Stochastic Bundling

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Abstract

We study the multiproduct monopoly price discrimination problem allowing for bundles with stochastic components. We determine conditions on the profit function under deterministic bundling guaranteeing that stochastic bundling is optimal. We show that the optimality of stochastic bundling is robust: the smallest tilt or shift of the uniform distribution of valuations makes stochastic bundling optimal. We show that the result extends to intertemporal price discrimination: hence we show that Stokey's [26] celebrated no discrimination across time result does not extend to multiple goods. We demonstrate that profitable time discounting takes the form of the marketing practice of dynamic cross-selling.

KEYWORDS: Multidimensional Mechanism Design; Second Degree Price Discrimination; Multidimensional Pricing; Bundling; Stochastic Delivery; Time Discounting; Haggling; Lotteries; Cross-sell.
JEL CLASSIFICATION: C72, D42, D82

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1 Introduction

This paper will determine when it is optimal for a multiple good monopolist to offer bundles in his/her selling strategy whose constituent parts are random. Both bundling and stochastic delivery are classical issues in the study of monopoly pricing. Bundling refers to the practice of selling combinations of goods for a single price so as to price discriminate across consumers. Stochastic delivery refers to the marketing strategy of selling lotteries over the goods for sale. This paper determines conditions under which it is optimal to use both.

Stochastic delivery captures mathematically many widespread selling techniques. Perhaps most prominently Riley and Zeckhauser [22] showed that stochastic delivery is equivalent to bargaining (or haggling); the randomness capturing the uncertainty of a successful deal from bargaining. Randomness is used directly in the sale of advertising space on the internet: publishers such as Yahoo! sell a distribution over readers to the advertiser; higher prices are payable for better targeted adverts which have a higher probability of engaging the reader and delivering a customer (Briest, Chawla, Kleinberg and Weinberg [3]). In industrial settings it is common for equipment prices to increase in return for reductions in failure probability. In retail settings, randomness is also present when price reductions are available for delayed and stochastic delivery times.

Optimising in such settings requires the seller to solve a multidimensional pricing problem. Further, the insights from the single good monopoly case are limited: Riley and Zeckhauser [22] showed that randomness is not required to maximise profits if a seller has only a single good to sell. In this paper we show that stochastic bundling, that is randomness, is part of the multiproduct seller's optimal mechanism: in our motivating example we demonstrate that the simplest case of two consumer types with unit demands for two goods requires the seller to include bundles whose constituent parts are delivered probabilisticaly to maximise profits. Guided by the optimal structure of stochastic bundles in such relatively simple settings, we study the general two discrete good optimal price discrimination problem.

This paper determines conditions on the profit function with deterministic bundles which guarantee that stochastic bundling is optimal. These conditions are open to economic intuition and allow us to show that stochastic bundling is robust: the smallest tilt or shift of the uniform distribution of valuations results in stochastic bundling being part of the optimal mechanism. Hence the optimality of deterministic bundling in the case of the uniform distribution (Manelli and Vincent [16], and numerically confirmed in Aguilera and Morin [1]) is a knife edge result.

We show that the introduction of stochastic bundles to further increase profits need not lower consumer surplus. We also show that the particular form of stochastic bundles necessary for profit improvements can be made isomorphic to temporal price discrimination. As a result we can show that the celebrated no price discrimination across time result of Stokey [26] does not extend to the multiple good seller. The profitable dynamic pricing mechanism we discover is new to the economics literature, though not to marketing practice: dynamic cross-selling. We show that a multiple good monopolist can, under conditions we characterise, increase his/her profits by holding the prices of individual products and (deterministic) bundles constant through time, but conditional on the purchase of one good, offer the single good buyers a reduction on a supplementary product (the cross-sell) after a period of delay.

Our approach to this classical problem is to use the calculus of variations to derive conditions under which deterministic bundles do not optimise the seller's problem – guaranteeing that stochastic bundling can generate a more profitable solution. To determine the appropriate variations to consider we first solve the monopoly problem with just two types of consumer and two goods. When the two consumer types are not strongly ordered, so that the consumers differ in the product they value the most, then stochastic bundles can be part of the optimal selling mechanism. The optimal stochastic bundle has a particular form: one good is delivered with certainty, randomness applies to the delivery of the supplementary good.

We apply this insight to the general problem of a two good monopolist facing consumers with unit demands and independent valuations for each good drawn from distributions on the positive reals. A monopolist who added a stochastic bundle which had one part of the bundle delivered with only positive probability would alter the participation regions of the consumers in a calculable way. The calculus of variations allows us to search over all such deformations to determine conditions under which such stochastic bundles increase the profits available from deterministic bundles. We derive three equivalent conditions under which a monopolist would prefer to use stochastic bundles. The conditions are expressed both in terms of the profit function, and in terms of the fundamental distribution of consumer valuations (Proposition 3, page 14). A direct economic intuition is available for the result that deterministic bundling cannot be optimal if the cross-partial derivative of profits $\partial^2\pi/\partial p_2 \partial p_B < 0$. We can show (discussion around Figure 6, page 19) that if this cross-partial derivative condition holds then one of two circumstances must apply. Either:

