game. If the agent favors late exercise, $b < 0$, there is full information revelation, independent of the agent's bias and the parameters μ, σ, and r. The decision is delayed by the factor $\frac{I-b}{I}$, which is also independent of these parameters. If the agent favors early exercise, $b > 0$, equilibrium exercise is unbiased, but there is loss of information, characterized by $\omega^* < 1$. The comparative statics of ω^* are presented in the next result.

Proposition 11. *Consider the case of an agent biased towards early exercise, $b > 0$. Then, ω^* decreases in b and increases in β, and hence decreases in σ and μ, and increases in r.*

The result that ω^* decreases in the agent's bias is in line with the result of Crawford and Sobel (1982) that less information is revealed if the misalignment of preferences is bigger. More interesting are the comparative statics results in μ, σ, and r. Proposition 11 shows that communication is less efficient when the option to wait is more valuable. For example, there is less information revelation (ω^* is lower) if the environment is more uncertain (σ is higher). Intuitively, higher uncertainty increases the value of the option to delay exercise and thus effectively increases the conflict of interest between the principal and the agent biased towards early exercise. Similarly, communication is less efficient in lower interest rate and higher growth rate environments.

We next analyze how the principal's choice between delegating the decision to the agent from the start and communicating with the agent depends on these parameters. As Section 4.2 shows, delegation is always weakly inferior if the agent favors late exercise. We therefore focus on the case of an early exercise bias. As Proposition 9 demonstrates, delegation is superior (inferior)



(a) (b)

Figure 5. Comparative statics of the delegation decision. The figure compares the principal's expected value from delegation and his expected value in the ω^*-equilibrium of the advising model. Figure (a) plots the two values as a function of the agent's bias b for $\underline{\theta} = 0$, $r = 0.15$, $\mu = 0.05$, $\sigma = 0.2$, and $I = 1$. Figure (b) plots the difference between the two values as a function of β for $\underline{\theta} = 0$, $I = 1$, $b = 0.1$.

to communication if the agent's bias is sufficiently small (large). In numerical analysis, we show that there exists a cutoff $\bar{b}$ such that the principal's value from delegation is higher if and only if $0 < b < \bar{b}$. Figure 5(a) illustrates this result.

The comparative statics of the delegation decision in σ, μ, and r are demonstrated in Figure 5(b). The figure plots the difference between the principal's value from delegation and his value from advising as a function of β, which increases in r and decreases in σ and μ. It shows that delegation dominates advising when β is sufficiently small (σ and μ are high, r is small), but is inferior when β is large. Intuitively, a small β corresponds to a high value of the option to wait. By Proposition 11, communication between the principal and the agent is less efficient when the value of the option to wait is higher. Thus, the principal prefers delegating the decision over retaining control and communicating with the agent when the option to wait is sufficiently valuable. As β increases, communication becomes more efficient and eventually dominates delegation.

5.2 Strategic choice of the agent

We next show that asymmetry of time has important implications for the strategic choice of an agent. We focus on the stationary case of $\underline{\theta} = 0$. The next result shows that if the principal needs to choose between an agent biased towards early exercise and an agent biased towards late exercise with the same (in absolute value) bias, he is better off choosing the agent with a late exercise bias, regardless of whether or not he has the option to delegate authority to the agent.

Proposition 12. *(i) Let $V_0(b)$ be the expected payoff of the principal at the initial date $t = 0$ in the most informative equilibrium of the advising game, given that the agent's bias is b. Then, $V_0(-b) \geq V_0(b)$ for any $b \geq 0$ and $V_0(-b) > V_0(b)$ for any $b \in (0, I)$.*

(ii) Let $\tilde{V}_0(b)$ be the expected payoff of the principal at the initial date $t = 0$ if the principal can choose between delegating authority to the agent and retaining authority and communicating with the agent. Then, $\tilde{V}_0(-b) \geq \tilde{V}_0(b)$ for any $b \geq 0$ and $\tilde{V}_0(-b) > \tilde{V}_0(b)$ for any $b \in (0, I)$.

The first result implies that between the two problems, poor communication but unbiased timing and full communication but delayed timing, the former is a bigger problem. Intuitively, the advising game features built-in commitment power of the principal when the agent is biased towards late exercise, but not when the agent is biased towards early exercise. Because of this, as shown in Proposition 7, the principal's payoff in the advising equilibrium coincides with his payoff in the optimal mechanism for $b < 0$, but is strictly smaller than in the optimal mechanism for $b > 0$. Because the principal's utility in the optimal mechanism only depends on the magnitude of the agent's bias and not on its direction, the principal is better off dealing with an agent biased towards late exercise. Allowing the principal to delegate authority to the agent does not change this result: although delegation can make the principal better off if the agent is biased towards early exercise, it does not allow him to implement the optimal mechanism.

Our results also imply that in an alternative setting, where the principal is biased towards early exercise (as in the case of an empire-building top manager), it is possible to ensure unbiased decision-making by having an unbiased agent, even if the principal has formal authority. For example, Proposition 1 suggests that if $\underline{\theta} = 0$, the principal will exercise the option at the agent's most-preferred threshold, and hence exercise will be unbiased. Thus, similarly to Landier, Sraer, and Thesmar (2009), divergence of preferences between the principal and his subordinate can enhance decision-making quality, although our mechanism is very different from theirs.

6 Robustness

6.1 Model with different discount rates

In our basic setup, the conflict of interest between the agent and the principal is modeled by the agent's bias b. Our results are similar in an alternative setup, where the conflict of interest arises because the agent and the principal have different discount rates. This section presents the summary of this analysis, and the full analysis is available from the authors upon request.

Suppose that the agent's discount rate is r_A, the principal's discount rate is r_P, and both

players' payoff from exercise at time t is $\theta X(t) - I$. Similar to the basic model, we can define β_A and β_P, where β_i is the positive root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta - r_i = 0$.

The case where the principal is more impatient than the agent ($r_P > r_A$, or equivalently, $\beta_P > \beta_A$) is similar to the case $b < 0$ in the basic model. We show that if $\underline{\theta} = 0$, then as long as $\beta_A > \frac{2\beta_P}{1+\beta_P}$, there exists an equilibrium with continuous exercise in which exercise occurs at the agent's most preferred threshold $\frac{\beta_A}{\beta_A - 1}\frac{I}{\theta}$. If $\underline{\theta} > 0$, the equilibrium features continuous exercise up to a cutoff. The case where the agent is more impatient than the principal ($r_P < r_A$) is similar to the case $b > 0$ in the basic model. We show that the equilibrium with continuous exercise does not exist and derive the analog of Proposition 4. Specifically, in the most informative stationary equilibrium, exercise is unbiased given the principal's information. This equilibrium is characterized by $\tilde{\omega}^* < 1$, which is the unique solution of

$$\frac{\left(1-\omega^{\beta_A}\right) I}{\omega\left(1-\omega^{\beta_A - 1}\right)} = \frac{\beta_P}{\beta_P - 1}\frac{2I}{\omega+1}.$$

In addition, for any $\omega \in [\underline{\tilde{\omega}}, \tilde{\omega}^*)$, where $0 < \underline{\tilde{\omega}} < \tilde{\omega}^*$, there is a unique ω-equilibrium where exercise happens with delay.

6.2 Simple compensation contracts

A reasonable question is whether simple compensation contracts, such as paying a fixed amount for exercise (if $b < 0$) or for the lack of exercise (if $b > 0$), can solve the problem and thus make the analysis less relevant. We show that this is not the case. Specifically, we allow the principal to offer the agent the following payment scheme. If the agent is biased towards late exercise ($b < 0$), the principal can promise the agent a lump-sum payment z that she will receive as soon as the option is exercised. A higher payment decreases the conflict of interest and speeds up option exercise. For example, if $z = \frac{-b}{2}$, the agent's and the principal's interests are aligned because each of them receives $\theta X - I + \frac{b}{2}$ upon exercise. However, a higher payment is also more expensive for the principal. Because of that, as the next result shows, it is always optimal for the principal to offer $z^* < \frac{-b}{2}$, and hence the conflict of interest will remain. Moreover, if the agent's bias is sufficiently small, the optimal payment is in fact zero.

Similarly, if the agent is biased towards early exercise ($b > 0$), the principal can promise the agent a flow of payments $\hat{z}dt$ up to the moment when the option is exercised. Higher $\hat{z}$ aligns the interests of the players but is expensive for the principal. The next result shows that if the initial value of the state process is sufficiently small, the optimal $\hat{z}$ is again zero. In numerical analysis, we also show that similarly to the late exercise bias case, the optimal payment is smaller than the payment that would eliminate the conflict of interest.

Proposition 13. *Suppose $b < 0$ and the principal can promise the agent a payment $z \geq 0$ upon exercise. Then the optimal z is always strictly smaller than $\frac{-b}{2}$ and equals zero if $b > \frac{-I}{\beta-1}$. Suppose $b > 0$ and the principal can promise the agent a flow of payments $\hat{z}dt \geq 0$ up to the moment of option exercise. Then the optimal $\hat{z}$ equals zero if $X(0)$ is sufficiently small.*

Thus, allowing simple compensation contracts often does not change the problem at all, and at most leads to an identical problem with a different bias b. We conclude that the problem and implications of our paper are robust to allowing simple compensation contracts.

6.3 Put option

So far, we have assumed that the decision problem is over the timing of exercise of a call option, such as the decision of when to invest. In this section, we show that if the decision problem is over the timing of exercise of a put option, such as the decision of when to liquidate a project, the analysis and economic insights are similar. The nature of the option, call or put, is irrelevant for the results. What matters is the asymmetric nature of time: Time moves forward and thereby creates a one-sided commitment device for the principal to follow the agent's recommendations.

Consider the model of Section 2 with the following change. The exercise of the option leads to the payoffs $\theta I - X(t)$ and $\theta(I+b) - X(t)$ for the principal and the agent, respectively. As before, θ is a random draw from a uniform distribution on $[\underline{\theta}, 1]$ and is privately learned by the agent at the initial date. If $\underline{\theta} = 0$, the model exhibits stationarity. For example, if the decision represents shutting down a project, $I\theta$ corresponds to the salvage value of the project, $b\theta$ represents the agent's private cost (if $b < 0$) or benefit (if $b > 0$) of liquidating the project, and $X(t)$ corresponds to the present value of the cash flows from keeping the project afloat. The solution of this model follows the same structure as the solution of the model with the call option. We summarize our findings below, and the full analysis is available from the authors upon request.

Suppose that we start with a high enough $X(0)$, so that immediate exercise does not happen. At the beginning of the Appendix, we show that if θ were known, the optimal exercise policy of each player would be given by a lower trigger on $X(t)$: $X_P^{**}(\theta) = \frac{\gamma}{\gamma+1} I\theta$, $X_A^{**}(\theta) = \frac{\gamma}{\gamma+1}(b+I)\theta$, where $-\gamma$ is the negative root of the quadratic equation that defined β. If $b > 0$, then $X_A^{**}(\theta) > X_P^{**}(\theta)$, i.e., the agent's preferred exercise policy is to exercise earlier than the principal. Similarly, if $b < 0$, the agent is biased towards late exercise.

Suppose that $\underline{\theta} = 0$ and consider the communication game like the one in Section 3. If $b \in (-\frac{I}{2}, 0)$, there is an equilibrium with full information revelation: The agent recommends to wait

as long as $X(t)$ exceeds her preferred exercise threshold $X_A^{**}(\theta)$ and recommends exercise at the first moment when $X(t)$ hits $X_A^{**}(\theta)$. Upon getting the recommendation to exercise, the principal realizes it is too late and finds it optimal to exercise immediately. Prior to that, the principal prefers to wait because the value of learning θ exceeds the cost of delay. If $b > 0$, this equilibrium does not exist, and all stationary equilibria are of the form $\{(\omega, 1), (\omega^2, \omega), ...\}$, where type $\theta \in (\omega^n, \omega^{n-1})$ recommends exercise at threshold $\omega^{n-1} Y_{put}(\omega)$, where $Y_{put}(\omega) = \frac{\omega - \omega^{\gamma+1}}{1-\omega^{\gamma+1}}(I + b)$.

7 Conclusion

This paper studies timing decisions in organizations. We consider a problem in which an uninformed principal is deciding when to exercise an option and has to rely on the information of a better-informed but biased agent. Depending on the application, the agent may be biased towards late or early exercise. We first analyze centralized decision-making, when the principal retains authority and repeatedly communicates with the agent via cheap talk. In contrast to the static cheap talk setting, where the decision variable is scale rather than stopping time, the properties of the equilibria are asymmetric in the direction of the agent's bias. When the agent favors late exercise, there is often full information revelation but suboptimal delay in option exercise. Conversely, when the agent favors early exercise, there is partial revelation of information, while exercise is either unbiased or delayed. The reason for this asymmetry is the asymmetric nature of time: While the principal can get advice and exercise the option at a later point in time, he cannot go back and exercise the option at an earlier point in time. When the agent is biased towards late exercise, the inability to go back in time creates an implicit commitment device for the principal to follow the agent's recommendation and often allows full information revelation. In contrast, when the agent is biased towards early exercise, time does not have built-in commitment, and only partial information revelation is possible. The analysis has implications for the informativeness and timeliness of option exercise decisions, depending on the direction of the agent's bias and the parameters of the stochastic environment, such as volatility, growth rate, and discount rate.

We next analyze the optimal allocation of authority for timing decisions by studying the principal's choice between centralized decision-making with communication and delegating the decision to the agent. We show that the optimal choice between delegation and centralization is also asymmetric in the direction of the agent's bias. Delegation is always weakly inferior when the agent favors late exercise, but is optimal when the agent favors early exercise and her bias is not very large. If the principal can time the delegation decision strategically, he can implement the second-best by delegating authority at the right time if the agent favors early exercise, but always finds it optimal to retain authority in the case of a late exercise bias.

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Appendix: Proofs

Derivation of the benchmark case for the call and put option. Let $V(X)$ be the value of the option to a risk-neutral player if the current value of $X(t)$ is X. Because the player is risk-neutral, the expected return from holding the option over a small interval dt, $\mathbb{E}\left[\frac{dV}{V}\right]$, must equal the riskless return rdt. By Itô's lemma,

$$dV(X(t)) = \left(V'(X(t))\,\mu X(t) + \frac{1}{2}V''(X(t))\,\sigma^2 X(t)^2\right)dt + \sigma V'(X(t))\,dB(t),$$

and hence

$$\mathbb{E}\left[\frac{dV(X(t))}{V(X(t))}\right] = rdt \Leftrightarrow \frac{1}{V(X(t))}\left(V'(X(t))\,\mu X(t) + \frac{1}{2}V''(X(t))\,\sigma^2 X(t)^2\right)dt = rdt,$$

which gives (10). This is a second-order linear homogeneous ordinary differential equation. The general solution to this equation is $V(X) = A_1 X^{\beta_1} + A_2 X^{\beta_2}$, where A_1 and A_2 are the constants to be determined, and $\beta_1 < 0 < 1 < \beta_2$ are the roots of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$. We denote the negative root by $-\gamma$, $\gamma > 0$, and the positive root by β, $\beta > 1$. To find A_1, A_2, we use two boundary conditions. If exercise of the option occurs at trigger $\bar{X}$ and gives a payoff $p(\bar{X})$, the first boundary condition is $V(\bar{X}) = p(\bar{X})$.

For the call option, the second boundary condition is $\lim_{X\to 0} V(X) = 0$ because zero is an absorbing barrier for the geometric Brownian motion. Hence, $A_1 = 0$. In addition, if θ is known to the principal, then $p_{call}(\bar{X}) = \theta\bar{X} - I$. Combining this with boundary conditions $A_1 = 0$ and $V(\bar{X}) = p(\bar{X})$, we get

$$V_{call}(X, \bar{X}) = \left(\frac{X}{\bar{X}}\right)^{\beta}(\theta\bar{X} - I). \tag{29}$$

Maximizing $V_{call}(X, \bar{X})$ with respect to $\bar{X}$ to derive the optimal call option exercise policy of the principal gives $\bar{X} = \frac{\beta}{\beta-1}\frac{I}{\theta}$, i.e. (14).

Similarly, for the put option, the second boundary condition is $\lim_{X\to\infty} V(X) = 0$, and hence $A_2 = 0$. Combining it with $V(\bar{X}) = p(\bar{X})$ and using $p_{put}(\bar{X}) = \theta I - \bar{X}$, gives $V_{put}(X, \bar{X}) = \left(\frac{X}{\bar{X}}\right)^{-\gamma}(\theta I - \bar{X})$. Maximizing $V_{put}(X, \bar{X})$ with respect to $\bar{X}$ to derive the optimal put option exercise policy of the principal gives $\bar{X} = \frac{\gamma}{\gamma+1}I\theta$. ■

Proof of Lemma 1. By contradiction, suppose that $\bar{X}(\theta_1) < \bar{X}(\theta_2)$ for some $\theta_2 > \theta_1$. Using the same arguments as in the derivation of (29) above but for $I - b$ instead of I, it is easy to see that if exercise occurs at a cutoff $\bar{X}$ and the current value of $X(t)$ is $X \le \bar{X}$, then the agent's expected utility is given by $\left(\frac{X}{\bar{X}}\right)^{\beta}(\theta\bar{X} - I + b)$, where $\beta > 1$ is defined by (13). Hence, because the message strategy of type θ_1 is feasible for type θ_2, the incentive compatibility (IC) condition of type θ_2 implies:

$$\left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}(\theta_2\bar{X}(\theta_2) - I + b) \ge \left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}(\theta_2\bar{X}(\theta_1) - I + b). \tag{30}$$

Similarly, because the message strategy of type θ_2 is feasible for type θ_1,

$$\left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}(\theta_1\bar{X}(\theta_1) - I + b) \ge \left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}(\theta_1\bar{X}(\theta_2) - I + b). \tag{31}$$

These inequalities imply

$$\theta_2\bar{X}(\theta_1)\left(1-\left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right)\leq(I-b)\left(1-\left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta}\right)$$
$$\leq\theta_1\bar{X}(\theta_1)\left(1-\left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right),$$

which is a contradiction, because $\theta_2>\theta_1$ and $\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}<1$. Thus, $\bar{X}(\theta_1)\geq\bar{X}(\theta_2)$ whenever $\theta_2\geq\theta_1$. ■

Proof of Lemma 2. Consider a threshold exercise equilibrium E with an arbitrary message space M^* and equilibrium message strategy m^*, in which exercise occurs at stopping time $\tau^*(\theta)=\inf\left\{t\geq 0|X(t)\geq\bar{X}(\theta)\right\}$ for some set of thresholds $\bar{X}(\theta)$, $\theta\in\Theta$. By Lemma 1, $\bar{X}(\theta)$ is weakly decreasing. Define $\theta_l(X)\equiv\inf\left\{\theta:\bar{X}(\theta)=X\right\}$ and $\theta_h(X)\equiv\sup\left\{\theta:\bar{X}(\theta)=X\right\}$ for any $X\in\mathcal{X}$. We will construct a different equilibrium, denoted by $\bar{E}$, which implements the same equilibrium exercise time $\tau^*(\theta)$ and has the structure specified in the formulation of the lemma. As we will see, it will imply that on the equilibrium path, the principal exercises the option at the first informative time $t\in\mathcal{T}$ at which he receives message $m(t)=1$, where the set $\mathcal{T}$ of informative times is defined as

$$\mathcal{T}\equiv\left\{t:X(t)=\bar{X}\text{ for some }\bar{X}\in\mathcal{X}\text{ and }X(s)<\bar{X}\ \forall s<t\ \right\},$$

i.e., the set of times when the process $X(t)$ reaches one of the thresholds in X for the first time.

For the collection of strategies (6) and (7) and the corresponding beliefs to be an equilibrium, we need to verify the IC conditions of the agent and the principal.

1 - IC of the agent. The IC condition of the agent requires that any type θ is better off sending a message $m(t)=1$ when $X(t)$ first reaches $\bar{X}(\theta)$ than following any other strategy. By Assumption 1, a deviation to sending $m(t)=1$ at any $t\notin\mathcal{T}$ does not lead the principal to change his beliefs, and hence, his behavior. Thus, it is without loss of generality to only consider deviations at $t\in\mathcal{T}$. There are two possible deviations: sending $m(t)=1$ before $X(t)$ first reaches $\bar{X}(\theta)$ and sending $m(t)=0$ at that moment and following some other strategy after that. Consider the first deviation: the agent of type θ can send $m(t)=1$ when $X(t)$ hits threshold $\bar{X}(\hat{\theta})$, $\hat{\theta}>\theta_h\left(\bar{X}(\theta)\right)$ for the first time, and then the principal will exercise immediately. Consider the second deviation: if type θ deviates to sending $m(t)=0$ when $X(t)$ hits threshold $\bar{X}(\theta)$, she can then either send $m(t)=1$ at one of the future $t\in\mathcal{T}$ or continue sending the message $m(t)=0$ at any future $t\in\mathcal{T}$. First, if the agent deviates to sending $m(t)=1$ at one of the future $t\in\mathcal{T}$, the principal will exercise the option at one of the thresholds $\hat{X}\in\mathcal{X}$, $\hat{X}>\bar{X}(\theta)$. Note that the agent can ensure exercise at any threshold $\hat{X}\in\mathcal{X}$ such that $\hat{X}\geq X(t)$ by adopting the equilibrium message strategy of type $\hat{\theta}$ at which $\bar{X}(\hat{\theta})=\hat{X}$. Second, if the agent deviates to sending $m(t)=0$ at all of the future $t\in\mathcal{T}$, there are two cases. If $\bar{X}(\underline{\theta})=\infty$, the principal will never exercise the option. If $\bar{X}(\underline{\theta})=\bar{X}_{\max}<\infty$, then the principal's belief when $X(t)$ first reaches $\bar{X}_{\max}$ is that $\theta=\underline{\theta}$, if $\bar{X}(\underline{\theta})\neq\bar{X}(\theta)$ $\forall\theta\neq\underline{\theta}$, or that $\theta\in\left[\underline{\theta},\theta_h\left(\bar{X}_{\max}\right)\right]$, otherwise. Upon receiving $m(t)=0$ at this moment, the principal does not change his belief by Assumption 1 and hence exercises the option at $\bar{X}_{\max}=\bar{X}(\underline{\theta})$. Finally, note that the agent cannot induce exercise at $\hat{X}\in\mathcal{X}$ if $\hat{X}<X(t)$: in this case, the principal's belief is that the agent's type is smaller than the type that could induce exercise at $\hat{X}$ and this belief cannot be reversed according to Assumption 1. Combining all possible deviations, at time t, the agent can deviate to exercise at any $\hat{X}\in\mathcal{X}$ as long as $\hat{X}\geq X(t)$. Using the same arguments as in the derivation of (29) above but for $I-b$ instead of I, it is easy to see that the agent's expected utility given exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\theta\bar{X}-I+b\right)$, where $\beta>1$ is given by (13). Hence, the IC condition of the agent is that

$$\left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I+b\right)\geq\max_{\hat{X}\in\mathcal{X},\hat{X}\geq X(t)}\left(\frac{X(t)}{\hat{X}}\right)^{\beta}\left(\theta\hat{X}-I+b\right).\tag{32}$$

Let us argue that it holds using the fact that E is an equilibrium. Suppose otherwise. Then, there

exists a pair $(\theta, \hat{X})$ with $\hat{X} \in \mathcal{X}$ such that

$$\frac{\theta\bar{X}(\theta) - I + b}{\bar{X}(\theta)^{\beta}} < \frac{\theta\hat{X} - I + b}{\hat{X}^{\beta}}. \tag{33}$$

However, (33) implies that in equilibrium E type θ is better off deviating from the message strategy $m^*(\theta)$ to the message strategy $m^*\left(\tilde{\theta}\right)$ of type $\tilde{\theta}$, where $\tilde{\theta}$ is any type satisfying $\bar{X}\left(\tilde{\theta}\right) = \hat{X}$ (since $\hat{X} \in \mathcal{X}$, at least one such $\tilde{\theta}$ exists). This is impossible, and hence (32) holds. Hence, if the principal plays strategy (7), the agent finds it optimal to play strategy (6).

Given Lemma 1 and the fact that the agent plays (6), the posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal). Next, consider the IC conditions of the principal. They are comprised of two parts, as evident from (7): we refer to the top line of (7) (exercising immediately when the principal "should" exercise) as the ex-post IC condition, and to the bottom line of (7) (not exercising when the principal "should" wait) as the ex-ante IC condition.

2 - "Ex-post" IC of the principal. First, consider the ex-post IC condition: we prove that the principal exercises immediately if the agent sends message $m(t) = 1$ at the first moment when $X(t)$ hits threshold $\hat{X}$ for some $\hat{X} \in \mathcal{X}$ (and sent message $m(t) = 0$ before). Given this message, the principal believes that $\theta \sim Uni[\theta_l(\hat{X}), \theta_h(\hat{X})]$. Because the principal expects the agent to play (6), the principal now expects the agent to send $m(t) = 1$ if $X(t) \geq \hat{X}$, and $m(t) = 0$ otherwise, regardless of $\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]$. Hence, the principal does not expect to learn any new information. This implies that the principal's problem is now equivalent to the standard option exercise problem with the option paying off $\frac{\theta_l(\hat{X})+\theta_h(\hat{X})}{2} X(t)$ upon exercise at time t. Using the same arguments as in the derivation of (29) above, the principal's expected payoff from exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\frac{\theta_l(\hat{X})+\theta_h(\hat{X})}{2}\bar{X} - I\right)$, which is an inverse U-shaped function with an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\theta_l(\hat{X})+\theta_h(\hat{X})}$. Thus, the solution of the problem is to exercise the option immediately if and only if

$$X(t) \geq \frac{\beta}{\beta - 1}\frac{2I}{\theta_l(\hat{X}) + \theta_h(\hat{X})}. \tag{34}$$

Let us show that any threshold $\hat{X} \in \mathcal{X}$ and the corresponding type cutoffs $\theta_l(\hat{X})$ and $\theta_h(\hat{X})$ in equilibrium E satisfy (34). Consider equilibrium E. For the principal to exercise at threshold $\bar{X}(\theta)$, the value that the principal gets upon exercise must be greater or equal than what he gets from delaying the exercise. The value from immediate exercise equals $\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\bar{X}(\theta) - I$, where $(\mathcal{H}_t, m(t))$ is any history of the sample path of $X(t)$ and equilibrium messages that leads to exercise at time t at threshold $\bar{X}(\theta)$ in equilibrium E. Because waiting until $X(t)$ hits a threshold $\tilde{X} > \bar{X}(\theta)$ and exercising then is a feasible strategy, the value from delaying exercise is greater or equal than the value from such a deviation, which equals $\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\tilde{X} - I\right)$. Hence, $\bar{X}(\theta)$ must satisfy

$$\bar{X}(\theta) \in \arg\max_{\tilde{X} \geq \bar{X}(\theta)} \left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\tilde{X} - I\right).$$

Using the fact that the unconditional maximizer of the right-hand side is $\tilde{X} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]}$ and that function $\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m_t]\tilde{X} - I\right)$ is inverted U-shaped in $\tilde{X}$, this condition can be equivalently re-written as

$$\bar{X}(\theta) \geq \frac{\beta}{\beta - 1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]},$$

for any history $(\mathcal{H}_t, m(t))$ with $X(t) = \bar{X}(\theta)$ and $m(s) = m_s^*(\mathcal{H}_s, \theta)$ for some $\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]$ and

$s \leq t$. Let $\mathbb{H}_t^*$ denote the set of such histories. Then,

$$\bar{X}(\theta) \geq \frac{\beta}{\beta-1} \max_{(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^*} \frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]},$$

or, equivalently,

$$\begin{aligned}
&\tfrac{\beta}{\beta-1}\tfrac{I}{\bar{X}(\theta)} \leq \min_{(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^*} \mathbb{E}[\theta|\mathcal{H}_t, m(t)] \\
&\leq \mathbb{E}\left[\mathbb{E}[\theta|\mathcal{H}_t, m(t)] \,|\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})], \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\theta|\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]\right] = \tfrac{\theta_l(\hat{X})+\theta_h(\hat{X})}{2},
\end{aligned}$$

where the inequality follows from the fact that the minimum of a random variable cannot exceed its mean, and the first equality follows from the law of iterated expectations. Therefore, when the principal obtains message $m = 1$ at threshold $\hat{X} \in \mathcal{X}$, his optimal reaction is to exercise immediately. Thus, the ex-post IC condition of the principal is satisfied.

3 - "Ex-ante" IC of the principal. Finally, consider the ex-ante IC condition of the principal stating that the principal is better off waiting following a history $\mathcal{H}_t$ with $m(s) = 0$, $s \leq t$, and $\max_{s \leq t} X(s) < \bar{X}(\underline{\theta})$. Given that the agent follows (7), for any such history $\mathcal{H}_t$, the principal's belief is that $\theta \sim Uni[\underline{\theta}, \theta_l(\hat{X})]$ for some $\hat{X} \in \mathcal{X}$. If the principal exercises immediately, her expected payoff is $\frac{\underline{\theta}+\theta_l(\hat{X})}{2} X(t) - I$. If the principal waits, her expected payoff is

$$\int_{\underline{\theta}}^{\theta_l(\hat{X})} \left(\frac{X(t)}{\bar{X}(\theta)}\right)^\beta \left(\theta\bar{X}(\theta) - I\right) \frac{1}{\theta_l(\hat{X}) - \underline{\theta}} d\theta.$$

Suppose that there exists a pair $\hat{X} \in \mathcal{X}$ and $\tilde{X} < \lim_{\theta \uparrow \theta_l(\hat{X})} \bar{X}(\theta)$ such that immediate exercise is optimal when $X(t) = \tilde{X}$:

$$\frac{\underline{\theta} + \theta_l(\hat{X})}{2}\tilde{X} - I > \int_{\underline{\theta}}^{\theta_l(\hat{X})} \left(\frac{\tilde{X}}{\bar{X}(\theta)}\right)^\beta \left(\theta\bar{X}(\theta) - I\right) \frac{1}{\theta_l(\hat{X}) - \underline{\theta}} d\theta. \tag{35}$$

We can re-write (35) as

$$\mathbb{E}_\theta\left[\left(\frac{1}{\tilde{X}}\right)^\beta \left(\theta\tilde{X} - I\right) |\theta < \theta_l(\hat{X})\right] > \mathbb{E}_\theta\left[\left(\frac{1}{\bar{X}(\theta)}\right)^\beta \left(\theta\bar{X}(\theta) - I\right) |\theta < \theta_l(\hat{X})\right]. \tag{36}$$

Let us show that if equilibrium E exists, then (36) must be violated. Consider equilibrium E, any type $\tilde{\theta} < \theta_l(\hat{X})$, time $t < \tau^*\left(\tilde{\theta}\right)$, and any history $(\mathcal{H}_t, m(t))$ such that $X(t) = \tilde{X}$, $\max_{s \leq t, s \in \mathcal{T}} X(s) = \hat{X}$, which is consistent with the equilibrium play of type $\tilde{\theta}$, i.e., with $m(s) = m_s^*\left(\tilde{\theta}, \mathcal{H}_s\right)$ $\forall s \leq t$. Let $\mathbb{H}_t^{**}(\tilde{\theta}, \tilde{X}, \hat{X})$ denote the set of such histories. Because the principal prefers waiting in equilibrium E, the payoff from immediate exercise in equilibrium E cannot exceed the payoff from waiting:

$$\begin{aligned}
&\mathbb{E}\left[\theta\tilde{X} - I|\mathcal{H}_t, m(t)\right] \leq \mathbb{E}\left[\left(\tfrac{\tilde{X}}{\bar{X}(\theta)}\right)^\beta \left(\theta\bar{X}(\theta) - I\right) |\mathcal{H}_t, m(t)\right] \Leftrightarrow \\
&\mathbb{E}\left[\left(\tfrac{1}{\bar{X}(\theta)}\right)^\beta \left(\theta\bar{X}(\theta) - I\right) - \left(\tfrac{1}{\tilde{X}}\right)^\beta \left(\theta\tilde{X} - I\right) |\mathcal{H}_t, m(t)\right] \geq 0.
\end{aligned}$$

This inequality must hold for all histories $(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^{**}(\tilde{\theta}, \tilde{X}, \hat{X})$. In any history $(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^{**}(\tilde{\theta}, \tilde{X}, \hat{X})$, the option is never exercised by time t if $\tilde{\theta} < \theta_l(\hat{X})$ and is exercised before time t if $\tilde{\theta} > \theta_l(\hat{X})$. Therefore, conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta} < \theta_l(\hat{X})$, the distribution of $\tilde{\theta}$ is independent of the sample path of $X(s), s \leq t$. Fixing $\tilde{X}$ and $\hat{X}$ and integrating over histories $(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^{**}(\tilde{\theta}, \tilde{X}, \hat{X})$ and then over

$\tilde{\theta} \in [\underline{\theta}, \theta_l(\hat{X}))$, we obtain that

$$\mathbb{E}_{\tilde{\theta}}\left[\mathbb{E}_{(\mathcal{H}_t, m(t))}\left[\left(\tfrac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right) - \left(\tfrac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X} - I\right) | (\mathcal{H}_t, m(t)) \in \mathbb{H}_t^{**}(\tilde{\theta}, \tilde{X}, \hat{X})\right] | \tilde{\theta} \in [\underline{\theta}, \theta_l(\hat{X})), \hat{X}, \tilde{X}\right]$$
$$\geq 0 \Leftrightarrow \mathbb{E}_{\theta}\left[\left(\tfrac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right) - \left(\tfrac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X} - I\right) | \theta < \theta_l(\hat{X})\right] \geq 0,$$

where we applied the law of iterated expectations and the conditional independence of the sample path of $X(t)$ and the distribution of $\tilde{\theta}$ (conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta} < \theta_l(\hat{X})$). Therefore, (36) cannot hold. Hence, the ex-ante IC condition of the principal is also satisfied.