1. The density of the consumers across the boundary between those who buy good 2 only, and those who buy the bundle, is locally downwards sloping. That is, there are more consumers who buy good 2 only and who lie close to the boundary at which they would buy the bundle, than there are bundle consumers who are at risk of dropping down to buy good 2 only. A stochastic bundle with good 2 delivered for certain and randomness applied to good 1 allows this boundary to be directly targeted. By offering such a stochastic bundle consumers along this boundary can be induced to swap to the lottery. As a result some bundle consumers pay less and will receive the bundle only with a positive probability. However some single good consumers pay more and will receive the bundle with some probability. The cross-partial derivative condition ensures the latter group outweighs the former, increasing profits. The tilted uniform example (Section 7) demonstrates this effect. Or;

2. There is a significant mass of consumers close to indifferent between buying just good 2 or the bundle, but who do not purchase at all at the deterministic bundle prices, and these consumers value the goods significantly above the costs of production. A stochastic bundle with good 2 delivered for certain and randomness applied to good 1 offered at a small discount on the actuarially fair value allows just these marginal consumers to be targeted whilst not lowering prices for the overwhelming majority of existing bundle or good 2 consumers. The shifted uniform example (Section 8) demonstrates this effect.

Standard deterministic bundling can allow these situations to persist as the prices must be

set with regards to numerous boundaries and so each represents an optimising average. These insights allow us to be constructive: profitable stochastic bundles can be identified, and they parallel those required for optimality in the two-type motivating example. We can also analyse directly the implications introducing such stochastic bundles has on consumer surplus. One might expect that as optimising stochastic bundles can remove a product from those who value it relatively more to deliver it, stochastically, to those who value it less, that consumer surplus would be harmed. This is not necessarily the case. Stochastic bundles can increase both profits and consumer surplus as the total mass of products delivered rises.

While we focus on the classical monopoly bundling problem, our analysis sheds new light on a second classical problem: optimal price discrimination through time. To this end we show that the stochastic bundling problem can be reinterpreted in a new way. A stochastic bundle in which one product is delivered with certainty, and a second with some probability, is shown to be isomorphic to a dynamic selling mechanism in which one product has its price fixed through time, while the second is offered conditional on the purchase of the first good, at a reduced price after a period of delay. This conditional time discounting exists in marketing practice and is known as cross-selling. Such dynamic discounting is clearly only possible for a multiple good seller. We exploit the isomorphism to stochastic bundling to derive conditions under which price discriminating through time is optimal for a multiple good seller. Since Stokey [26] demonstrated that temporal price discrimination is not optimal for a single good monopolist, the literature has not been able to determine whether or not the result extended to multiple goods – we know now it does not.

We present the literature related to our study in Section 2. The two type case is solved as a motivating example in Section 3. Section 4 describes the general model we use to study multiproduct stochastic bundling, in Section 5 we use the calculus of variations to begin our categorisation of optimal pricing, and in Section 6 we derive our sufficiency conditions for the optimality of stochastic bundles. The optimality of stochastic bundles in the tilted uniform case is presented in Section 7, and in the shifted uniform case in Section 8. The intuition for the optimality of stochastic bundling and the construction of profitable stochastic bundles is demonstrated in Section 9. The isomorphism to temporal price discrimination is developed in Section 10. Section 11 concludes with omitted proofs collected in Appendix A.

2 Literature Review

We have noted in the Introduction that stochastic delivery is widespread in reality. It is manifested in seller haggling, the sale of internet advertising, price increases for reductions in failure probability, and price reductions for delayed and stochastic delivery times. This paper contributes to our understanding of when stochastic delivery is part of the solution to the monopolist's price discrimination problem.

This literature began with the path-breaking work of Riley and Zeckhauser [22] who showed that stochastic delivery was not part of the optimal selling strategy for a single good monopolist. Not withstanding the many examples in reality, if one accurately balanced the gain in revenues from introducing stochastic delivery against the loss from higher valuation consumers selecting cheaper products, Riley and Zeckhauser demonstrated that introducing lotteries was always

profit reducing for the single good monopolist who has set the optimal (deterministic) take-it-or-leave-it price. McAfee and McMillan [18] then claimed that, under a general regularity condition on consumers' demand, stochastic delivery was not optimal for a multiproduct monopolist also. This result was however incorrect, a counter-example being provided for the case of two goods in Thanassoulis [28].

Following Thanassoulis [28] a large number of authors have discovered isolated numerical examples in which stochastic bundling is part of the multiproduct monopolist's optimal selling mechanism: Manelli and Vincent ([16], [17]); Pycia [21]; Hart and Reny [12]. However these examples have not yielded general conditions characterising when stochastic bundling is part of the multiproduct monopolist's optimal selling mechanism. This question is not easy to address as Rochet and Choné [23] have shown that the fully optimal selling mechanism for a multiproduct monopolist cannot, in general, be determined analytically.

In answer to this research question, Thanassoulis [28], in analysis of the case of perfect substitutes, determined that a stochastic bundle adds profit to the optimal set of prices for deterministic delivery if the lower bound of the support of consumer valuations is moved sufficiently far above the costs of production. In this case a stochastic bundle can be used to serve those consumers close to indifferent between the two substitutable goods, but who would be unserved under deterministic delivery prices; and this can be made profitable enough to outweigh losses from the high value consumers who swap to the stochastic bundle. Returning to the standard case of additive valuations for the goods, Pycia [21], in unpublished work, argues that even if the support of valuations is not translated north-east, the set of density functions for which stochastic bundling is optimal is dense: one can add a discrete distribution of valuations, similar to the numerical examples noted above, and so construct a nearby distribution of valuations for which stochastic bundling is marginally more profitable. However beyond these two contributions positive results yielding conditions for the optimality of stochastic bundling do not exist. Our contribution is here. We derive conditions on the profit function under deterministic bundling for general distribution functions of consumer demand which ensure that stochastic bundling is more profitable. Further we demonstrate that elaborate valuation density constructions are not required: the smallest tilt or shift of the uniform distribution leads to the optimality of stochastic bundling. In addition the economics underlying our stochastic bundling optimality condition is interpretable by appropriate consideration of the meaning of the cross-partial derivative of the profit function with respect to price.

Other scholars have asked the reverse question of when stochastic bundling can be ruled out. Manelli and Vincent [16] demonstrate that to rule stochastic bundling out, one would have to augment the McAfee and McMillan valuation density conditions by multiple further restrictions on the distribution function and the profit function under deterministic bundling (see their Theorem 3).[1] These conditions are not easily interpretable, as the authors acknowledge, though they have the implication that stochastic bundles are not optimal if the distribution of valuations is uniform on $[0,1]^2$. Our work on the tilted uniform (Proposition 4) demonstrates that this is

[1] The result requires a stronger form of the hazard rate condition (Manelli and Vincent [16], Assumption 2); an integral version of the hazard rate condition at optimal deterministic delivery prices (Manelli and Vincent, Assumption 3); conditions on optimal deterministic delivery prices of sub-modularity and all bundle combinations being bought (referred to as ABS in Manelli and Vincent); and a further condition on the profit function (Manelli and Vincent, Assumption 4).

a knife-edge result. If one confines oneself to the valuation density conditions used by McAfee and McMillan [18] then Pavlov [20] has demonstrated that, given the support of valuations is weakly above cost, as here, the optimal tariff in the case of two goods can contain any bundles drawn from the north-east boundary of the feasible stochastic bundle probabilities: $[0,1]^2$. Hence stochastic bundles cannot be ruled out, nor are they ruled in by this result. Our work provides conditions when stochastic bundles are ruled in, and generically constructs stochastic bundles which are profit increasing.

One might be tempted to hope that, given the difficulty in determining when stochastic bundles are optimal, restricting to deterministic bundling would not be too costly in terms of lost revenues. Particularly as the optimisation of prices under deterministic bundling, though difficult analytically, is quite possible numerically (Cai and Daskalakis, [4]). However this is very much not the case. Briest, Chawla, Kleinberg and Weinberg [3] show through examples that, for two goods, profits can be trebled by the use of stochastic bundles; if there are three or more goods the increase in profits possible by using stochastic bundles can be essentially unbounded. Such large gains in profits very much depend upon the distributions of consumer valuations (Chawla, Malek and Sivan [6]).

As stochastic bundling can be very important to profits, there has been renewed focus in the mathematics community on deriving the optimal stochastic price discrimination mechanism by numerical optimisation allied to raw computational power. The problem of the optimal stochastic bundles is complicated however as the consumer's utility function, which must be convex, need not be piecewise linear, unlike under deterministic bundling. Choné and Le Meur [7] demonstrate that in general convex functions need not be the limit of piecewise linear functions on a mesh; thus standard finite element numerical methods cannot be guaranteed to find the optimal mechanism. The literature has therefore proposed two broad approaches. The first approach is to endogenise the shape of the mesh as the optimisation progresses. Ekeland and Moreno [10] proceed in this way and show that the optimal utility function can be found as the supremum over affine functions satisfying the convexity constraints. The second approach is to maintain a rigid mesh as the basis for optimisation, but consider a richer family of convex functions on that mesh. Carlier, Lachland-Robert, and Maury [5] show that quadratic objective functions can be solved over the family of convex Lagrange polynomials; however this restriction on the objective functions is too restrictive for the monopolist's problem. Aguilera and Morin [1] show that if the Hessian is discretised over the grid then the resultant utility functions are rich enough to solve the price discrimination problem. The findings of our study may provide some insight into how these optimisation processes can be further refined to allow the most profitable stochastic bundles to be determined.

By establishing an isomorphism between stochastic bundling and temporal price discrimination our analysis sheds new light on the economic and marketing literature studying price reductions over time. In celebrated work Stokey [26] established that, in general settings, a single good monopolist who could commit to a time-path for prices would choose not to lower prices over time: delayed price reductions to capture low value consumers would encourage too many high value consumers to delay their purchases. This trade-off is analogous to that exploited by Riley and Zeckhauser [22] in their study of stochastic delivery for the single good monopolist.

Given that the optimality of time discounting *per se* had apparently been settled, the literature on dynamic pricing has since explored how prices would change in the absence of seller commitment (they would collapse: Stokey [27], Gul, Sonneschein and Wilson [11]), and explored how a single product monopolist could commit to keep prices above cost (Stokey [27], Conlisk, Gertner and Sobel [9], Board and Pycia [2]). Our work demonstrates that the no temporal price discrimination result does not extend to multiple good settings, a finding new to the economics literature.