Thus, if there exists a threshold exercise equilibrium E where $\tau^*(\theta) = \inf\{t \geq 0 | X(t) \geq \bar{X}(\theta)\}$ for some threshold $\bar{X}(\theta)$, then there exists a threshold exercise equilibrium $\bar{E}$ of the form specified in the lemma, in which the option is exercised at the same time. Finally, let us show that on the equilibrium path, the option is indeed exercised at the first informative time t at which the principal receives message $m(t) = 1$. Because any message sent at $t \notin \mathcal{T}$ does not lead to updating of the principal's beliefs and because of the second part of (7), the principal never exercises the option prior to the first informative time $t \in \mathcal{T}$ at which she receives message $m(t) = 1$. Consider the first informative time $t \in \mathcal{T}$ at which the principal receives $m(t) = 1$. By Bayes' rule, the principal believes that θ is distributed uniformly over $(\theta_l(X(t)), \theta_h(X(t)))$. Equilibrium strategy of the agent (6) implies $X(t) = \bar{X}(\theta)\ \forall\theta \in (\theta_l(X(t)), \theta_h(X(t)))$. Therefore, in equilibrium the principal exercises the option immediately. ■

Derivation of the principal's value function in the ω-equilibrium, $V_P(X(t), 1; \omega)$. It satisfies

$$rV_P(X, 1; \omega) = \mu X V_{P,X}(X, 1; \omega) + \frac{1}{2}\sigma^2 X^2 V_{P,XX}(X, 1; \omega). \tag{37}$$

The value matching condition is:

$$V_P(Y(\omega), 1; \omega) = \int_{\omega}^{1}(\theta Y(\omega) - I)\,d\theta + \omega V_P(Y(\omega), \omega; \omega). \tag{38}$$

The intuition behind (38) is as follows. With probability $1 - \omega$, θ is above ω. In this case, the agent recommends exercise, and the principal follows the recommendation. The payoff of the principal, given θ, is $\theta Y(\omega) - I$. With probability ω, θ is below ω, so the agent recommends against exercise, and the option is not exercised. The continuation payoff of the principal in this case is $V_P(Y(\omega), \omega; \omega)$. Solving (37) subject to (38), we obtain

$$V_P(X, 1; \omega) = \left(\frac{X}{Y(\omega)}\right)^{\beta}\left(\int_{\omega}^{1}(\theta Y(\omega) - I)\,d\theta + \omega V_P(Y(\omega), \omega; \omega)\right). \tag{39}$$

By stationarity,

$$V_P(Y(\omega), \omega; \omega) = V_P(\omega Y(\omega), 1; \omega). \tag{40}$$

Evaluating (39) at $X = \omega Y(\omega)$ and using the stationarity condition (40), we obtain:

$$V_P(\omega Y(\omega), 1; \omega) = \omega^{\beta}\left[\frac{1}{2}\left(1 - \omega^2\right)Y(\omega) - (1 - \omega)I\right] + \omega^{\beta+1}V_P(\omega Y(\omega), 1; \omega).$$

Therefore,

$$V_P(\omega Y(\omega), 1; \omega) = \frac{\omega^{\beta}(1 - \omega)}{1 - \omega^{\beta+1}}\left[\frac{1}{2}(1 + \omega)Y(\omega) - I\right]. \tag{41}$$

Plugging (41) into (39), we obtain the principal's value function (21). ■

Proof of Proposition 1. Part 1: Existence of equilibrium with continuous exercise. As shown in the text, the principal's ex-post IC is satisfied, and hence we only need to check the ex-ante IC (27). We show that (27) is satisfied if and only if $b \geq -I$. Using (26), (27) is equivalent to

$$\frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1} \geq \max_{X\in(0,X_A^*(1)]} X^{-\beta}\left(\frac{1}{2}X-I\right). \tag{42}$$

The function $X^{-\beta}\left(\frac{1}{2}X-I\right)$ is inverse U-shaped with a maximum at $\bar{X}_u \equiv \frac{\beta}{\beta-1}2I$, where $\bar{X}_u > X_A^*(1) \Leftrightarrow b > -I$. First, suppose that $b < -I$, and hence $\bar{X}_u < X_A^*(1)$. Then, (42) is equivalent to

$$\frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1} \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right) \Leftrightarrow \frac{1}{\beta+1}(I-b)^{-\beta}(I-\beta b) \geq (2I)^{-\beta} I. \tag{43}$$

Consider $f(b) \equiv (I-b)^{-\beta}(I-\beta b) - (\beta+1)(2I)^{-\beta} I$. Note that $f(-I) = 0$ and $f'(b) > 0$. Hence, $f(b) \geq 0 \Leftrightarrow b \geq -I$, and hence (42) is violated when $b < -I$.

Second, suppose that $b \geq -I$, and hence (43) is satisfied. Since, in this case, $\bar{X}_u \geq X_A^*(1)$, then

$$\max_{X\in(0,X_A^*(1)]} X^{-\beta}\left(\frac{1}{2}X-I\right) \leq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right),$$

and hence the inequality (42) follows from the fact that inequality (43) is satisfied.

Part 2. Existence of ω-equilibria. We first show that if $b < 0$, then for any positive $\omega < 1$, the principal's ex-post IC is strictly satisfied, i.e., $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$. Define:

$$G(\omega) \equiv \frac{\left(1-\omega^{\beta}\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)} - \frac{\beta}{\beta-1}\frac{2(I-b)}{1+\omega}.$$

Note that $G(\omega) = \frac{2(I-b)}{1+\omega} g(\omega)$, where $g(\omega) \equiv \frac{\left(1-\omega^{\beta}\right)(1+\omega)}{2\left(\omega-\omega^{\beta}\right)} - \frac{\beta}{\beta-1}$.We have:

$$\lim_{\omega\to 1} g(\omega) = \lim_{\omega\to 1}\frac{1-\omega^{\beta}-\beta\omega^{\beta-1}(1+\omega)}{2\left(1-\beta\omega^{\beta-1}\right)} - \frac{\beta}{\beta-1} = 0,$$
$$g'(\omega) = \frac{\beta\left(\omega^{\beta-1}-\omega^{\beta+1}\right)+\omega^{2\beta}-1}{2\left(\omega-\omega^{\beta}\right)^2},$$

where the first limit holds by l'Hopital's rule. Denote the numerator of $g'(\omega)$ by $h(\omega) \equiv \omega^{2\beta} - \beta\omega^{\beta+1} + \beta\omega^{\beta-1} - 1$. Function $h(\omega)$ is a generalized polynomial. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883),[15] the number of positive roots of $h(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $h(\omega)$, i.e., three. Because $\omega = 1$ is the root of $h(\omega)$ of order three and $h(0) < 0$, then $h(\omega) < 0$ for all $\omega \in [0,1)$, and hence $g'(\omega) < 0$ for all $\omega \in [0,1)$. Combined with $\lim_{\omega\to 1} g(\omega) = 0$, this implies $g(\omega) > 0$ and hence $G(\omega) > 0$ for all $\omega \in [0,1)$. Thus, $Y(\omega) > \frac{\beta}{\beta-1}\frac{2(I-b)}{1+\omega} > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$, where the second inequality follows from the fact that $b < 0$.

Hence, the ex-post IC condition of the principal is satisfied for any $\omega < 1$, which implies that the ω-equilibrium exists if and only if the ex-ante IC (24) is satisfied, where $V_P(X,1;\omega)$ is given by (21). Because $X^{-\beta}V_P(X,1;\omega)$ does not depend on X, we can rewrite (24) as

$$X^{-\beta}V_P(X,1;\omega) \geq \max_{X\in(0,Y(\omega)]} X^{-\beta}\left(\frac{1}{2}X-I\right). \tag{44}$$

We pin down the range of ω that satisfies this condition in the following steps.

Step 1: *If $b < 0$, $V_P(X,1;\omega)$ is strictly increasing in ω for any $\omega \in (0,1)$.*

[15] See Theorem 3.1 in Jameson (2006).

Because $V_P(X,1;\omega)$ is proportional to X^β, it is enough to prove the statement for $X=1$. We can re-write $V_P(1,1;\omega)$ as $2^{-\beta} f_1(\omega) f_2(\omega)$, where

$$f_1(\omega) \equiv \frac{(1-\omega)(1+\omega)^\beta}{1-\omega^{\beta+1}} \text{ and } f_2(\omega) \equiv \frac{\frac{1}{2}(1+\omega)Y(\omega) - I}{\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^\beta}. \tag{45}$$

Since, as shown above, $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$, then $\frac{1}{2}(1+\omega)Y(\omega) > \frac{\beta}{\beta-1}I > I$, and hence $f_2(\omega) > 0$ for $\omega < 1$. Because $f_1(\omega) > 0$ and $f_2(\omega) > 0$ for any $\omega \in (0,\omega^*)$, a sufficient condition for $V_P(1,1;\omega)$ to be increasing is that both $f_1(\omega)$ and $f_2(\omega)$ are increasing for $\omega \in (0,\omega^*)$.

First, consider $f_2(\omega)$. As an auxiliary result, we prove that $(1+\omega)Y(\omega)$ is strictly decreasing in ω. This follows from the fact that

$$\frac{\partial\left((1+\omega)Y(\omega)\right)}{\partial\omega} = (I-b)\frac{-1+\beta\omega^{\beta-1} - \beta\omega^{\beta+1} + \omega^{2\beta}}{\left(\omega-\omega^\beta\right)^2}$$

and that as shown above, the numerator, $h(\omega)$, is strictly negative for all $\omega \in [0,1)$. Next,

$$f_2'(\omega) = \frac{(\beta-1)(1+\omega)}{4\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega} - Y(\omega)\right)\frac{\partial\left((1+\omega)Y(\omega)\right)}{\partial\omega}.$$

Because $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$ as shown above, and because $(1+\omega)Y(\omega)$ is strictly decreasing in ω, $f_2'(\omega) > 0$ for any $\omega \in (0,\omega^*)$.

Second, consider $f_1(\omega)$. Note that

$$f_1'(\omega) = \frac{(1+\omega)^{\beta-1}}{1-\omega^{\beta+1}}\frac{\beta-1-(\beta+1)\omega+(\beta+1)\omega^\beta-(\beta-1)\omega^{\beta+1}}{1-\omega^{\beta+1}}.$$

Denote the numerator of the second fraction by $d(\omega) \equiv -(\beta-1)\omega^{\beta+1} + (\beta+1)\omega^\beta - (\beta+1)\omega + \beta - 1$. By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $d(\omega)$ does not exceed the number of sign changes of coefficients of $d(\omega)$, i.e., three. It is easy to show that $d(1) = d'(1) = d''(1) = 0$. Hence, $\omega = 1$ is the root of $d(\omega) = 0$ of order three, and $d(\omega)$ does not have roots other than $\omega = 1$. Since $d(0) = \beta - 1 > 0$, this implies that for any $\omega \in (0,1)$, $d(\omega) > 0$. Hence, $f_1'(\omega) > 0$, which completes the proof of this step.

Step 2: $\lim_{\omega\to1} V_P(X,1;\omega) = V_P^c(X,1)$.

According to (45), $V_P(X,1;\omega) = 2^{-\beta}X^\beta f_1(\omega) f_2(\omega)$. By l'Hopital's rule, $\lim_{\omega\to1} f_1(\omega) = \frac{2^\beta}{\beta+1}$, $\lim_{\omega\to1} Y(\omega) = \frac{\beta}{\beta-1}(I-b)$, and hence $\lim_{\omega\to1} f_1(\omega) = (\frac{\beta}{\beta-1}(I-b) - I)(\frac{\beta}{\beta-1}(I-b))^{-\beta}$.Using (26), it is easy to see that $\lim_{\omega\to1} V_P(X,1;\omega) = 2^{-\beta}X^\beta \lim_{\omega\to1} f_1(\omega) \lim_{\omega\to1} f_1(\omega) = V_P^c(X,1)$.

Step 3. *Suppose $-I < b < I$. For ω close enough to zero, the ex-ante IC condition (44) does not hold.*

The function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is inverse U-shaped and has a maximum at $\bar{X}_u \equiv \frac{\beta}{\beta-1}2I$. When ω is close to zero, $Y(\omega) = \frac{\left(1-\omega^\beta\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)} \to +\infty$, and hence $\max_{X\in(0,Y(\omega)]} X^{-\beta}\left(\frac{1}{2}X - I\right) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. Hence, we can rewrite (44) as $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$, and it is easy to show that it is equivalent to

$$\left(\omega-\omega^\beta\right)^{\beta-1} H(\omega) \geq (I-b)^\beta\left(1-\omega^{\beta+1}\right)\left(1-\omega^\beta\right)^\beta, \tag{46}$$

where

$$H(\omega) \equiv 2^{\beta-1}\beta^\beta\left(\frac{I}{\beta-1}\right)^{\beta-1}(1-\omega)\left(I(1-\omega)\left(1+\omega^\beta\right) - b(1+\omega)\left(1-\omega^\beta\right)\right).$$

Since $H(0) > 0$, then as $\omega \to 0$, the left-hand side of (46) converges to zero, while the right-hand side converges to $(I-b)^\beta > 0$. Hence, for ω close enough to 0, the ex-ante IC condition is violated.

Step 4. *Suppose $-I < b < I$. Then (44) is satisfied for any $\omega \geq \bar{\omega}$, where $\bar{\omega}$ is the unique solution to $Y(\omega) = \bar{X}_u$. For any $\omega < \bar{\omega}$, (44) is satisfied if and only if $X^{-\beta} V_P(X, 1; \omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.*

Note that for any $b > -I$, $\lim_{\omega \to 1} Y(\omega) = \frac{\beta(I-b)}{\beta - 1} < \frac{\beta}{\beta-1} 2I = \bar{X}_u$, and hence there exists a unique $\bar{\omega}$ such that $Y(\omega) \leq \bar{X}_u \Leftrightarrow \omega \geq \bar{\omega}$. Hence, (44) becomes

$$X^{-\beta} V_P(X, 1; \omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \text{ for } \omega \leq \bar{\omega}, \tag{47}$$

$$X^{-\beta} V_P(X, 1; \omega) \geq Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \text{ for } \omega \geq \bar{\omega}. \tag{48}$$

Suppose that (48) is satisfied for some $\tilde{\omega} \geq \bar{\omega}$. Because $Y(\omega)$ is decreasing, $Y(\tilde{\omega}) \geq Y(\omega)$ for $\omega \geq \tilde{\omega}$. Because $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is increasing for $X \leq \bar{X}_u$ and because $Y(\omega) \leq Y(\tilde{\omega}) \leq Y(\bar{\omega}) = \bar{X}_u$ for $\omega \geq \tilde{\omega} \geq \bar{\omega}$, we have $Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \leq Y(\tilde{\omega})^{-\beta}\left(\frac{1}{2}Y(\tilde{\omega}) - I\right)$ for any $\omega \geq \tilde{\omega}$. On the other hand, according to Step 1, $X^{-\beta}V_P(X, 1; \tilde{\omega}) \leq X^{-\beta}V_P(X, 1; \omega)$ for any $\omega \geq \tilde{\omega}$. Hence, if (48) is satisfied for $\tilde{\omega} \geq \bar{\omega}$, it is also satisfied for any $\omega \in [\tilde{\omega}, 1)$. Hence, to prove that (44) is satisfied for any $\omega \geq \bar{\omega}$, it is sufficient to prove (48) for $\omega = \bar{\omega}$. Using (21) and the fact that $Y(\bar{\omega}) = \bar{X}_u$, (48) for $\omega = \bar{\omega}$ is equivalent to

$$\begin{aligned}\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}}\bar{X}_u^{-\beta}\left(\frac{1}{2}(1+\bar{\omega})\bar{X}_u - I\right) &\geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \Leftrightarrow \frac{1}{2}\bar{X}_u\left(\frac{1-\bar{\omega}^2}{1-\bar{\omega}^{\beta+1}} - 1\right) \geq I\left(\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}} - 1\right) \\ &\Leftrightarrow \frac{1}{2I}\bar{X}_u \leq \frac{\bar{\omega} - \bar{\omega}^{\beta+1}}{\bar{\omega}^2 - \bar{\omega}^{\beta+1}} \Leftrightarrow \frac{\beta}{\beta-1} \leq \frac{\bar{\omega} - \bar{\omega}^{\beta+1}}{\bar{\omega}^2 - \bar{\omega}^{\beta+1}}\end{aligned} \tag{49}$$

Consider the function $Q(\omega) \equiv \frac{\omega - \omega^{\beta+1}}{\omega^2 - \omega^{\beta+1}}$. Note that $Q'(\omega) < 0 \Leftrightarrow q(\omega) \equiv (\beta - 1)\omega^{\beta} - \beta\omega^{\beta-1} + 1 > 0$. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $q(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $q(\omega)$, i.e., two. Since $q(1) = q'(1) = 0$, $q(\omega)$ does not have any roots on $(0, \infty)$ other than 1. Since $q''(1) > 0$, we have $q(\omega) > 0$ for all $\omega \in (0, 1)$, and hence $Q'(\omega) < 0$. By l'Hopital's rule, $\lim_{\omega \to 1} Q(\omega) = \frac{\beta}{\beta-1}$, and hence $\frac{\beta}{\beta-1} \leq Q(\omega)$ for any $\omega \in (0, 1)$, which proves (49).

Step 5. Combining the four steps above yields the proposition. First, if $b \leq -I$, then $I - b \geq 2I$, and hence $\lim_{\omega \to 1} Y(\omega) = \frac{\beta(I-b)}{\beta-1} \geq \bar{X}_u$. Since $Y(\omega)$ is decreasing, it implies that $Y(\omega) > \bar{X}_u$ for any $\omega < 1$, and hence (44) is equivalent to

$$X^{-\beta} V_P(X, 1; \omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right). \tag{50}$$

According to Steps 1 and 2, for any $\omega < 1$, $V_P(X, 1; \omega) < \lim_{\omega \to 1} V_P(X, 1; \omega) = V_P^c(X, 1)$. As shown in the proof of the equilibrium with continuous exercise above, $X^{-\beta}V_P^c(X, 1) \leq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$ for $b \leq -I$, and hence (50) is violated. Hence, there is no ω-equilibrium in this case.

Second, if $0 > b > -I$, then according to Step 4, (44) is satisfied for any $\omega \geq \bar{\omega}$, and for any $\omega \leq \bar{\omega}$ (44) is satisfied if and only if $X^{-\beta}V_P(X, 1; \omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. The left-hand side of this inequality is increasing in ω according to Step 1, while the right-hand side is constant. Hence, if (44) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega \geq \tilde{\omega}$. According to Step 3, for ω close to 0, (44) does not hold. Together, this implies that there exists a unique $\underline{\omega} \in (0, \bar{\omega})$ such that the principal's ex-ante IC (44) holds if and only if $\omega \geq \underline{\omega}$, and that $X^{-\beta}V_P(X, 1; \underline{\omega}) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. ■

Proof of Proposition 3. First, we prove that the principal's ex-ante IC constraint is violated in the equilibrium with continuous exercise, and hence such equilibrium does not exist. To show this, we prove that when the current belief of the principal is that $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, where $\hat{\theta}$ is sufficiently close to $\underline{\theta}$, then the principal is strictly better off exercising immediately at $X_A^*(\hat{\theta})$. Indeed, the value from waiting if the current

value of $X(t)$ is $X_A^*(\hat{\theta})$ is given by

$$\int_{\underline{\theta}}^{\hat{\theta}} \frac{1}{\hat{\theta}-\underline{\theta}} \left(\frac{X_A^*(\hat{\theta})}{X_A^*(\theta)}\right)^{\beta} \left(X_A^*(\theta)\theta - I\right) d\theta,$$

while the value from immediate exercise at $X_A^*(\hat{\theta})$ is $X_A^*(\hat{\theta})\frac{\hat{\theta}+\underline{\theta}}{2} - I$. Hence, we want to show that for a sufficiently small $\hat{\theta}$,

$$\begin{gathered} \int_{\underline{\theta}}^{\hat{\theta}} X_A^*(\theta)^{-\beta}\left(X_A^*(\theta)\theta - I\right) d\theta < X_A^*(\hat{\theta})^{-\beta}\left(\hat{\theta}-\underline{\theta}\right)\left(X_A^*(\hat{\theta})\frac{\hat{\theta}+\underline{\theta}}{2} - I\right) = \int_{\underline{\theta}}^{\hat{\theta}} X_A^*(\hat{\theta})^{-\beta}\left(X_A^*(\hat{\theta})\theta - I\right) d\theta \Leftrightarrow \\ \int_{\underline{\theta}}^{\hat{\theta}} \left[\phi\left(X_A^*(\theta),\theta\right) - \phi\left(X_A^*(\hat{\theta}),\theta\right)\right] d\theta < 0, \end{gathered} \tag{51}$$

where $\phi(X,\theta) \equiv X^{-\beta}(X\theta - I)$. We next show that (51) is satisfied for any $\hat{\theta} \in \left(\underline{\theta}, \frac{I-b}{I}\underline{\theta}\right)$. Indeed, for any such $\hat{\theta}$, we have $\hat{\theta} < \frac{I-b}{I}\underline{\theta} \Leftrightarrow X_P^*(\underline{\theta}) < X_A^*(\hat{\theta})$, and hence $X_A^*(\hat{\theta}) > X_P^*(\theta)$ for any $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$. The function $\phi(X,\theta)$ is inverse U-shaped with a maximum at $X_P^*(\theta)$, and since $X_P^*(\theta) < X_A^*(\hat{\theta}) < X_A^*(\theta)$ for any $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, then $\phi\left(X_A^*(\theta)\right) < \phi\left(X_A^*(\hat{\theta})\right)$, which implies (51).

Next, we consider the existence of the equilibrium with continuous exercise up to a cutoff. First, suppose that $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Note that this implies $b > -I$, and hence $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} \Leftrightarrow b + I > (I-b)\underline{\theta} \Leftrightarrow \hat{\theta}^* \equiv \frac{I-b}{I+b}\underline{\theta} < 1$. Given that the principal plays the strategy stated in the proposition, it is clear that the strategy of any type θ of the agent is incentive-compatible. Indeed, for any type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$, exercise occurs at her most preferred time. Therefore, no type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$ can benefit from a deviation. Any type $\theta < \left(\frac{I-b}{I+b}\right)\underline{\theta}$ cannot benefit from a deviation either: the agent would lose from inducing the principal to exercise earlier because she is biased towards late exercise, and it is not feasible for her to induce the principal to exercise later because the principal exercises at threshold $\hat{X}$ regardless of the recommendation.

We next show that the principal's ex-post IC constraint is satisfied. If the agent sends a message to exercise when $X(t) < \hat{X}$, the principal learns the agent's type θ and realizes that it is already too late ($X_P^*(\theta) < X_A^*(\theta)$) and thus does not benefit from delaying exercise even further. If the agent sends a message to exercise when $X(t) = \hat{X}$, the principal infers that $\theta \leq \hat{\theta}^*$ and that he will not learn any additional information by waiting more. Given the belief that $\theta \in \left[\underline{\theta}, \hat{\theta}^*\right]$, the optimal exercise threshold for the principal is given by

$$\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\hat{\theta}^*} = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\left(\frac{I-b}{I+b}\right)\underline{\theta}} = \frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}} = \hat{X},$$

and hence the ex-post IC constraint is satisfied.

Next, we consider the principal's ex-ante IC constraint. Let $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ denote the expected value to the principal in the equilibrium with continuous exercise up to a cutoff if the current value of $X(t)$ is X and the current belief is that $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$ for some $\hat{\theta} > \hat{\theta}^*$. If the agent's type is $\theta > \hat{\theta}^*$, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. If $\theta < \hat{\theta}^*$, exercise occurs at threshold $\hat{X}$. Hence,

$$\left(\hat{\theta}-\underline{\theta}\right)V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right) = X^{\beta}\int_{\underline{\theta}}^{\hat{\theta}^*} \hat{X}^{-\beta}\left(\theta\hat{X} - I\right)d\theta + X^{\beta}\int_{\hat{\theta}^*}^{\hat{\theta}} \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right)d\theta.$$

Given belief $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, the principal can either wait and get $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ or exercise immediately and

get $X\frac{\underline{\theta}+\hat{\theta}}{2}-I$. The current value of $X(t)$ satisfies $X(t)\le X_A^*(\hat{\theta})$ because otherwise, the principal's belief would not be that $\theta\in\left[\underline{\theta},\hat{\theta}\right]$. Hence, the ex-ante IC condition requires that for any $\hat{\theta}>\hat{\theta}^*$, $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\ge X\frac{\underline{\theta}+\hat{\theta}}{2}-I$ for any $X\le X_A^*(\hat{\theta})$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, this condition is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\ge\max_{X\in(0,X_A^*(\hat{\theta})]}\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).\tag{52}$$

The function $\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right)$ is inverse U-shaped and has an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\hat{\theta}}$, which is strictly greater than $X_A^*(\hat{\theta})$ for any $\hat{\theta}>\frac{I-b}{I+b}\underline{\theta}=\hat{\theta}^*$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, (52) is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\ge X_A^*(\hat{\theta})^{-\beta}\left(X_A^*(\hat{\theta})\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).$$

Suppose there exists $\hat{\theta}$ for which the ex-ante IC constraint is violated, i.e.,

$$\int_{\underline{\theta}}^{\hat{\theta}^*}\hat{X}^{-\beta}\left(\theta\hat{X}-I\right)d\theta+\int_{\hat{\theta}^*}^{\hat{\theta}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta<\left(\hat{\theta}-\underline{\theta}\right)X_A^*(\hat{\theta})^{-\beta}\left(X_A^*(\hat{\theta})\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).\tag{53}$$

We show that this implies that the contract derived in Lemma 4 cannot be optimal, which is a contradiction. In particular, denote the contract from the second part of Lemma 4 by $\hat{X}_-(\theta)$. Then (53) implies that the contract $\hat{X}_-(\theta)$ is dominated by the contract with continuous exercise at $X_A^*(\theta)$ for $\theta\ge\hat{\theta}$ and exercise at $X_A^*(\hat{\theta})$ for $\theta\le\hat{\theta}$. Indeed, the principal's expected utility under the contract $\hat{X}_-(\theta)$, divided by $X(0)^\beta$, is

$$\int_{\underline{\theta}}^{\hat{\theta}^*}\hat{X}^{-\beta}\left(\theta\hat{X}-I\right)d\theta+\int_{\hat{\theta}^*}^{1}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta.\tag{54}$$

Similarly, the principal's expected utility under the modified contract (also divided by $X(0)^\beta$), where $\frac{I-b}{I+b}\underline{\theta}$ in $\hat{X}_-(\theta)$ is replaced by $\hat{\theta}$, and the cutoff $\frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}}$ in $\hat{X}_-(\theta)$ is replaced by $X_A^*(\hat{\theta})=\frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, is given by

$$\int_{\underline{\theta}}^{\hat{\theta}}X_A^*(\hat{\theta})^{-\beta}\left(\theta X_A^*(\hat{\theta})-I\right)d\theta+\int_{\hat{\theta}}^{1}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta.\tag{55}$$

Combining (54) and (55), it is easy to see that the contract with continuous exercise up to the cutoff $\hat{\theta}$ dominates the contract $\hat{X}_-(\theta)$ if and only if (53) is satisfied. Hence, the ex-ante IC constraint is indeed satisfied.

Finally, consider $b\le-\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. According to Lemma 4, if $b\le-\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the optimal contract is characterized by $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$, i.e., the uninformed exercise threshold of the principal. Denote V_u the expected utility of the principal under this contract. Consider any equilibrium of the communication game, and note that the payoff of the principal in this equilibrium cannot be higher than V_u: otherwise, the contract $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ would not be optimal. It cannot be lower than V_u either: otherwise, the principal would be better off deviating to exercising at $\bar{X}=\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Hence, the payoff of the principal in any equilibrium is exactly V_u. According to the proof of Lemma 4, when $\underline{\theta}>0$, then for any b, there is a unique exercise policy $\hat{X}(\theta)$ that maximizes the principal's expected utility, and hence any equilibrium must be characterized by the principal exercising at $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. ■

Proof of Lemma 3. We can rewrite this equation as

$$\omega=\frac{(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}\Leftrightarrow\frac{2\beta I\left(\omega-\omega^{\beta}\right)+(\beta-1)(I-b)\left(\omega^{\beta+1}-\omega-1+\omega^{\beta}\right)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}=0.$$

Denote the left-hand side of the second equation as a function of ω by $l(\omega)$. The denominator of $l(\omega)$, $l_d(\omega)$, is nonnegative on $\omega\in[0,1]$ and equals zero only at $\omega=1$. This follows from $l_d(0)=2\beta I-(\beta-1)(I-b)>0$, $l_d(1)=0$, and $l_d'(\omega)=\beta(\beta-1)\omega^{\beta-2}(-2I+\omega(I-b))<0$.Therefore, $l(\omega)=0$ if and only if the numerator of $l(\omega)$, $l_n(\omega)$, equals zero at $\omega\in(0,1)$. Since $b\in(0,I)$, then $l_n(0)=-(\beta-1)(I-b)<0$,

$$\begin{aligned} l_n'(\omega) &= 2\beta I\left(1-\beta\omega^{\beta-1}\right)+(\beta-1)(I-b)\left((\beta+1)\omega^{\beta}-1+\beta\omega^{\beta-1}\right), \\ l_n''(\omega) &= -2\beta^2(\beta-1)I\omega^{\beta-2}+(\beta-1)(I-b)\left(\beta(\beta+1)\omega^{\beta-1}+\beta(\beta-1)\omega^{\beta-2}\right), \end{aligned}$$

and

$$l_n''(\omega)<0\Leftrightarrow(I-b)((\beta+1)\omega+\beta-1)<2\beta I\Leftrightarrow\omega<\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)}.$$

Since $\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)}>1$, $l_n''(\omega)<0$ for any $\omega\in[0,1]$. Since $l_n'(0)=2\beta I-(\beta-1)(I-b)>0$ and $l_n'(1)=-2\beta(\beta-1)b<0$, there exists $\hat{\omega}\in(0,1)$ such that $l_n(\omega)$ increases to the left of $\hat{\omega}$ and decreases to the right. Since $\lim_{\omega\to1}l_n(\omega)=0$, then $l_n(\hat{\omega})>0$, and hence $l_n(\omega)$ has a unique root ω^* on $(0,1)$.

Since the function $l_n(\omega)$ increases in b and is strictly increasing at the point ω^*, then ω^* decreases in b. To prove that $\lim_{b\to0}\omega^*=1$, it is sufficient to prove that for any small $\varepsilon>0$, there exists $b(\varepsilon)>0$ such that $l_n(1-\varepsilon)<0$ for $b<b(\varepsilon)$. Since $l_n(\omega)>0$ on $(\omega^*,1)$, this would imply that $\omega^*\in(1-\varepsilon,1)$, i.e., that ω^* is infinitely close to 1 when b is close to zero. Using the expression for $l_n(\omega)$, $l_n(\omega)<0$ is equivalent to

$$\frac{2\beta}{\beta-1}\frac{\omega}{\omega+1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}<1-\frac{b}{I}. \tag{56}$$

Denote the left-hand side of (56) by $L(\omega)$. Note that $L(\omega)$ is increasing on $(0,1)$. Indeed, differentiating $L(\omega)$ and simplifying, $L'(\omega)>0\Leftrightarrow\lambda(\omega)\equiv1-\omega^{2\beta}-\beta\omega^{\beta-1}+\beta\omega^{\beta+1}>0$. The function $\lambda(\omega)$ is decreasing on $(0,1)$ because $\lambda'(\omega)<0\Leftrightarrow\varphi(\omega)\equiv-2\omega^{\beta+1}-(\beta-1)+(\beta+1)\omega^2<0$, where $\varphi'(\omega)>0$ and $\varphi(1)=0$. Since $\lambda(\omega)$ is decreasing and $\lambda(1)=0$, then, indeed, $\lambda(\omega)>0$ and hence $L'(\omega)>0$ for all $\omega\in(0,1)$. In addition, by l'Hopital's rule, $\lim_{\omega\to1}L(\omega)=1$. Hence, $L(1-\varepsilon)<1$ for any $\varepsilon>0$, and thus $l_n(1-\varepsilon)<0$ for $b\in[0,I(1-L(1-\varepsilon)))$.

Finally, to prove that $\lim_{b\to I}\omega^*=0$, it is sufficient to prove that for any small $\varepsilon>0$, there exists $b(\varepsilon)$ such that $l_n(\varepsilon)>0$ for $b>b(\varepsilon)$. Since $l_n(0)<0$, this would imply that $\omega^*\in(0,\varepsilon)$ for $b>b(\varepsilon)$, i.e., that ω^* is infinitely close to zero when b is close to I. Based on (56), $l_n(\omega)>0\Leftrightarrow L(\omega)>1-\frac{b}{I}$. Then, for any $\varepsilon>0$, if $b>I(1-L(\varepsilon))$, we get $1-\frac{b}{I}<L(\varepsilon)\Leftrightarrow l_n(\varepsilon)>0$, which completes the proof. ■

Proof of Proposition 4. Since the agent's IC condition is guaranteed by (18), we only have to ensure that the principal's ex-post and ex-ante IC conditions are satisfied. First, we check the principal's ex-post IC condition (20). To see this, we start with proving that $Y(\omega)$ is strictly decreasing in ω for $\omega\in(0,1)$. Note that

$$\frac{\partial Y(\omega)}{\partial\omega}=\frac{(I-b)}{\omega\left(\omega-\omega^{\beta}\right)^2}\left[-(\beta-1)\omega^{\beta+1}+\beta\omega^{\beta}-\omega\right],$$

where $\frac{(I-b)}{\omega(\omega-\omega^{\beta})^2}>0$. Thus, we need to show that $k(\omega)\equiv-(\beta-1)\omega^{\beta+1}+\beta\omega^{\beta}-\omega<0$. According to an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $k(\omega)=0$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., two. Since $k(1)=0$, $k'(1)=0$, and $k"(1)=-\beta(\beta-1)<0$, $\omega=1$ is a root of order two, and there are no other positive roots. Further, $k(0)=0$ and $k'(0)=-1<0$. It follows that $k(0)=k(1)=0$ and $k(\omega)<0$ for all $\omega\in(0,1)$, and hence, indeed, $\frac{\partial Y(\omega)}{\partial\omega}<0$.

Since $\lim_{\omega \to 0} Y(\omega) = +\infty$, and $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ has only one solution $\omega = \omega^*$ according to Lemma 3, it follows that the principal's ex-post IC condition is equivalent to $\omega \leq \omega^*$.

Next, we check the principal's ex-ante IC condition (24), which is equivalent to (44), where $V_P(X, 1; \omega)$ is given by (21). We pin down the range of ω that satisfies this condition in three steps, which are similar to the steps used in the proof of Proposition 1.

Step 1: *If $b > 0$, $V_P(X, 1; \omega)$ is strictly increasing in ω for any $\omega \in (0, \omega^*)$.*

The proof of this step is the same as the proof of Step 1 in the proof of Proposition 1 with the only difference: instead of relying on the inequality $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega \in (0,1)$ as in the proof of Proposition 1 (which holds for $b < 0$), we rely on the inequality $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega \in (0, \omega^*)$, which was proved above.

Step 2: *If $0 < b < I$, then the ex-ante IC condition (44) holds as a strict inequality for $\omega = \omega^*$.*

Using (21) and $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$, we can rewrite $V_P(X, 1; \omega^*)$ as $X^\beta K(\omega^*)$, where

$$K(\omega) \equiv \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{\beta}{\beta-1} \frac{2I}{\omega+1} \right)^{-\beta} \frac{I}{\beta-1}.$$

Note that $K(0) = \bar{X}_u^{-\beta} \left(\frac{1}{2}\bar{X}_u - I \right)$ and that

$$K'(\omega) > 0 \Leftrightarrow \kappa(\omega) \equiv -(\beta-1)\omega^{\beta+1} + (\beta+1)\omega^\beta - (\beta+1)\omega + \beta - 1 > 0.$$

By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $\kappa(\omega)$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., three. Note that $\omega = 1$ is the root of $\kappa(\omega)$ of order three: $\kappa(1) = \kappa'(1) = \kappa''(1) = 0$, and hence there are no other roots. Since $\kappa(0) = \beta - 1 > 0$, it follows that $\kappa(\omega) > 0$ and hence $K'(\omega) > 0$ for all $\omega \in [0, 1)$. Therefore, $K(\omega)$ is strictly increasing in ω, which implies

$$X^{-\beta} V_P(X, 1; \omega^*) = K(\omega^*) > K(0) = \bar{X}_u^{-\beta} \left(\frac{1}{2}\bar{X}_u - I \right). \tag{57}$$

Because the function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ achieves its global maximum at the point $\bar{X}_u$, (57) implies that (44) holds as a strict inequality for $\omega = \omega^*$, which completes the proof of step 2.