Exploiting the isomorphism between stochastic bundling and temporal price discrimination further, we can establish profitable mechanisms for time discounting: temporal price discrimination conditional on the prior purchase of an alternative product. This structure for price discrimination over time is only possible when selling multiple goods and its study is new, we believe, to the economics literature. The practice is known in the marketing profession as *cross-selling*, and is the subject of marketing books targeted at sales executives (for example Kumar [15]). The marketing literature has focused on how data-mining techniques can be used to identify which consumers should be the subject of cross-sell sales promotions (Cohen [8], Salazar, Harrison, Ansell [25], Ngobo [19]). By deriving conditions on the profit and consumer utility functions which determine that price discounting is optimal, our work may offer an entirely new route by which the seller's cross-sell problem can be addressed.

3 Motivating Example: Optimal Stochastic Bundling

Consider two discrete types of consumer. Though this is a relatively standard setting, we will show that the solution is not standard: stochastic bundling is a part of the optimal selling mechanism.

Denote the two consumer types a and b who each wish to purchase only a single unit of each of good 1 and of good 2. The consumers' valuations are additive across goods and given by positive vectors, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Thus consumer a values a unit of good 1 at a_1, and the bundle of both goods at $a_1 + a_2$. There is a proportion λ^a of type a consumers; the rest (proportion λ^b) are type b. Assume the consumers are risk neutral and have quasilinear utility.

The seller wishes to determine the couple of, potentially stochastic, bundles $\left\{q^a, q^b\right\} \in [0,1]^2$ and the prices p^a, p^b to deliver to consumers a and b which maximise his/her profits, subject to incentive compatibility and participation constraints. The stochastic bundle $q^a = (q_1^a, q_2^a)$, for example, implies that good 1 is delivered with probability q_1^a, while good 2 is delivered with independent probability q_2^a. The costs of production are normalised to zero so profit maximisation is equivalent to revenue maximisation.

An implication of the normalisation of costs to zero is that the consumers' valuations for both goods weakly exceed cost, so the first best allocation is to provide the full bundle with certain delivery, $q = (1, 1)$, to both of these consumers. However, when types are private information this is not necessarily the allocation that maximizes the seller's profits. Consider the case in which the consumers are not strongly ordered so that good 1 is valued more by type b than by type a; and similarly good 2 is valued more by type a than by type b:

$$a_1 - b_1 < 0 < a_2 - b_2. \tag{1}$$

Up to a possible relabelling of types and goods, we can always assume that type b values the full bundle weakly more than type a :

$$a_1 + a_2 \leq b_1 + b_2 \tag{2}$$

This distribution of consumer types is depicted graphically in panel (a) of Figure 1.

Denote consumer rents by $U^a = a \cdot q^a - p^a$ and similarly for consumer b. To solve the seller's problem we first characterise the optimal rents to assign to each consumer for given bundles $\left\{q^a, q^b\right\}$. Then we solve for the optimal bundles in a second stage.

Given bundles $\left\{q^a, q^b\right\}$ the seller maximises $\sum_{k \in \{a,b\}} \lambda^k (k \cdot q^k - U^k)$ under the participation and incentive compatibility constraints which can be written:

$$\begin{aligned} U^k &\geq 0, &(3)\\ (b-a) \cdot q^a &\leq U^b - U^a \leq (b-a) \cdot q^b. &(4) \end{aligned}$$

The feasible set in $\left(U^a, U^b\right)$ space takes the form of an upward sloping band as shown in panel (b) of Figure 1. Analysis yields:

Proposition 1 *If there are two types of consumer whose valuations are not strongly ordered (condition (1) holds) then the optimal selling strategy is:*

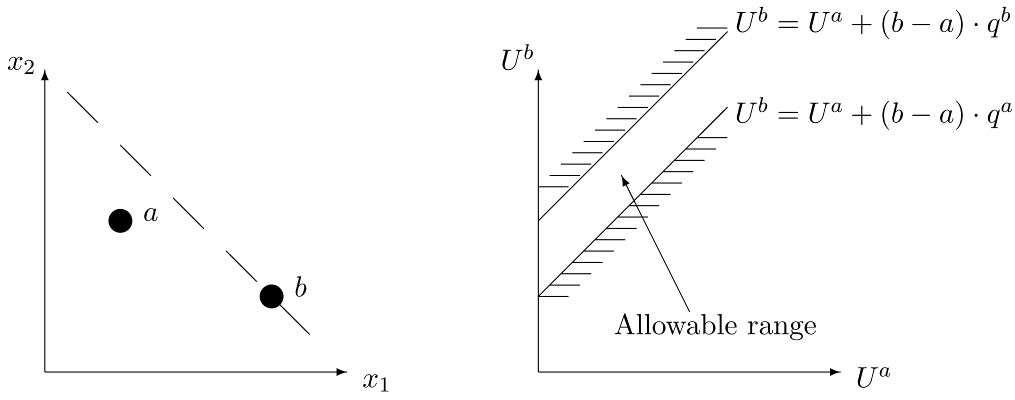
1. *Sell the bundle* $(1,1)$ *to consumer* b *and the stochastic bundle* $q^a = \left(\frac{a_2 - b_2}{b_1 - a_1}, 1\right)$ *to consumer* a *if* $\lambda^b b_1 > a_1$.
2. *Sell the bundle* $(1,1)$ *to both consumers if* $\lambda^b b_1 \leq a_1$.