Step 3: Combining the steps above yields the proposition. Suppose $b < I$. As shown above, the ex-post IC condition holds if and only if $\omega \leq \omega^*$. Recall that $Y(\omega^*) = \frac{\beta}{\beta-1}\frac{2I}{\omega^*+1} < \frac{\beta}{\beta-1}2I = \bar{X}_u$, and hence $\omega^* > \bar{\omega}$. According to Step 4 from the proof of Proposition 1, the ex-ante IC condition (44) is satisfied for any $\omega \geq \bar{\omega}$, and for any $\omega \leq \bar{\omega}$ (44) is satisfied if and only if $X^{-\beta}V_P(X, 1; \omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. The left-hand side of this inequality is increasing in ω for $\omega \leq \omega^*$ according to Step 1, while the right-hand side is constant. Together, this implies that if (44) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega \geq \tilde{\omega}$. According to Step 3 from the proof of Proposition 1, for ω close to 0, (44) does not hold. Hence, there exists a unique $\underline{\omega} \in (0, \bar{\omega})$ such that the principal's ex-ante IC (44) holds if and only if $\omega \geq \underline{\omega}$, and $X^{-\beta}V_P(X, 1; \underline{\omega}) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. Because, $\underline{\omega} < \bar{\omega}$ and $\bar{\omega} < \omega^*$, we have $\underline{\omega} < \omega^*$. We conclude that both the ex-post and the ex-ante IC conditions hold if and only if $\omega \in [\underline{\omega}, \omega^*]$. Finally, consider $b \geq I$. In this case, all types of agents want immediate exercise, which implies that the principal must exercise the option at the optimal uninformed threshold $\bar{X}_u = \frac{\beta}{\beta-1}2I$. ■

Proof of Proposition 5. The expected utility of the principal in the ω-equilibrium is $V_P(X, 1; \omega)$, given by (21). As shown in Step 1 in the proof of Proposition 4, $V_P(X, 1; \omega)$ is strictly increasing in ω for $\omega \in (0, \omega^*)$. Hence, $V_P(X, 1; \omega^*) > V_P(X, 1; \omega)$ for any $\omega < \omega^*$.

Denote the ex-ante expected utility of the agent (before the agent's type is realized) by $V_A(X, 1; \omega)$. Repeating the derivation of the principal's value function $V_P(X, 1; \omega)$ in the appendix above, it is easy to

see that

$$V_A(X,1;\omega) = \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X}{Y(\omega)}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega) - (I-b)\right).$$

The only difference of this expression from the expression for $V_P(X,1;\omega)$ given by (21) is that I in the second bracket of (21) is replaced by $(I-b)$. To prove that $V_A(X,1;\omega^*) > V_A(X,1;\omega)$ for any $\omega < \omega^*$, we prove that $V_A(X,1;\omega)$ is strictly increasing in ω for $\omega \in (0,\omega^*)$. The proof repeats the arguments of Step 1 in the proof of Proposition 4. In particular, we can re-write $V_A(X,1;\omega)$ as $2^{-\beta}X^{\beta}f_1(\omega)\tilde{f}_2(\omega)$, where

$$f_1(\omega) \equiv \frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}} \text{ and } \tilde{f}_2(\omega) \equiv \frac{\frac{1}{2}(1+\omega)Y(\omega) - (I-b)}{\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta}}.$$

As shown in Step 1 in the proof of Proposition 4, $f_1(\omega) > 0$ and $f_1'(\omega) > 0$. In addition, $\tilde{f}_2(\omega) > 0$ because $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega} > \frac{2(I-b)}{1+\omega}$ for any $\omega < \omega^*$, and $\tilde{f}_2'(\omega) > 0$ for the same reasons why $f_2'(\omega) > 0$ in the proof of Proposition 4. Hence, $V_A'(X,1;\omega) > 0$ for any $\omega \in (0,\omega^*)$, which completes the proof. ■

Proof of Proposition 6. First, consider the case $b < 0$. Proposition 1 shows that in the dynamic communication game, there exists an equilibrium with continuous exercise, where for each type θ, the option is exercised at the threshold $X_A^*(\theta)$. No such equilibrium exists in the static communication game. Indeed, continuous exercise requires that the principal perfectly infers the agent's type. However, since the principal gets this information at time 0, he will exercise the option at $X_P^*(\theta) \neq X_A^*(\theta)$.

We next show that no stationary equilibrium with partitioned exercise exists in the static communication game either. To see this, note that for such an equilibrium to exist, the following conditions must hold. First, the boundary type ω must be indifferent between exercise at $\bar{X}$ and at $\frac{\bar{X}}{\omega}$. Repeating the derivations in Section 3, this requires that (18) holds: $\bar{X} = \frac{(1-\omega^{\beta})(I-b)}{\omega(1-\omega^{\beta-1})}$. Second, given that the exercise threshold $\bar{X}$ is optimally chosen by the principal given the belief that $\theta \in [\omega, 1]$, it must satisfy $\bar{X} = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. Combining these two equations, ω must be the solution to (28), which can be rewritten as

$$2\beta I\left(\omega - \omega^{\beta}\right) - (\beta-1)(I-b)(1+\omega)\left(1-\omega^{\beta}\right) = 0. \tag{58}$$

We next show that the left-hand side of (58) is negative for any $b < 0$ and $\omega < 1$. Since $b < 0$, it is sufficient to prove that

$$2\beta\left(\omega-\omega^{\beta}\right) < (\beta-1)(1+\omega)\left(1-\omega^{\beta}\right) \Leftrightarrow s(\omega) \equiv 2\beta\left(\omega-\omega^{\beta}\right) + (\beta-1)\left(\omega^{\beta+1}-\omega-1+\omega^{\beta}\right) < 0.$$

It is easy to show that $s'(1) = 0$ and that $s''(\omega) < 0 \Leftrightarrow \omega < 1$, and hence $s'(\omega) > 0$ for any $\omega < 1$. Since $s(1) = 0$, then, indeed, $s(\omega) < 0$ for all $\omega < 1$.

Next, consider the case $b > 0$. As argued above, for ω-equilibrium to exist in the static communication game, ω must satisfy (28). According to Lemma 3, for $b > 0$, this equation has a unique solution, denoted by ω^*. Thus, among equilibria with $\omega \in [\underline{\omega}, \omega^*]$, which exist in the dynamic communication game, only equilibrium with $\omega = \omega^*$ is an equilibrium of the static communication game. ■

For Lemma 4, we prove the following lemma, which characterizes the structure of any incentive-compatible decision-making rule and is an analogue of Proposition 1 in Melumad and Shibano (1991) for the payoff specification in our model:

Lemma A.1. *An incentive-compatible threshold schedule $\hat{X}(\theta)$ must satisfy the following conditions:*

1. *$\hat{X}(\theta)$ is weakly decreasing in θ.*
2. *If $\hat{X}(\theta)$ is strictly decreasing on (θ_1, θ_2), then $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$.*

3. *If $\hat{X}(\theta)$ is discontinuous at $\hat{\theta}$, then the discontinuity satisfies*

$$\hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right) = \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right), \tag{59}$$

$$\hat{X}(\theta) = \begin{cases} \hat{X}^-(\hat{\theta}), & \forall\theta \in \left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\hat{\theta}\right), \\ \hat{X}^+(\hat{\theta}), & \forall\theta \in \left(\hat{\theta},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right], \end{cases} \tag{60}$$

$$\hat{X}(\hat{\theta}) \in \left\{\hat{X}^-(\hat{\theta}),\hat{X}^+(\hat{\theta})\right\}, \tag{61}$$

where $\hat{X}^-(\hat{\theta}) \equiv \lim_{\theta\uparrow\hat{\theta}}\hat{X}(\theta)$ and $\hat{X}^+(\hat{\theta}) \equiv \lim_{\theta\downarrow\hat{\theta}}\hat{X}(\theta)$.

Proof of Lemma A.1. **Proof of Part 1**. The first part of the lemma can be proven by contradiction. Suppose there exist $\theta_1,\theta_2 \in \Theta$, $\theta_2 > \theta_1$, such that $\hat{X}(\theta_2) > \hat{X}(\theta_1)$. Note that $\hat{U}_A(\hat{X},\theta) \equiv X(0)^\beta \hat{X}^{-\beta}(\theta\hat{X} - I + b)$ and $\hat{U}_D(\hat{X},\theta) \equiv X(0)^\beta \hat{X}^{-\beta}(\theta\hat{X} - I)$. The agent's IC constraint for $\theta = \theta_1$ and $\hat{\theta} = \theta_2$, $\hat{U}_A\left(\hat{X}(\theta_1),\theta_1\right) \geq \hat{U}_A\left(\hat{X}(\theta_2),\theta_1\right)$, can be written in the integral form:

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)} \left(\frac{X(0)}{\hat{X}}\right)^\beta \frac{-(\beta-1)\theta_1\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} \leq 0. \tag{62}$$

Because $\theta_2 > \theta_1$ and $\beta > 1$, (62) implies

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)} \left(\frac{X(0)}{\hat{X}}\right)^\beta \frac{-(\beta-1)\theta_2\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} < 0,$$

or, equivalently, $\hat{U}_A(\hat{X}(\theta_1),\theta_2) > \hat{U}_A(\hat{X}(\theta_2),\theta_2)$. However, this violates the agent's incentive compatibility constraint $\hat{U}_A(\hat{X}(\theta_2),\theta_2) \geq \hat{U}_A(\hat{X}(\theta_1),\theta_2)$ for $\theta = \theta_2$ and $\hat{\theta} = \theta_1$. Thus, $\hat{X}(\theta)$ is weakly decreasing in θ.

Proof of Part 2. To prove the second part of the lemma, note that $\hat{U}_A\left(\hat{X},\theta\right)$ is differentiable in θ for all $\hat{X} \in (X(0),\infty)$. Because $\hat{U}_A(\hat{X},\theta)$ is linear in θ, it satisfies the Lipschitz condition and hence is absolutely continuous in θ for all $\hat{X} \in (X(0),\infty)$. Also, $\frac{\partial\hat{U}_A(\hat{X},\theta)}{\partial\theta} = \left(\frac{X(0)}{\hat{X}}\right)^\beta \hat{X}$, and hence $\sup_{\hat{X}\in\mathbf{X}}\left|\frac{\partial\hat{U}_A(\hat{X},\theta)}{\partial\theta}\right|$ is integrable on $\theta \in \Theta$. By the generalized envelope theorem (see Corollary 1 in Milgrom and Segal, 2002), the equilibrium utility of the agent in any mechanism implementing exercise at thresholds $\hat{X}(\theta)$, $\theta \in \Theta$, denoted $V_A(\theta)$, satisfies the integral condition,

$$V_A(\theta) = V_A(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left(\frac{X(0)}{\hat{X}(s)}\right)^\beta \hat{X}(s)\,ds.$$

On the other hand, $V_A(\theta) = \hat{U}_A\left(\hat{X}(\theta),\theta\right)$. At any point θ at which $\hat{X}(\theta)$ is strictly decreasing, we have

$$\frac{dV_A(\theta)}{d\theta} = \frac{d\hat{U}_A(\hat{X}(\theta),\theta)}{d\theta} \Leftrightarrow \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\hat{X}(\theta) = \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\hat{X}(\theta) - \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\frac{(\beta-1)\theta\hat{X}(\theta) - \beta(I-b)}{\hat{X}(\theta)}\frac{d\hat{X}(\theta)}{d\theta}.$$

Because $d\hat{X}(\theta) < 0$, it must be that $(\beta-1)\theta\hat{X}(\theta) - \beta(I-b) = 0$. Thus, $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, which proves the second part of the lemma.

Proof of Part 3. Finally, consider the third part of the lemma. Eq. (59) follows from (60), continuity of $\hat{U}_A(\cdot)$, and incentive compatibility of the contract. Otherwise, for example, if $\hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right) > \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right)$, then $\hat{U}_A\left(\hat{X}(\hat{\theta}-\varepsilon),\hat{\theta}-\varepsilon\right) = \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}-\varepsilon\right) < \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}-\varepsilon\right)$ for a sufficiently

small ε, and hence types close enough to $\hat{\theta}$ from below would benefit from a deviation to $\hat{X}^{+}(\hat{\theta})$, i.e., from mimicking types slightly above $\hat{\theta}$.

Next, we prove (60). First, note that, (60) is satisfied at the boundaries. Indeed, denote $\theta_1^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}$ and suppose that $\hat{X}(\theta_1^*) \neq \hat{X}^{-}(\hat{\theta})$. Then, by the first part of the lemma, $\hat{X}(\theta_1^*) > \hat{X}^{-}(\hat{\theta})$. Because $\hat{X}^{-}(\hat{\theta}) \equiv \lim_{\theta\uparrow\hat{\theta}} \hat{X}(\theta)$, there exists $\varepsilon > 0$ such that $\hat{X}(\theta_1^*) > \hat{X}(\hat{\theta}-\varepsilon) \geq \hat{X}^{-}(\hat{\theta})$. Because the function $\hat{U}_A(\hat{X}, \theta_1^*)$ has a maximum at $\hat{X}^{-}(\hat{\theta})$ and is strictly decreasing for $\hat{X} > \hat{X}^{-}(\hat{\theta})$, this would imply $\hat{U}_A(\hat{X}(\theta_1^*), \theta_1^*) < \hat{U}_A(\hat{X}\left(\hat{\theta}-\varepsilon\right), \theta_1^*)$, and hence would contradict the IC condition for type θ_1^*. The proof for the boundary $\theta_2^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}$ is similar.

We next prove (60) for interior values of θ. First, suppose that $\hat{X}(\theta) \neq \hat{X}^{-}(\hat{\theta})$ for some $\theta \in \left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \hat{\theta}\right)$. By part 1 of the lemma, $\hat{X}(\theta) > \hat{X}^{-}(\hat{\theta})$. By incentive compatibility, $\hat{U}_A(\hat{X}(\theta), \theta) \geq \hat{U}_A(\hat{X}^{-}(\hat{\theta}), \theta)$, which can be written in the integral form as:

$$\int_{\hat{X}^{-}(\hat{\theta})}^{\hat{X}(\theta)} \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \frac{-(\beta-1)\theta\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} \geq 0.$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}^{-}(\hat{\theta}), \hat{X}(\theta))$ is non-empty. Thus, we can replace θ by $\tilde{\theta} < \theta$ under the integral and get a strict inequality: $\hat{U}_A(\hat{X}(\theta), \tilde{\theta}) > \hat{U}_A(\hat{X}^{-}(\hat{\theta}), \tilde{\theta})$ for every $\tilde{\theta} \in [\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \theta)$. However, this contradicts $\hat{X}^{-}(\hat{\theta}) = \arg\max_{\hat{X}} \hat{U}_A(\hat{X}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})})$. Second, suppose that $\hat{X}(\theta) \neq \hat{X}^{+}(\hat{\theta})$ for some $\theta \in \left(\hat{\theta}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right)$. By part 1 of the lemma, $\hat{X}(\theta) < \hat{X}^{+}(\hat{\theta})$. By incentive compatibility, $\hat{U}_A\left(\hat{X}(\theta), \theta\right) \geq \hat{U}_A\left(\hat{X}^{+}(\theta), \theta\right)$, which can be written as

$$\int_{\hat{X}(\theta)}^{\hat{X}^{+}(\theta)} \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \frac{-(\beta-1)\theta\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} \leq 0.$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}(\theta), \hat{X}^{+}(\hat{\theta}))$ is non-empty. Therefore, we can replace θ by $\tilde{\theta} > \theta$ under the integral and get a strict inequality, $\hat{U}_A\left(\hat{X}(\theta), \tilde{\theta}\right) > \hat{U}_A\left(\hat{X}^{+}(\theta), \tilde{\theta}\right)$, for every $\tilde{\theta} \in \left(\theta, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right]$. However, this contradicts $\hat{X}^{+}(\hat{\theta}) = \arg\max_{\hat{X}} \hat{U}_A\left(\hat{X}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right)$.

Finally, (61) follows from the continuity of $\hat{U}_A(\cdot)$ and incentive compatibility of $\hat{X}(\theta)$. Because $\hat{\theta} \in (\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})})$, every policy with thresholds strictly below $\hat{X}^{-}(\hat{\theta})$ or strictly above $\hat{X}^{+}(\hat{\theta})$ is strictly dominated by $\hat{X}^{-}(\hat{\theta})$ and $\hat{X}^{+}(\hat{\theta})$, respectively, and thus cannot be incentive-compatible. Suppose that $\hat{X}(\hat{\theta}) \in \left(\hat{X}^{-}(\hat{\theta}), \hat{X}^{+}(\hat{\theta})\right)$. Incentive compatibility and (59) imply $\hat{U}_A\left(\hat{X}(\hat{\theta}), \hat{\theta}\right) \geq \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \hat{\theta}\right) = \hat{U}_A\left(\hat{X}^{+}(\hat{\theta}), \hat{\theta}\right)$. Because $\hat{U}_A\left(\hat{X}, \hat{\theta}\right)$ is strictly increasing in $\hat{X}$ for $\hat{X} < \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$ and strictly decreasing in $\hat{X}$ for $\hat{X} > \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, the inequality must be strict: $\hat{U}_A\left(\hat{X}(\hat{\theta}), \hat{\theta}\right) > \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \hat{\theta}\right) = \hat{U}_A\left(\hat{X}^{+}(\hat{\theta}), \hat{\theta}\right)$. However, this together with (60) and continuity of $\hat{U}_A(\cdot)$ implies that types close enough to $\hat{\theta}$ benefit from a deviation to threshold $\hat{X}(\hat{\theta})$. Hence, it must be that $\hat{X}(\hat{\theta}) \in \left\{\hat{X}^{-}(\hat{\theta}), \hat{X}^{+}(\hat{\theta})\right\}$. ■

Proof of Lemma 4. We show that for all parameter values, except the case $b = -I$ and $\underline{\theta} = 0$, there exists a unique optimal contract, and it takes the form specified in the lemma. When $b = -I$ and $\underline{\theta} = 0$, the optimal contract is not unique, but the flat contract specified in the lemma is optimal. To prove the lemma, we consider three cases: $b \geq I$, $b \in [-I, I)$, and $b < -I$. Denote the flat contract from the first part of the lemma by $\hat{X}_{flat}(\theta)$, the contract from the second part by $\hat{X}_{-}(\theta)$, and the contract from the

third part by $\hat{X}_+(\theta)$.

Case 1: $b \geq I$. In this case, all types of agents want to exercise the option immediately. This means that any incentive-compatible contract must be flat. Among flat contracts $\hat{X}(\theta) = \bar{X}$, the one that maximizes the payoff to the principal solves

$$\arg\max_{\bar{X}} \int_{\underline{\theta}}^{1} \frac{\theta\bar{X} - I}{\bar{X}^{\beta}} d\theta = \frac{2\beta}{\beta - 1} \frac{I}{1 + \underline{\theta}}. \tag{63}$$

Case 2: $b \in [-I, I)$. The proof for this case proceeds in two steps. First, we show that the optimal contract cannot have discontinuities, except the case $b = -I$. Second, we show that the optimal continuous contract is as specified in the lemma.

Step 1: *If $b > -I$, the optimal contract is continuous.* Indeed, by contradiction, suppose that the optimal contract $C = \left\{\hat{X}(\theta), \theta \in \Theta\right\}$ has a discontinuity at some point $\hat{\theta} \in (\underline{\theta}, 1)$. By Lemma A.1, the discontinuity must satisfy (59)–(61). In particular, (60) implies that there exist $\theta_1 < \hat{\theta}$ and $\theta_2 > \hat{\theta}$ such that $\hat{X}(\theta) = X_A^*(\theta_1)$ for $\theta \in \left[\theta_1, \hat{\theta}\right)$ and $\hat{X}(\theta) = X_A^*(\theta_2)$ for $\theta_2 \in \left(\hat{\theta}, \theta_2\right]$. For any $\tilde{\theta}_2 \in (\hat{\theta}, \theta_2]$, consider a contract $C_1 = \left\{\hat{X}_1(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_1(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta_1), & \text{if } \theta \in \left[\theta_1, \tilde{\theta}\right), \\ X_A^*\left(\tilde{\theta}_2\right), & \text{if } \theta \in \left(\tilde{\theta}, \tilde{\theta}_2\right], \\ X_A^*(\theta), & \text{if } \theta \in \left(\tilde{\theta}_2, \theta_2\right), \end{cases}$$

where $\tilde{\theta} = \tilde{\theta}\left(\tilde{\theta}_2\right)$ satisfies

$$\frac{\tilde{\theta} X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^{\beta}} = \frac{\tilde{\theta} X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}. \tag{64}$$

Because $X^{-\beta}(\theta X - I + b)$ is maximized at $X_A^*(\theta)$, the function $\pi(\theta) \equiv \frac{\theta X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^{\beta}} - \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}$ satisfies $\pi(\theta_1) > 0 > \pi\left(\tilde{\theta}_2\right)$, and hence, by continuity of $\pi(\theta)$, there exists $\tilde{\theta} \in \left(\theta_1, \tilde{\theta}_2\right)$ such that $\pi\left(\tilde{\theta}\right) = 0$, i.e., (64) is satisfied. Intuitively, contract C_1 is the same as contract C, except that it substitutes a subset $\left[\tilde{\theta}_2, \theta_2\right]$ of the flat region with a continuous region where $\hat{X}_1(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Because contract C is incentive-compatible and $\tilde{\theta}$ satisfies (64), contract C_1 is incentive-compatible too. Below we show that the payoff to the principal from contract C_1 exceeds the payoff to the principal from contract C for $\tilde{\theta}_2$ very close to θ_2. Because $\hat{X}_1(\theta) = \hat{X}(\theta)$ for $\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_1 in this range, divided by $X(0)^{\beta} \frac{1}{1-\underline{\theta}}$, is

$$\int_{\theta_1}^{\tilde{\theta}(\tilde{\theta}_2)} \frac{\theta X_A^*(\theta_1) - I}{X_A^*(\theta_1)^{\beta}} d\theta + \int_{\tilde{\theta}(\tilde{\theta}_2)}^{\tilde{\theta}_2} \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}} d\theta + \int_{\tilde{\theta}_2}^{\theta_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^{\beta}} d\theta. \tag{65}$$

The derivative of (65) with respect to $\tilde{\theta}_2$, after the application of (64) and Leibniz's integral rule, is

$$\int_{\tilde{\theta}}^{\tilde{\theta}_2} \frac{\beta I-(\beta-1)\,\theta X_A^*\left(\tilde{\theta}_2\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}} X_A^{*\prime}\left(\tilde{\theta}_2\right) d\theta + b\left(\frac{1}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}-\frac{1}{X_A^*\left(\theta_1\right)^{\beta}}\right)\frac{d\tilde{\theta}}{d\tilde{\theta}_2}. \tag{66}$$

Because $X_A^{*\prime}(\theta)=-\frac{X_A^*(\theta)}{\theta}$, the first term of (66) can be simplified to

$$\frac{(\beta-1)\,X_A^*\left(\tilde{\theta}_2\right)\frac{\tilde{\theta}_2^2-\tilde{\theta}^2}{2}-\beta I\left(\tilde{\theta}_2-\tilde{\theta}\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}}\frac{X_A^*\left(\tilde{\theta}_2\right)}{\tilde{\theta}_2}=\beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left[\frac{I-b}{\tilde{\theta}_2}\frac{\tilde{\theta}_2+\tilde{\theta}}{2}-I\right]. \tag{67}$$

From (64), $\frac{d\tilde{\theta}}{d\tilde{\theta}_2}=(\frac{\theta_1^{\beta}}{\theta_1}-\frac{\tilde{\theta}_2^{\beta}}{\tilde{\theta}_2})^{-1}(\beta-1)\,\tilde{\theta}_2^{\beta-2}(\tilde{\theta}-\tilde{\theta}_2)$.Using this and (64), the second term of (66) can be simplified to

$$\frac{b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}\left(1-\left(\frac{\tilde{\theta}_2}{\theta_1}\right)^{-\beta}\right)\frac{d\tilde{\theta}}{d\tilde{\theta}_2}=\beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left(\frac{\tilde{\theta}}{\tilde{\theta}_2}\right)b. \tag{68}$$

Adding (67) and (68), the derivative of the principal's payoff with respect to $\tilde{\theta}_2$ is $-\beta\frac{(\tilde{\theta}_2-\tilde{\theta})^2}{2\tilde{\theta}_2^2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}(I+b)$, which is strictly negative for any $b>-I$. By the mean value theorem, if $U_P\left(\tilde{\theta}_2\right)$ stands for the expected principal's utility from contract C, then $\frac{U_P\left(\tilde{\theta}_2\right)-U_P(\theta_2)}{\hat{\theta}_2-\theta_2}=U_P'(\hat{\theta}_2)<0$ for some $\hat{\theta}_2\in\left(\tilde{\theta}_2,\theta_2\right)$, and hence a deviation from contract C to contract C_1 is beneficial for the principal. Hence, contract C cannot be optimal for $b>-I$.

Next, suppose $b=-I$. In this case, the derivative of (65) with respect to $\tilde{\theta}_2$ is zero for any $\tilde{\theta}_2\in(\hat{\theta},\theta_2]$. It can be similarly shown that if, instead, we replace a subset $\left[\theta_1,\tilde{\theta}_1\right]$ of the flat region $[\theta_1,\theta_2]$ with a continuous region where $\hat{X}_1(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, then the derivative of the principal's utility with respect to $\tilde{\theta}_1$ is zero for any $\tilde{\theta}_1\in[\theta_1,\hat{\theta})$. Combining the two arguments, contract C gives the principal the same expected utility as the contract where the flat region $[\theta_1,\theta_2]$ is replaced by a continuous region with $\hat{X}_1(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the rest of the contract is unchanged. Thus, if a discontinuous contract is optimal, then there exists an equivalent continuous contract, which contains a strictly decreasing region and which is also optimal.

Step 2: *Optimal continuous contract.* We prove that among continuous contracts satisfying Lemma A.1, the one specified in Lemma 4 maximizes the payoff to the principal. By Lemma A.1 and continuity of the contract, it is sufficient to restrict attention to contracts that are combinations of, at most, one downward sloping part $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ and two flat parts: any contract that has at least two disjoint regions with $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ will exhibit discontinuity. Consider a contract such that $\hat{X}(\theta)$ is flat for $\theta\in[\underline{\theta},\theta_1]$, is downward-sloping with $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ for $\theta\in[\theta_1,\theta_2]$, and is again flat for $\theta\in[\theta_2,1]$, for some $\theta_1\in[0,\theta_2]$ and $\theta_2\in[\theta_1,1]$. This consideration allows for all possible cases, because it can be that $\theta_1=\underline{\theta}$ and/or $\theta_2=1$, or $\theta_1=\theta_2$). The payoff to the principal, divided by $X(0)^{\beta}\frac{1}{1-\underline{\theta}}$, is

$$P=\int_{\underline{\theta}}^{\theta_1}\frac{\theta X_A^*(\theta_1)-I}{X_A^*(\theta_1)^{\beta}}d\theta+\int_{\theta_1}^{\theta_2}\frac{\theta X_A^*(\theta)-I}{X_A^*(\theta)^{\beta}}d\theta+\int_{\theta_2}^{1}\frac{\theta X_A^*(\theta_2)-I}{X_A^*(\theta_2)^{\beta}}d\theta. \tag{69}$$

Since $X_A^*(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, the derivative with respect to θ_1 is

$$\frac{\partial P}{\partial\theta_1}=\int_{\underline{\theta}}^{\theta_1}\frac{\beta I-(\beta-1)\theta X_A^*(\theta_1)}{X_A^*(\theta_1)^{\beta+1}}X_A^{*\prime}(\theta_1)\,d\theta=-\frac{\beta}{X_A^*(\theta_1)^{\beta}}\left[\frac{I+b}{2}-I\frac{\underline{\theta}}{\theta_1}+\left(\frac{\underline{\theta}}{\theta_1}\right)^2\frac{I-b}{2}\right].$$

First, suppose $\underline{\theta}>0$. Then $x=\frac{\underline{\theta}}{\theta_1}$ takes values between $\frac{\underline{\theta}}{\theta_2}$ and 1. Since $b\in[-I,I)$, the function $x^2\frac{I-b}{2}-Ix+\frac{I+b}{2}$ is U-shaped and has two roots, 1 and $\frac{I+b}{I-b}$, which coincide for $b=0$. If $b\in[0,I)$, this function is strictly positive for $x<1$ because $\frac{I+b}{I-b}\geq 1$. Hence, $\frac{\partial P}{\partial\theta_1}<0$ for $\theta_1>\underline{\theta}$, which implies that (69) is maximized at $\theta_1=\underline{\theta}$. If $-I<b<0$, then $0<\frac{I+b}{I-b}<1$ and hence $\frac{\partial P}{\partial\theta_1}<0$ when $\frac{\underline{\theta}}{\theta_1}<\frac{I+b}{I-b}$ or $\frac{\underline{\theta}}{\theta_1}>1$, and $\frac{\partial P}{\partial\theta_1}>0$ when $\frac{\underline{\theta}}{\theta_1}\in\left(\frac{I+b}{I-b},1\right)$. Because $\frac{\underline{\theta}}{\theta_1}\leq 1$, we conclude that (69) is increasing in θ_1 in the range $\theta_1<\frac{I-b}{I+b}\underline{\theta}$ and decreasing in θ_1 in the range $\theta_1>\frac{I-b}{I+b}\underline{\theta}$. Therefore, if $-I<b<0$, (69) reaches its maximum at $\theta_1=\min\left\{\frac{I-b}{I+b}\underline{\theta},1\right\}$. In particular, the maximum is achieved at $\theta_1=\frac{I-b}{I+b}\underline{\theta}$ if $b\in[-\frac{1-\underline{\theta}}{1+\underline{\theta}}I,0)$, and $\theta_1=\theta_2$ if $-I<b\leq-\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Finally, if $b=-I$, then $\frac{I+b}{I-b}=0$ and hence $\frac{\partial P}{\partial\theta_1}>0$, i.e., (69) is increasing in θ_1. Thus, (69) is also maximized at $\theta_1=\theta_2$.

Next, suppose $\underline{\theta}=0$. Then $\frac{\partial P}{\partial\theta_1}<0$ if $-I<b<I$ and $\frac{\partial P}{\partial\theta_1}=0$, otherwise. Hence, for $-I<b<I$ and $\underline{\theta}=0$, (69) is maximized at $\theta_1=0=\underline{\theta}$. If $b=-I$ and $\underline{\theta}=0$, the principal's utility does not depend on θ_1.

Next, the derivative of (69) with respect to θ_2 is

$$\frac{\partial P}{\partial\theta_2}=\int_{\theta_2}^{1}\frac{\beta I-(\beta-1)\theta X_A^*(\theta_2)}{X_A^*(\theta_2)^{\beta+1}}X_A^{*\prime}(\theta_2)\,d\theta=\frac{\beta(1-\theta_2)}{2\theta_2^2X_A^*(\theta_2)^{\beta}}\left(I-b-(I+b)\theta_2\right). \tag{70}$$

1) If $b\in[-I,0)$, then $I-b-(I+b)\theta_2\geq I-b-(I+b)>0$, and hence (70) is positive for any $\theta_2\in[\underline{\theta},1)$. Therefore, (69) is maximized at $\theta_2=1$. Combining this with the conclusions for θ_1 above, we get:

1a) For $\underline{\theta}>0$: If $b\in[-\frac{1-\underline{\theta}}{1+\underline{\theta}}I,0]$, then $\theta_1^*=\frac{I-b}{I+b}\underline{\theta}$ and $\theta_2^*=1$, which together with continuity of the contract gives $\hat{X}_-(\theta)$. If $b\in[-I,-\frac{1-\underline{\theta}}{1+\underline{\theta}}I]$, then $\theta_1^*=\theta_2$ and $\theta_2^*=1$, i.e., the optimal contract is flat. As shown above, among flat contracts, the one that maximizes the principal's payoff is $\hat{X}_{flat}(\theta)$. Note that this result implies that the optimal contract is unique among both continuous and discontinuous contracts even if $b=-I$. Indeed, Step 1 shows that the principal's utility in any discontinuous contract is the same as in a continuous contract with a strictly decreasing region. Because the optimal contract among continuous contracts is unique and is strictly flat, the principal's utility in any discontinuous contract is strictly smaller than in the flat contract, which proves uniqueness.

1b) For $\underline{\theta}=0$: If $b\in(-I,0)$, then $\theta_1^*=0$ and $\theta_2^*=1$, i.e., the optimal contract is $X_A^*(\theta)$ for all θ, consistent with $\hat{X}_-(\theta)$. If $b=-I$, then $\theta_2^*=1$ and $\theta_1^*\in[0,1]$, i.e., multiple optimal contracts exist (including some discontinuous contracts, as shown before). The flat contract given by $\hat{X}_{flat}(\theta)$ is one of the optimal contracts.

2) If $b\in[0,I)$, we have shown that $\theta_1^*=\underline{\theta}$ for any $\underline{\theta}\geq 0$, and hence we need to choose $\theta_2\in[\underline{\theta},1]$. According to (70), $\frac{\partial P}{\partial\theta_2}>0$ for $\theta_2<\frac{I-b}{I+b}$ and $\frac{\partial P}{\partial\theta_2}<0$ for $\theta_2>\frac{I-b}{I+b}$. Since $b\geq 0$, $\frac{I-b}{I+b}<1$. Also, $\frac{I-b}{I+b}\geq\underline{\theta}\Leftrightarrow b\leq\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Hence, if $b\geq\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $\frac{\partial P}{\partial\theta_2}<0$ for $\theta_2>\underline{\theta}$, and hence (69) is maximized at $\theta_2=\underline{\theta}$. Thus, for $b\geq\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the optimal contract is flat, which gives $\hat{X}_{flat}(\theta)$. Finally, if $b\in(0,\frac{1-\underline{\theta}}{1+\underline{\theta}}I]$, then (69) is increasing in θ_2 up to $\frac{I-b}{I+b}$ and decreasing after that. Hence, (69) is maximized at $\theta_2=\frac{I-b}{I+b}$. Combined with $\theta_1=\underline{\theta}$ and continuity of the contract, this gives $\hat{X}_+(\theta)$.