The optimality of stochastic delivery cannot happen in the one dimensional context (Riley and Zeckhauser [22]). Proposition 1 demonstrates that in the simplest two good settings, stochastic bundling can be optimal. The form of the stochastic bundle type a receives has one of the goods delivered with certainty, while the other good is delivered with positive probability. In this paper we will characterise when such a stochastic bundling structure is more profitable than deterministic bundling generally.

4 The Model

A seller produces two discrete goods at marginal cost normalised to zero. We consider a population of consumers, each characterised by their valuations for the goods: $x \equiv (x_1, x_2) \in \mathbb{R}^2_+$. The variable x_i $(i \in \{1,2\})$ is the valuation for a single unit of good i. Consumers desire only one unit of each good and are assumed to have additive valuations: thus the value of the bundle to consumer x is $x_1 + x_2$. Consumers have quasilinear utility and are risk neutral.[2] Each consumer knows their valuation vector, but the seller only knows the distribution of valuations in the population. We denote the marginal density of the valuations of good i by $f_i(x_i)$, and the cumulative density function is $F_i(x_i)$. We assume the distribution of valuations across goods is independent so that the density of valuations x is given by $f(x) = f_1(x_1) f_2(x_2)$.

[2]Thus utility is additive over monetary payments and linear in the probability of delivery.



(a) Consumer Valuations. (b) Consumer Utility Space.

Figure 1: Graphical Representation of Motivating Example.
Notes: In panel (a) we have the two consumer valuations. The consumers are not strongly ordered (condition (1) holds). The dotted line has a gradient of -1 so, in the case depicted, consumer a values the full bundle strictly less than consumer type b (condition (2) holds with strict inequality).
In panel (b) we plot the feasible region in consumer utility space given stochastic bundles $\{q^a, q^b\}$. The feasible region lies within the upwards sloping band.

The seller is able to bundle the products and so can offer the menu of take-it-or-leave-it prices $\{p_1, p_2, p_B\}$ with p_B being the price for the bundle of both goods. In this study we additionally allow the seller to offer a menu of bundles $(q_1, q_2) \in [0,1]^2$ with probabilistic delivery: we call these stochastic bundles. The bundle $q \equiv (q_1, q_2)$ is a lottery which delivers good 1 with probability q_1 and good 2 with independent probability q_2. If such a bundle were bought at a price $p(q)$ then consumer x would derive utility $u(x) = q \cdot x - p(q)$. The seller wishes to select the menu of, potentially stochastic, bundles and associated prices to maximise her profits.

4.1 Optimal Deterministic Bundles

Before searching for optimal stochastic bundles, we first capture some characteristics of the optimal deterministic bundle prices. Under deterministic bundling the total profit of the seller is the sum of profits from the sale of good 1, good 2 and the bundle:

$$\pi = \pi_1(p_1, p_B) + \pi_2(p_2, p_B) + \pi_B(p_1, p_2, p_B) \tag{5}$$

The assignment of consumers to products under deterministic bundling is depicted graphically in Figure 2.

Inspection of Figure 2 and using the independence of valuations across products implies

$$\begin{aligned} \pi_1(p_1, p_B) &= p_1 [1 - F_1(p_1)] F_2(p_B - p_1), \ \pi_2(p_2, p_B) = p_2[1 - F_2(p_2)]F_1(p_B - p_2), \quad (6) \\ \text{and } \pi_B(p_1, p_2, p_B) &= p_B \left\{ \int_{p_B - p_2}^{p_1} f_1(z) [1 - F_2(p_B - z)] \, dz + [1 - F_2(p_B - p_1)] [1 - F_1(p_1)] \right\}. \end{aligned}$$

Further analysis yields:

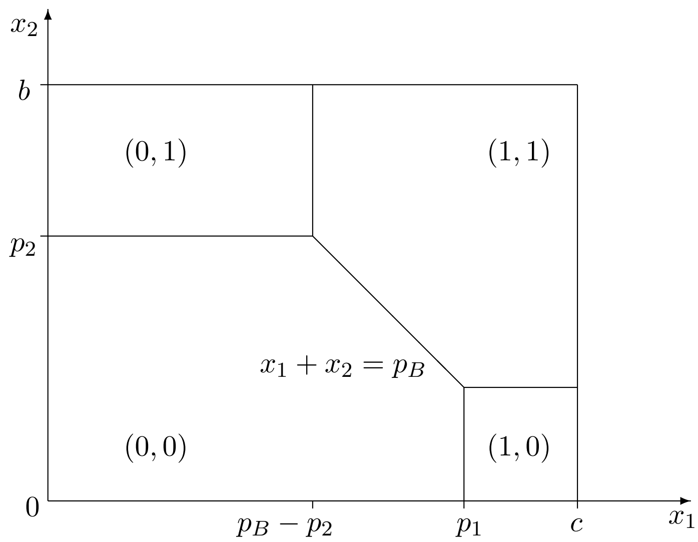


Figure 2: Consumer Participation Regions Under Deterministic Bundling

Notes: The consumers are depicted with support on $[0, c] \times [0, b] \subset \mathbb{R}^2_+$. The deterministic bundle $(0, 1)$ which delivers good 2 with certainty is purchased by consumers in the north-west. The bundles bought in the other three regions of the square are labelled similarly.