Case 3: $b<-I$. We show that the optimal contract is flat with $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. The proof proceeds in three steps. First, we show that the optimal contract cannot have any strictly decreasing regions and hence can only consist of flat regions. Second, we show that any contract with two flat regions is strictly dominated by a completely flat contract. Third, we show that any contract with at least three flat regions cannot be optimal. Combined, these steps imply that the optimal contract can only have one flat region,

i.e., is completely flat. Combining this with (63) gives $\hat{X}_{flat}(\theta)$ and completes the proof of this case.
Step 1: *The optimal contract cannot have any strictly decreasing regions.*

Consider a contract with a strictly decreasing region. According to Lemma A.1, any strictly decreasing region is characterized by $\hat{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Consider θ_1 and θ_2 such that $\hat{X}(\theta) = X_A^*(\theta)$ for $\theta \in [\theta_1, \theta_2]$. For any $\hat{\theta}_2 \in (\theta_1, \theta_2)$, consider a contract $C_2 = \left\{\hat{X}_2(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_2(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta), & \text{if } \theta \in \left[\theta_1, \hat{\theta}_2\right), \\ X_A^*(\hat{\theta}_2), & \text{if } \theta \in \left(\hat{\theta}_2, \hat{\theta}\right], \\ X_A^*(\theta_2), & \text{if } \theta \in \left(\hat{\theta}, \theta_2\right), \end{cases}$$

where $\hat{\theta} = \hat{\theta}(\hat{\theta}_2)$ satisfies

$$\frac{\hat{\theta}X_A^*(\hat{\theta}_2) - I + b}{X_A^*(\hat{\theta}_2)^{\beta}} = \frac{\hat{\theta}X_A^*(\theta_2) - I + b}{X_A^*(\theta_2)^{\beta}}. \tag{71}$$

(such $\hat{\theta}$ always exists and lies between $\hat{\theta}_2$ and $\hat{\theta}_1$ for the same reason as in contract C_1). Intuitively, contract C_2 is the same as contract C, except that it substitutes a subset $\left[\hat{\theta}_2, \theta_2\right]$ of the decreasing region with a piecewise flat region with a discontinuity at $\hat{\theta}$. Because contract C is incentive-compatible and $\hat{\theta}$ satisfies (71), contract C_2 is incentive-compatible too. Below we show that the payoff to the principal from contract C_2 exceeds the payoff to the principal from contract C for $\hat{\theta}_2$ very close to θ_2. Because $\hat{X}_2(\theta) = \hat{X}(\theta)$ for $\theta \le \theta_1$ and $\theta \ge \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_2 in this range, divided by $X(0)^{\beta}\frac{1}{1-\underline{\theta}}$, is

$$\int_{\theta_1}^{\hat{\theta}_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^{\beta}} d\theta + \int_{\hat{\theta}_2}^{\hat{\theta}(\hat{\theta}_2)} \frac{\theta X_A^*(\hat{\theta}_2) - I}{X_A^*(\hat{\theta}_2)^{\beta}} d\theta + \int_{\hat{\theta}(\hat{\theta}_2)}^{\theta_2} \frac{\theta X_A^*(\theta_2) - I}{X_A^*(\theta_2)^{\beta}} d\theta. \tag{72}$$

Following the same arguments as for the derivative of (65) with respect to $\tilde{\theta}_2$ above, we can check that the derivative of (72) with respect to $\hat{\theta}_2$ is given by $\beta\frac{(\hat{\theta}-\hat{\theta}_2)^2}{2\hat{\theta}_2^2}X_A^*(\hat{\theta}_2)^{-\beta}(I+b)$, which is strictly negative at any point $\hat{\theta}_2 < \theta_2$ for $b < -I$. By the mean value theorem, if $U_P(\hat{\theta}_2)$ stands for the expected principal's utility from contract C, then $\frac{U_P(\hat{\theta}_2) - U_P(\theta_2)}{\hat{\theta}_2 - \theta_2} = U_P'\left(\tilde{\theta}_2\right) < 0$ for some $\tilde{\theta}_2 \in \left(\hat{\theta}_2, \theta_2\right)$, and hence a deviation from contract C to contract C_2 is beneficial for the principal. Hence, contract C cannot be optimal for $b < -I$. This result implies that any optimal contract must consist only of flat regions.
Step 2: *Any contract with two flat regions is dominated by a contract with one flat region.*

Consider a contract with two flat regions: Types $\left[\underline{\theta}, \hat{\theta}\right]$ pick exercise at $\hat{X}_L$, and types $\left[\hat{\theta}, 1\right]$ pick exercise at $\hat{X}_H < \hat{X}_L$. Type $\hat{\theta} \in (\underline{\theta}, 1)$ satisfies

$$\frac{\hat{\theta}\hat{X}_L - I + b}{\hat{X}_L^{\beta}} = \frac{\hat{\theta}\hat{X}_H - I + b}{\hat{X}_H^{\beta}}. \tag{73}$$

Consider an alternative contract with $\hat{X}(\theta) = \hat{X}_H$ for all θ. The difference between the principal's value under this pooling contract and his value under the original contract, divided by $X(0)^{\beta}$, is given by

$$\begin{aligned} \Delta U &= \int_{\underline{\theta}}^{1} \frac{\theta\hat{X}_H - I}{\hat{X}_H^{\beta}}\frac{d\theta}{1-\underline{\theta}} - \left[\int_{\underline{\theta}}^{\hat{\theta}} \frac{\theta\hat{X}_L - I}{\hat{X}_L^{\beta}}\frac{d\theta}{1-\underline{\theta}} + \int_{\hat{\theta}}^{1} \frac{\theta\hat{X}_H - I}{\hat{X}_H^{\beta}}\frac{d\theta}{1-\underline{\theta}}\right] = \int_{\underline{\theta}}^{\hat{\theta}} \left(\frac{\theta\hat{X}_H - I}{\hat{X}_H^{\beta}} - \frac{\theta\hat{X}_L - I}{\hat{X}_L^{\beta}}\right)\frac{d\theta}{1-\underline{\theta}} \\ &= \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}}\left(\frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^{\beta}} - \frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^{\beta}}\right) = \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}}\left(\frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^{\beta}} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^{\beta}} + \frac{\underline{\theta}}{2}\left(\frac{1}{\hat{X}_H^{\beta-1}} - \frac{1}{\hat{X}_L^{\beta-1}}\right)\right) \end{aligned} \tag{74}$$

Using (73) and the fact that $b \leq -I$,

$$\frac{\hat{\theta}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\hat{\theta}\hat{X}_L - I}{\hat{X}_L^\beta} = b\left(\frac{1}{\hat{X}_L^\beta} - \frac{1}{\hat{X}_H^\beta}\right) \geq I\left(\frac{1}{\hat{X}_H^\beta} - \frac{1}{\hat{X}_L^\beta}\right) \Leftrightarrow \frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta} \geq 0,$$

and the inequalities are strict if $b < -I$. Combining this with $\hat{X}_H < \hat{X}_L$ and using (74), implies that $\Delta U \geq 0$ and $\Delta U > 0$ if at least one of $b < -I$ or $\underline{\theta} > 0$ holds. Thus, the contract with two flat regions is dominated by a contract with one flat region.

Step 3: *Any contract with at least three flat regions cannot be optimal.*

The proof of this step is similar to the proof of Step 3 in Proposition 4 in Melumad and Shibano (1991) for the payoff specification in our model. Suppose, on the contrary, that the optimal contract $X(\theta)$ has at least three flat regions. Consider three adjacent steps, $X_L > X_M > X_H$, of the assumed optimal contract. Types $\left(\hat{\theta}_0, \hat{\theta}_1\right)$ pick exercise at X_L, types $\left(\hat{\theta}_1, \hat{\theta}_2\right)$ pick exercise at X_M, and types $\left(\hat{\theta}_2, \hat{\theta}_3\right)$ pick exercise at X_H, where $\underline{\theta} \leq \hat{\theta}_0 < \hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3 \leq 1$. Incentive compatibility implies that types $\hat{\theta}_1$ and $\hat{\theta}_2$ satisfy

$$\frac{\hat{\theta}_1 X_L - I + b}{X_L^\beta} = \frac{\hat{\theta}_1 X_M - I + b}{X_M^\beta} \Leftrightarrow \hat{\theta}_1 = \frac{(I-b)\left(X_M^{-\beta} - X_L^{-\beta}\right)}{X_M^{1-\beta} - X_L^{1-\beta}}, \tag{75}$$

$$\frac{\hat{\theta}_2 X_M - I + b}{X_M^\beta} = \frac{\hat{\theta}_2 X_H - I + b}{X_H^\beta} \Leftrightarrow \hat{\theta}_2 = \frac{(I-b)\left(X_H^{-\beta} - X_M^{-\beta}\right)}{X_H^{1-\beta} - X_M^{1-\beta}}. \tag{76}$$

Consider an alternative contract with $\tilde{X}(\theta) = X_L$ for types $(\hat{\theta}_0, y)$, $\tilde{X}(\theta) = X_H$ for types $(y, \hat{\theta}_3)$, and $\tilde{X}(\theta) = X(\theta)$ otherwise, where $y \in (\hat{\theta}_1, \hat{\theta}_2)$ satisfies

$$\frac{yX_L - I + b}{X_L^\beta} = \frac{yX_H - I + b}{X_H^\beta} \Leftrightarrow y = \frac{(I-b)\left(X_H^{-\beta} - X_L^{-\beta}\right)}{X_H^{1-\beta} - X_L^{1-\beta}}. \tag{77}$$

This contract is incentive-compatible. The difference between the principal's value under this contract and the original contract, divided by $X(0)^\beta \frac{1}{1-\underline{\theta}}$, is given by

$$\begin{aligned}\Delta &= \int_{\hat{\theta}_1}^{y}\left(\tfrac{\theta X_L - I}{X_L^\beta} - \tfrac{\theta X_M - I}{X_M^\beta}\right)d\theta + \int_{y}^{\hat{\theta}_2}\left(\tfrac{\theta X_H - I}{X_H^\beta} - \tfrac{\theta X_M - I}{X_M^\beta}\right)d\theta \\ &= \left(y - \hat{\theta}_1\right)\left(\tfrac{\frac{y+\hat{\theta}_1}{2}X_L - I}{X_L^\beta} - \tfrac{\frac{y+\hat{\theta}_1}{2}X_M - I}{X_M^\beta}\right) + \left(\hat{\theta}_2 - y\right)\left(\tfrac{\frac{y+\hat{\theta}_2}{2}X_H - I}{X_H^\beta} - \tfrac{\frac{y+\hat{\theta}_2}{2}X_M - I}{X_M^\beta}\right).\end{aligned}$$

Using the left equalities of (75) and (76), we can rewrite Δ as

$$\begin{aligned}\Delta &= \left(y - \hat{\theta}_1\right)\left(\tfrac{y}{2}\left(\tfrac{X_L}{X_L^\beta} - \tfrac{X_M}{X_M^\beta}\right) + \tfrac{-I+b}{2X_M^\beta} - \tfrac{-I+b}{2X_L^\beta} - I\left(\tfrac{1}{X_L^\beta} - \tfrac{1}{X_M^\beta}\right)\right) \\ &+ \left(\hat{\theta}_2 - y\right)\left(\tfrac{y}{2}\left(\tfrac{X_H}{X_H^\beta} - \tfrac{X_M}{X_M^\beta}\right) - I\left(\tfrac{1}{X_H^\beta} - \tfrac{1}{X_M^\beta}\right) + \tfrac{-I+b}{2X_M^\beta} - \tfrac{-I+b}{2X_H^\beta}\right).\end{aligned}$$

Plugging in the values for y, $\hat{\theta}_1$, and $\hat{\theta}_2$ from the right equalities of (75), (76), and (77), we get

$$\begin{aligned}\Delta &= \tfrac{I-b}{X_M^{1-\beta} - X_L^{1-\beta}}\tfrac{\Sigma}{X_H^{1-\beta} - X_L^{1-\beta}}\left(\tfrac{\frac{(I-b)}{2}\left(X_H^{-\beta} - X_L^{-\beta}\right)\left(X_L^{1-\beta} - X_M^{1-\beta}\right)}{X_H^{1-\beta} - X_L^{1-\beta}} + \tfrac{b+I}{2}\left(X_M^{-\beta} - X_L^{-\beta}\right)\right) \\ &+ \tfrac{I-b}{X_H^{1-\beta} - X_M^{1-\beta}}\tfrac{\Sigma}{X_H^{1-\beta} - X_L^{1-\beta}}\left(\tfrac{\frac{(I-b)}{2}\left(X_H^{-\beta} - X_L^{-\beta}\right)\left(X_H^{1-\beta} - X_M^{1-\beta}\right)}{X_H^{1-\beta} - X_L^{1-\beta}} + \tfrac{b+I}{2}\left(X_M^{-\beta} - X_H^{-\beta}\right)\right),\end{aligned}$$

where $\Sigma = X_H^{-\beta} X_M^{-\beta} (X_M - X_H) + X_L^{-\beta} X_H^{-\beta} (X_H - X_L) + X_M^{-\beta} X_L^{-\beta} (X_L - X_M)$. Rearranging, Δ equals

$$\frac{\Sigma (I-b)(b+I)}{2(X_H^{1-\beta} - X_L^{1-\beta})} \left(\frac{X_M^{-\beta} - X_L^{-\beta}}{X_M^{1-\beta} - X_L^{1-\beta}} + \frac{X_M^{-\beta} - X_H^{-\beta}}{X_H^{1-\beta} - X_M^{1-\beta}} \right) = \frac{\frac{1}{2}\Sigma \left(I^2 - b^2\right)}{(X_H^{1-\beta} - X_L^{1-\beta})} \frac{-\Sigma}{(X_M^{1-\beta} - X_L^{1-\beta})(X_H^{1-\beta} - X_M^{1-\beta})},$$

which is strictly positive because $I^2 - b^2 < 0$ and $X_L^{1-\beta} < X_M^{1-\beta} < X_H^{1-\beta}$. Thus, contract $X(\theta)$ is strictly dominated by contract $\tilde{X}(\theta)$ and hence cannot be optimal. ■

Proof of Proposition 9. Let $VD(X,b)$ denote the expected value to the principal under delegation if the current value of $X(t)$ is X. If the decision is delegated to the agent, exercise occurs at threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. Hence,

$$VD(X,b) = \int_0^1 X^\beta \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta} \left(\frac{\beta}{\beta-1}(I-b) - I\right) d\theta = \frac{X^\beta}{\beta+1} \left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta} \left(\frac{\beta}{\beta-1}(I-b) - I\right).$$

Let $VA(X,b)$ denote the expected value to the principal in the most informative equilibrium of the advising game if the current value of $X(t)$ is X. Using (21) and $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$,

$$VA(X,b) = X^\beta \frac{1-\omega^*(b)}{1-\omega^*(b)^{\beta+1}} \left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega^*(b)}\right)^{-\beta} \frac{I}{\beta-1},$$

where $\omega^*(b)$ is the unique solution to (28), given b. Because X^β enters as a multiplicative factor in both $VD(X,b)$ and $VA(X,b)$, it is sufficient to compare $VD(b)$ and $VA(b)$, where $VD(b) \equiv X^{-\beta} VD(X,b)$ and $VA(b) \equiv X^{-\beta} VA(X,b)$.

First, consider the behavior of $VA(b)$ and $VD(b)$ around $b = I$. Since $\lim_{b\to I}\omega^*(b) = 0$, $\lim_{b\to I} VD(b) = -\infty$ and $\lim_{b\to I} VA(b) = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}$. By continuity of $VD(b)$ and $VA(b)$ in b, this implies that there exists $\bar{b} \in (0,I)$, such that for any $b > \bar{b}$, $VA(b) > VD(b)$. In other words, advising dominates delegation if the conflict of interest between the agent and the principal is big enough.