Lemma 1 *The first derivatives of the components of the profit function with respect to the good 2 price are:*

$$\frac{\partial \pi_2}{\partial p_2} = [1 - F_2(p_2) - p_2 f_2(p_2)]F_1(p_B - p_2) - p_2[1 - F_2(p_2)]f_1(p_B - p_2) \tag{7}$$

$$\frac{\partial \pi_B}{\partial p_2} = p_B f_1(p_B - p_2)[1 - F_2(p_2)] \tag{8}$$

If a positive measure of consumers buys each combination of products, then the optimal deterministic bundling prices $\{p_1, p_2, p_B\}$ satisfy:

$$\frac{p_2 f_2(p_2)}{1 - F_2(p_2)} - \frac{(p_B - p_2) f_1(p_B - p_2)}{F_1(p_B - p_2)} = 1. \tag{9}$$

Proof. Using (6) and differentiating with respect to p_2 yields (7) and (8). For given (p_1, p_B), if a positive measure of consumers buys each combination of products, then we have an interior solution and so the optimal p_2 satisfies $\partial \pi_2/\partial p_2 + \partial \pi_B/\partial p_2 = 0$. Summing (7) and (8) and setting to zero yields a necessary condition for optimality, giving (9). ■

Further characterisation of the optimal deterministic bundling pricing is not, in general, tractable, and explicit solutions are only available in special cases. Nonetheless, Lemma 1 will prove to be sufficient structure on the deterministic prices to allow us to characterise when deterministic bundles are dominated by stochastic bundles.

4.2 The Example of Uniform Consumer Valuations

To fix ideas, the optimal deterministic bundles are tractable when consumers' valuations are uniformly distributed on the rectangle $[0, c] \times [0, b]$ as depicted in Figure 2. In this case we have

$$f_1(x_1) = \frac{1}{c},\ F_1(x_1) = \frac{x_1}{c},\ \text{and } f_2(x_2) = \frac{1}{b},\ F_2(x_2) = \frac{x_2}{b}$$

substitution into (9) of Lemma 1 yields optimal individual good prices

$$p_1 = \frac{2c}{3} \text{ and } p_2 = \frac{2b}{3}.$$

The optimal bundle price requires explicit calculation, even in this relatively simple case. For the interested reader the bundle price when all subsets of the two products are purchased by consumers of positive measure (so that the participation regions are as in Figure 2) is offered as Claim 1 and proved in the appendix.

Claim 1 *If the support of the uniform distribution* $[0,c] \times [0,b]$ *satisfies* $2c > b > c/2$ *then the optimal deterministic bundle price is given by*

$$p_B = \frac{2}{3}\left[c + b - \sqrt{\frac{cb}{2}}\right] \tag{10}$$

Claim 1 ensures that a positive measure of consumers buys all subsets of products with the prices satisfying $p_B < p_1 + p_2$, $p_B > p_1$, and $p_B > p_2$.

5 Introducing Stochastic Bundles

In the motivating example of Section 3 we demonstrated that optimality required the seller to include an offer of a stochastic bundle of the form $(q_1, 1)$ under which the buyer always receives one good (good 2), and will in addition receive a second good (good 1) with probability $q_1 > 0$. We use this insight to guide our search for profitable stochastic bundles in this richer setting of consumers with continuous and independent valuations over two goods. Our approach is to structure our search in such a way that we can apply the Calculus of Variations to determine the form of a profitable deviation to the optimal menu of deterministic prices.

In our analysis we restrict attention to the case in which all product combinations are purchased by a positive measure of consumers under optimal deterministic bundle prices. Following Manelli and Vincent [16] we refer to this assumption as ABS – All Bundles Sold.[3]

ABS We assume that at the optimal deterministic prices non-zero volumes of all bundles are sold.

Assumption ABS guarantees that the consumer with valuation $(p_B - p_2, p_2)$ lies strictly within the support of the consumer type space. Under optimal deterministic prices a consumer x in the north-west corner of the valuation space who purchases good 2 receives utility $x_2 - p_2$. We consider a perturbation in the north-west corner for $x_1 \leq \tilde{x}$ with $\tilde{x} \in [p_B - p_2, p_1]$. The perturbation is given by the one-dimensional function $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$ such that in the region of the perturbation the utility of a consumer x who participates is

$$u(x) := x_2 - \varphi(x_1) \tag{11}$$

An example of such a perturbation is plotted in Figure 3.

[3] This is a standard technical assumption which allows us to avoid arguments which apply only for sets of zero measure (see Manelli and Vincent [16] where the assumption is referred to as ABS).

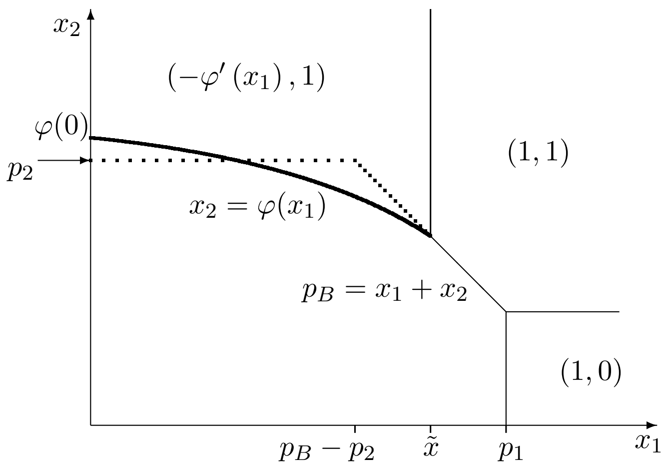


Figure 3: Perturbed menu of stochastic bundles to be optimised

Notes: The figure depicts the product allocations in consumer valuation space (x_1, x_2) from the family of stochastic bundles which the seller optimises over. The seller delivers utility $u(x) = x_2 - \varphi(x_1)$ to those consumers lying above the thick curved line. This curve is the stochastic bundle participation constraint: $x_2 = \varphi(x_1)$. Using Lemma 2 this utility assignment implies delivery of stochastic bundle $(-\varphi'(x_1), 1)$ to those in the north-west corner. The deterministic bundle $(1, 1)$ is assigned to those in the north-east, and the single product $(1, 0)$ to those in the south-east. The remaining consumers are not served. The dotted lines depict the deterministic pricing participation constraints which the new stochastic bundles have replaced.

We first consider the restrictions on the perturbation $\varphi(\cdot)$ to ensure that there exists a menu of stochastic bundles which can deliver the utility $u(x)$ defined by (11) in an incentive compatible way. Suppose therefore there exists a full suite of stochastic bundles with prices $P(q) : [0,1]^2 \to \mathbb{R}_+$ with $P(0) = 0$ to ensure individual rationality for those receiving the empty bundle. A consumer x would derive utility from this tariff of

$$u(x) = \max_q [x \cdot q - P(q)] \tag{12}$$

Lemma 2 *Utility maximisation requires that*

$$\begin{aligned} u(0) &= 0 \text{ if } 0 \in supp\ x && (13)\\ \nabla u(x) &= q(x) \in [0,1]^2 && (14)\\ u \text{ convex} & && (15) \end{aligned}$$

Further any utility function u satisfying (13)-(15) is implementable through a suite of stochastic bundles.

Proof. As $P(q) \geq 0$, (13) is immediate from (12). Applying the envelope theorem to (12) gives (14). Convexity (condition 15) is immediate as $u(x)$ is the supremum of affine (and therefore convex) functions of x (Rockafellar [24], Theorem 5.5).

For the converse implication, given a utility function $u(x)$ satisfying (13)-(15) define the stochastic bundle price function as $P(q) \equiv \max_y [y \cdot q - u(y)]$. This is the Fenchel transformation or convex conjugate: $P(q) = u^*(q)$. We have $P(q) : [0,1]^2 \to \mathbb{R}_+$ and $P(0) = 0$. This is therefore an allowable price schedule. Faced with this price schedule, consumer x optimises to

secure utility

$$\tilde{u}(x) = \max_q [x \cdot q - P(q)] = \max_q [x \cdot q - u^*(q)] = u^{**}(x) = u(x)$$

Where the third equality follows as, by definition, we have the convex conjugate of $u^*(q)$. The final equality then follows from Fenchel duality (see Rockafellar [24], Theorem 12.2), yielding the required result. ■

Applying Lemma 2 we see that for the perturbation to utilities $u(x) = x_2 - \varphi(x_1)$ to be allowable we require

$$\varphi \text{ concave } \left(\varphi'' \leq 0\right) \tag{16}$$

$$\varphi'(0) \leq 0 \text{ and } \varphi'(\tilde{x}) \geq -1 \tag{17}$$

$$\varphi(\tilde{x}) = p_B - \tilde{x} \tag{18}$$

Condition (13) is satisfied given any consumer at $x = 0$ does not participate. Condition (18) ensures the participation constraint in the perturbed region pastes onto the participation constraint for the bundle purchasers at $x_1 = \tilde{x}$. Condition (16) with the pasting condition (18) implies (15); and using (16), (17) implies (14).

The permutation of utilities we have postulated is one-dimensional, rendering its analysis tractable. In the perturbed region, the consumer x buys the stochastic bundle $\bigtriangledown u(x) = (-\varphi'(x_1), 1)$ (using condition (14)). The consumer pays $x \cdot \bigtriangledown u(x) - u(x) = \varphi(x_1) - x_1\varphi'(x_1)$ where we have used (11). The assignment of these stochastic bundles to consumers is depicted in Figure 3.