Second, consider the behavior of $VA(b)$ and $VD(b)$ for small but positive b. By l'Hopital's rule,

$$\lim_{b\to 0+} VD(b) = \lim_{b\to 0+} VA(b) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta}\frac{I}{\beta-1}.$$

Note that $VD'(b) = -\frac{\beta b}{(\beta+1)(I-b)}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}$. In particular, $\lim_{b\to 0+} VD'(b) = 0$ and $\lim_{b\to 0}\frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}$. The derivative of $VA(b)$ with respect to b can be found as

$$VA'(b) = C\frac{d\omega^*(b)}{db}\left[\frac{(1-\omega)(1+\omega)^\beta}{1-\omega^{\beta+1}}\right]' |_{\omega=\omega^*(b)}, \tag{78}$$

where $C \equiv \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}$. Recall that $\omega^*(b)$ solves (28), which is equivalent to

$$\frac{2I}{I-b}\frac{\beta}{\beta-1} = \left(\frac{1}{\omega}+1\right)\frac{1-\omega^\beta}{1-\omega^{\beta-1}}. \tag{79}$$

Differentiating this equation, we get

$$\frac{2I}{(I-b)^2}\frac{\beta}{\beta-1}db=\frac{-\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega\left(-\beta\omega^{\beta-1}\left(1-\omega^{\beta-1}\right)+(\beta-1)\,\omega^{\beta-2}\left(1-\omega^{\beta}\right)\right)}{\omega^2\left(1-\omega^{\beta-1}\right)^2}d\omega. \tag{80}$$

Because (79) is equivalent to $\frac{1}{I-b}=\frac{1}{2I}\frac{\beta-1}{\beta}\frac{1+\omega}{\omega}\frac{1-\omega^{\beta}}{1-\omega^{\beta-1}}$, we can rewrite the left-hand side of (80) as

$$2I\frac{\beta}{\beta-1}\left(\frac{1}{2I}\right)^2\left(\frac{\beta-1}{\beta}\right)^2\frac{(1+\omega)^2}{\omega^2}\frac{\left(1-\omega^{\beta}\right)^2}{\left(1-\omega^{\beta-1}\right)^2}db.$$

Substituting this into (80) and simplifying, we get

$$\frac{d\omega}{db}|_{\omega=\omega^*(b)}=\frac{1}{2I}\frac{\beta-1}{\beta}\frac{(1+\omega)^2\left(1-\omega^{\beta}\right)^2}{-\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)}. \tag{81}$$

Plugging (81) and

$$\left[\frac{(1-\omega)\,(1+\omega)^{\beta}}{1-\omega^{\beta+1}}\right]'=\frac{(1+\omega)^{\beta-1}}{\left(1-\omega^{\beta+1}\right)^2}\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right],$$

into (78), we get

$$VA'(b)=-D\frac{(1+\omega)^{\beta+1}\left(1-\omega^{\beta}\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right]}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]},$$

where $D\equiv\frac{C}{2I}\frac{\beta-1}{\beta}$. To find $\lim_{b\to 0}\frac{VA'(b)}{b}$, we express $\frac{1}{b}$ from (79) as

$$\frac{1}{b}=\frac{(\beta-1)\,(1+\omega)\left(1-\omega^{\beta}\right)}{I\left[(\beta-1)\,(1+\omega)\left(1-\omega^{\beta}\right)-2\beta\omega\left(1-\omega^{\beta-1}\right)\right]},$$

and hence

$$\begin{aligned}\frac{VA'(b)}{b}&=-D\frac{(1+\omega)^{\beta+1}\left(1-\omega^{\beta}\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right]}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]}\frac{(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)}{I\left[(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)-2\beta\omega\left(1-\omega^{\beta-1}\right)\right]}\\&=-\frac{(\beta-1)D}{I}\frac{(1+\omega)^{\beta+2}\left(1-\omega^{\beta}\right)^3}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]}.\end{aligned}$$

Hence,

$$\lim_{b\to 0}\frac{VA'(b)}{b}=-\frac{(\beta-1)\,2^{\beta+2}D}{I}\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{1-\omega^{\beta+1}}\right]^2\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)}\right].$$

By l'Hopital's rule, the first limit equals $(\frac{\beta}{\beta+1})^2$, and the second limit equals ∞. Therefore,

$$\lim_{b\to 0}\frac{VA'(b)}{b}=-\infty<\lim_{b\to 0}\frac{VD'(b)}{b}=-\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}.$$

By continuity of $VA'(b)$ and $VD'(b)$ for $b>0$, there exists $\underline{b}>0$ such that $VA'(b)<VD'(b)$ for any

$b < \underline{b}$. Because $VA(0) = VD(0)$, then

$$VD(b) - VA(b) = \int_0^b (VD'(y) - VA'(y))\, dy > 0$$

for any $b \in (0, \underline{b}]$. In other words, delegation dominates advising when the agent is biased towards early exercise but the bias is low enough. ■

Proof of Proposition 10. Note that the following three inequalities are equivalent: $b \leq \frac{1-\underline{\theta}}{1+\underline{\theta}} I \Leftrightarrow \frac{I-b}{I+b} \geq \underline{\theta} \Leftrightarrow \frac{\beta}{\beta-1}(I+b) \leq \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Hence, there are two cases. If $b < \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, delegation occurs at the threshold $\frac{\beta}{\beta-1}(I+b) = X_A^*\left(\frac{I-b}{I+b}\right)$, where $\frac{I-b}{I+b} > \underline{\theta}$. If $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, then $\frac{I-b}{I+b} \leq \underline{\theta}$ and delegation occurs at the principal's uninformed exercise threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$.

We prove that neither the agent nor the principal wants to deviate from the specified strategies. First, consider the agent's strategy. Given Assumption 1, sending a message $m = 1$ is never beneficial because it does not change the principal's belief and hence his strategy. Hence, the agent cannot induce exercise before she is given authority. After the agent is given authority, her optimal strategy is to: 1) exercise immediately if $b \geq I$, or if $b < I$ and $X_d \geq X_A^*(\theta)$; 2) exercise when $X(t)$ first reaches $X_A^*(\theta)$ if $b < I$ and $X_d < X_A^*(\theta)$. Consider two cases. If $0 < b < \frac{1-\underline{\theta}}{1+\underline{\theta}} I$ $(\leq I)$, then $X_d = X_A^*\left(\frac{I-b}{I+b}\right)$, and hence $X_d < X_A^*(\theta)$ if and only if $\theta < \frac{I-b}{I+b}$. Thus, types below $\frac{I-b}{I+b}$ exercise at $X_A^*(\theta)$ and types above $\frac{I-b}{I+b}$ exercise immediately at X_d, consistent with the equilibrium strategy. Second, if $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, the agent finds it optimal to exercise immediately at X_d regardless of her type: if $b \geq I$, this is always the case, and if $\frac{1-\underline{\theta}}{1+\underline{\theta}} I \leq b < I$, this is true because $X_A^*(\theta) \leq X_A^*(\underline{\theta}) = \frac{\beta}{\beta-1}\frac{I-b}{\underline{\theta}} \leq \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1} = X_d$. Since $\frac{I-b}{I+b} \leq \underline{\theta}$, this strategy again coincides with the equilibrium strategy. Hence, the agent does not want to deviate.

Next, consider the principal's strategy. The above arguments show that the equilibrium exercise times coincide with the exercise times under the optimal contract in Lemma 4 for all b. Hence, the principal's expected utility in this equilibrium equals his expected utility in the optimal contract. Consider possible deviations of the principal, taking into account that the agent's messages are uninformative and hence the principal does not learn new information by waiting. First, the principal can exercise the option himself, before or after $X(t)$ first reaches X_d. Because a contract with such an exercise policy is incentive-compatible, the principal's utility from such a deviation cannot exceed his utility under the optimal contract and hence his equilibrium utility. Thus, such a deviation cannot be strictly profitable. Second, the principal can deviate by delegating authority to the agent before or after $X(t)$ first reaches X_d. An agent who receives authority at some point t will exercise immediately if $b \geq I$, or if $b < I$ and $X(t) \geq X_A^*(\theta)$, and will exercise when $X(t)$ first reaches $X_A^*(\theta)$ otherwise. Because a contract with such an exercise schedule is incentive-compatible, the principal's utility from this deviation cannot exceed his utility under the optimal contract and hence his equilibrium utility. Hence, the principal does not want to deviate either. ■

Proof of Proposition 11. The fact that ω^* decreases in b has been proved in the proof of Lemma 3. We next show that ω^* increases in β. From (28), ω^* solves $F(\omega, \beta) = 0$, where

$$F(\omega, \beta) = \frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}\frac{2I}{I-b} - 1 - \frac{1}{\omega}.$$

Denote the unique solution by $\omega^*(\beta)$. Function $F(\omega, \beta)$ is continuously differentiable in both arguments on $\omega \in (0,1)$, $\beta > 1$. Differentiating $F(\omega^*(\beta), \beta)$ in β, $\frac{\partial \omega^*}{\partial \beta} = -\frac{F_\beta(\omega^*(\beta),\beta)}{F_\omega(\omega^*(\beta),\beta)}$. Since $F(0,\beta) < 0$, $F(1,\beta) = \frac{2b}{I-b} > 0$, and ω^* is the unique solution of $F(\omega,\beta) = 0$ in $(0,1)$, we know that $F_\omega(\omega^*(\beta), \beta) > 0$. Hence, it is sufficient to prove that $F_\beta(\omega, \beta) < 0$. Differentiating $F(\omega, \beta)$ with respect to β and reorganizing the

terms, we obtain that $F_\beta(\omega,\beta)<0$ is equivalent to

$$\frac{\left(1-\omega^{\beta-1}\right)\left(1-\omega^{\beta}\right)}{\omega^{\beta-1}(1-\omega)}+\beta(\beta-1)\ln\omega>0.$$

Denote the left-hand side as a function of β by $N(\beta)$. Because $N(1)=0$, a sufficient condition for $N(\beta)>0$ for any $\beta>1$ is that $N'(\beta)>0$ for $\beta>1$. Differentiating $N(\beta)$:

$$N'(\beta)=\ln\omega\left[-\frac{\omega^{1-\beta}-\omega^{\beta}}{1-\omega}+2\beta-1\right].$$

Because $\ln\omega<0$ for any $\omega\in(0,1)$, condition $N'(\beta)>0$ is equivalent to $n(\beta)\equiv\frac{\omega^{1-\beta}-\omega^{\beta}}{1-\omega}-2\beta+1>0$. Note that $\lim_{\beta\to1}n(\beta)=0$ and $n'(\beta)=-\left(\omega^{1-\beta}+\omega^{\beta}\right)\frac{\ln\omega}{1-\omega}-2\equiv\eta(\beta)$. Note that

$$\eta(\beta)=\eta(1)+\int_1^\beta\eta'(x)\,dx=-\frac{(1+\omega)\ln\omega}{1-\omega}-2+\frac{(\ln\omega)^2}{1-\omega}\int_1^\beta\left(\left(\frac{1}{\omega}\right)^{2x-1}-1\right)\omega^x dx. \tag{82}$$

The second term of (82) is positive, because $\left(\frac{1}{\omega}\right)^{2x-1}-1>0$, since $\frac{1}{\omega}>1$ and $2x-1>1$ for any $x>1$. The first term of (82) is positive, because

$$\begin{array}{l}\lim_{\omega\to1}\left(-\frac{(1+\omega)\ln\omega}{1-\omega}-2\right)=\lim_{\omega\to1}\left(\ln\omega+\frac{1+\omega}{\omega}\right)-2=0\\ \text{and }\frac{\partial\left(-\frac{(1+\omega)\ln\omega}{1-\omega}-2\right)}{\partial\omega}=\frac{-2\ln\omega-\frac{1}{\omega}+\omega}{(1-\omega)^2}<0,\end{array}$$

where the first row is by l'Hopital's rule, and the second row is because $\left(-2\ln\omega-\frac{1}{\omega}+\omega\right)'=\frac{(1-\omega)^2}{\omega^2}>0$ and $-2\ln\omega-\frac{1}{\omega}+\omega$ equals zero at $\omega=1$. Thus, $\eta(\beta)>0$ and hence $n'(\beta)>0$ for any $\beta>1$, which together with $n(1)=0$ implies $n(\beta)>0$, which in turn implies that $N(\beta)>0$ for any $\beta>1$. Hence, $F_\beta(\omega,\beta)<0$. Therefore, ω^* is strictly increasing in $\beta>1$. Finally, a standard calculation shows that $\frac{\partial\beta}{\partial\sigma}<0$, $\frac{\partial\beta}{\partial\mu}<0$, and $\frac{\partial\beta}{\partial r}>0$. Therefore, ω^* is decreasing in β and μ and increasing in r. ■

Proof of Proposition 12. (i) To prove this proposition, we use the solution for the optimal contract under full commitment power (Lemma 4). Suppose that the current value of the state process is 1, and let $VC(b)$ denote the principal's ex-ante utility under commitment as a function of b. We start by showing that $VC(b)=VC(-b)$ for any $b>0$. First, consider $b\notin(-I,I)$. In this case, the exercise trigger is $\frac{2\beta}{\beta-1}I$, regardless of θ and b. Hence, $VC=VC(-b)$ for any $b\geq I$. Second, consider $b\in(-I,I)$. For $b\in(-I,0]$, the exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Thus, the expected payoff of the principal is:

$$VC(b)=\int_0^1\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta=\frac{(\beta-1)^{\beta-1}}{(\beta(I-b))^{\beta}}\frac{I-\beta b}{\beta+1}=\frac{(\beta-1)^{\beta-1}}{(\beta(I+|b|))^{\beta}}\frac{I+\beta|b|}{\beta+1}.$$

Next, consider $b\in[0,I)$. The exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ if $\theta\leq\frac{I-b}{I+b}$ and $\frac{\beta}{\beta-1}(I+b)$ otherwise. Thus, the utility of the principal is:

$$VC(b)=\int_0^{\frac{I-b}{I+b}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta+\int_{\frac{I-b}{I+b}}^1\left(\frac{\beta}{\beta-1}(I+b)\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I+b)\theta-I\right)d\theta,$$

which can be shown to be equal to $\frac{(\beta-1)^{\beta-1}}{(\beta(I+b))^{\beta}}\frac{I+\beta b}{\beta+1}$. Thus, $VC(b)=VC(-b)$ for any $b\in[0,I)$. Combining the two cases, we conclude that $VC(b)=VC(-b)$ for any $b\geq0$.

We now use this property to prove the statement of the proposition. Recall that for $b\in(-I,0)$, the prin-

cipal's expected utility in the advising equilibrium coincides with his expected utility under commitment. Hence, $V_0(-b) = VC(-b)$ for $b \in (0, I)$. Recall also that the principal strictly benefits from commitment when $b > 0$: $V_0(b) < VC(b)$ for $b > 0$. Since, as shown above, $VC(b) = VC(-b)$, we conclude that $V_0(-b) = VC(-b) = VC(b) > V_0(b)$ for $b \in (0, I)$.

Next, consider $b \geq I$. Denote by V_u the principal's utility from following the optimal uniformed exercise strategy $\bar{X}_u = \frac{\beta}{\beta-1}2I$. As shown in the proof of Proposition 1, when $b = -I$, the principal's utility in the equilibrium with continuous exercise equals V_u. For $b < -I$, only babbling equilibria exist and hence the principal's utility is again V_u. Similarly, if $b \geq I$, there is no informative equilibrium in the advising game. Hence, the principal exercises the option at the uninformed threshold $\bar{X}_u = \frac{\beta}{\beta-1}2I$, and his utility is also given by V_u. Thus, $V_0(-b) = V_0(b) = V_u$ for any $b \geq I$, which completes the proof.

(ii) Consider $b \in (0, I)$. According to Proposition 8, $\tilde{V}_0(-b) = V_0(-b)$, and hence $\tilde{V}_0(-b) = VC(-b) = VC(b)$. For an agent biased towards early exercise, the exercise schedule under both delegation and communication is different from the exercise schedule in the optimal mechanism of Lemma 4. Hence, regardless of the principal's choice between delegation and retaining authority, $\tilde{V}_0(b) < VC(b) = \tilde{V}_0(-b)$. If $|b| \geq I$, then for any direction of the agent's bias, the principal's utility in the advising equilibrium is V_u, which coincides with his utility in the optimal mechanism. Hence, $\tilde{V}_0(b) = V_u = \tilde{V}_0(-b)$ for $b \geq I$. ■

Proof of Proposition 13. First, consider $b < 0$. The payoffs of the principal and the agent upon exercise are given by $\theta X - I - z$ and $\theta X - I + b + z$, respectively. Hence, the problem is equivalent to the problem of the basic model with $I' \equiv I + z$ and $b' = 2z + b$. The interests of the principal and the agent become aligned if $b' = 0$, i.e., if $z = \frac{-b}{2}$. Note that it is never optimal to have $z > 0$ if $b' < -I'$: in this case, the equilibrium will feature uninformed exercise and hence would give the principal the same expected utility as if he did not make any payments. Similarly, it is never optimal to have $b' > 0$. Hence, we can restrict attention to $b' \in [-I, 0]$. Then, the most informative equilibrium of the advising game features continuous exercise, and according to (26), the principal's expected utility as a function of z is

$$V(z) = \frac{X(0)^{\beta}}{\beta+1}\left(\frac{\beta}{\beta-1}(I'-b')\right)^{-\beta}\frac{I'-\beta b'}{\beta-1} = \frac{X(0)^{\beta}}{\beta^2-1}\left(\frac{\beta}{\beta-1}\right)^{-\beta}(I-b-z)^{-\beta}\left(I-\beta b+z(1-2\beta)\right).$$

Note that $V'(z) > 0 \Leftrightarrow z < z^*$, where $z^* = \frac{-(I-b)(2\beta-1)+\beta I-\beta^2 b}{(\beta-1)(2\beta-1)}$. It is easy to show that $z^* > 0 \Leftrightarrow b < \frac{-I}{\beta-1}$ and that $z^* < -\frac{b}{2} \Leftrightarrow (\beta-1)(b-2I) < 0$, which holds for any $b < 0$. This completes the proof of the first statement.

Next, consider $b > 0$. If the principal makes flow payoffs $\hat{z}dt$ before exercise, then upon exercise the agent loses $\frac{\hat{z}}{r}$, which is the present value of continuation payments at that moment. Thus, the principal's and agent's effective payoffs upon exercise are $\theta X(t) - I + \frac{\hat{z}}{r}$ and $\theta X(t) - I + b - \frac{\hat{z}}{r}$, respectively. Hence, we can consider the communication game with $I' = I - \frac{\hat{z}}{r}$ and $b' = b - 2\frac{\hat{z}}{r}$. The interests of the principal and the agent become aligned if $b = 2\frac{z}{r}$, i.e., if $z = \frac{rb}{2}$. Similarly to the case $b < 0$, it is never optimal to have $\hat{z} > 0$ if $b' \geq I'$ or $b' < 0$, and hence we can restrict attention to $b' \in [0, I')$. Denoting $\tilde{z} \equiv \frac{\hat{z}}{r}$ and using (21), the payoff of the principal at the initial date is

$$V(\tilde{z}) = -\tilde{z} + \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X(0)}{Y(\omega,\tilde{z})}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega,\tilde{z}) - I + \tilde{z}\right),$$

where by (18), $Y(\omega,\tilde{z}) = \frac{(1-\omega^{\beta})(I-b+\tilde{z})}{\omega(1-\omega^{\beta-1})}$. By (28), the most informative equilibrium of this game is characterized by $\omega = \frac{1}{\frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}\frac{2(I-\tilde{z})}{I-b+\tilde{z}}-1}$. If $X(0) \to 0$, $V'(\tilde{z}) \to -1$, and hence $\tilde{z} = 0$ is optimal, which completes the proof. ■