Using Figure 3 the perturbation captured by $\varphi(\cdot)$ alters the profit from consumers with $x_1 \leq \tilde{x}$ to

$$\Pi(\varphi) \equiv \int_0^{\tilde{x}} [1 - F_2(\varphi(x_1))][\varphi(x_1) - x_1\varphi'(x_1)]f_1(x_1)dx_1.$$

Given the optimal deterministic prices, the seller therefore solves

$$\max_{\tilde{x},\varphi(\cdot)} \Pi(\varphi) \text{ subject to (16)-(18).} \tag{19}$$

The research question is therefore to find conditions under which the profit $\Pi(\varphi)$ is not maximised by setting $\tilde{x} = p_B - p_2$ and $\varphi(x) = p_2$. In this case the seller can improve upon the optimal menu of deterministic bundles.

The seller's optimisation problem (19) is a variations calculus problem with a monotonicity constraint on $\varphi'(\cdot)$. Using the techniques developed by Hellwig ([13], [14]) it can be shown that its solution exists and can be found by the Lagrangian technique. To this end we define the function $\lambda(x_1)$ as the Lagrangian multiplier applying to the concavity constraints for $x_1 \in [0, \tilde{x}]$ (condition (16)). The pointwise constraints (17), (18) are given Lagrangian multipliers $\{\mu_1, \mu_2, \mu_3\}$. The Lagrangian of this problem for given $\tilde{x}$ is therefore:

$$\begin{aligned} L(\tilde{x}) \quad = \quad & \int_0^{\tilde{x}} \{[1 - F_2(\varphi)][\varphi - x_1\varphi']f_1(x_1) - \lambda(x_1)\varphi''(x_1)\}dx_1 \\ & -\mu_1\varphi'(0) - \mu_2[\varphi(\tilde{x}) - p_B + \tilde{x}] + \mu_3\left[\varphi'(\tilde{x}) + 1\right] \end{aligned} \tag{20}$$

This Lagrangian must be optimised over the function $\varphi(\cdot)$ as well as the variable $\tilde{x}$.

Proposition 2 *The seller's optimisation problem (19) has a solution. At the solution the Lagrangian multiplier function $\lambda(x_1) \geq 0$ on the concavity constraint must satisfy*

$$\begin{aligned} \lambda'(0) &= 0 && (21) \\ \lambda''(x_1) &= [x_1 f_1(x_1)]'[1 - F_2(\varphi)] + [1 - F_2(\varphi) - f_2(\varphi)\varphi] f_1(x_1) && (22) \end{aligned}$$

And

$$\text{if } \varphi'(\tilde{x}) > -1 \text{ then } \lambda(\tilde{x}) = \lambda'(\tilde{x}) = 0. \tag{23}$$

The seller's task is to optimise the Lagrangian, L, over the set of twice differentiable functions. This is achieved by taking the directional derivative of L at any proposed perturbation $\varphi(\cdot)$. If the seller's objective function is optimised by the function $\varphi(\cdot)$ then this derivative must vanish. Hence the conditions contained in Proposition 2 can be established.

6 Sufficient Conditions For Stochastic Bundling To Be Optimal

We present our first substantive result:

Proposition 3 *Stochastic bundling is part of the optimal selling strategy if at optimal deterministic prices $\{p_1, p_2, p_B\}$ one of the following equivalent conditions hold:*

1. $$\frac{f_2(p_2)p_2}{[1 - F_2(p_2)]} - \frac{(p_B - p_2)\, f_1'(p_B - p_2)}{f_1(p_B - p_2)} > 2 \tag{24}$$

2. $$\frac{\partial^2 \pi}{\partial p_2 \partial p_B} < 0 \tag{25}$$

3. $$\frac{d}{dx}\left[\frac{x f_1(x)}{F_1(x)}\right]\Bigg|_{x = p_B - p_2} < 0 \tag{26}$$

Proof. Suppose for a contradiction that the optimal selling strategy is deterministic and so requires $\varphi(x_1) \equiv p_2$ and $\tilde{x} = p_B - p_2$. As $\varphi' = 0 > -1$, (23) applies: $\lambda(\tilde{x}) = 0 = \lambda'(\tilde{x})$. Thus if $\lambda''(\tilde{x}) < 0$ then $\lambda'(\tilde{x} - \varepsilon) > 0$ for small ε which implies that $\lambda(\tilde{x} - \varepsilon) < 0$ which is a contradiction. From (22):

$$\lambda''(p_B - p_2) = [1 - F_2(p_2)] f_1(p_B - p_2) \left\{ 2 + \frac{(p_B - p_2)\, f_1'(p_B - p_2)}{f_1(p_B - p_2)} - \frac{f_2(p_2)p_2}{[1 - F_2(p_2)]} \right\}$$

Hence we have a contradiction if

$$\frac{f_2(p_2)p_2}{[1 - F_2(p_2)]} - \frac{(p_B - p_2)\, f_1'(p_B - p_2)}{f_1(p_B - p_2)} > 2$$

which gives condition (24).

For condition (25), using (7) and (8):

$$\begin{aligned}
\frac{\partial^2 \pi}{\partial p_2 \partial p_B} &= \frac{\partial^2 \pi_2}{\partial p_2 \partial p_B} + \frac{\partial^2 \pi_B}{\partial p_2 \partial p_B} \\
&= [1 - F_2(p_2) - p_2 f_2(p_2)] f_1(p_B - p_2) - p_2 [1 - F_2(p_2)] f_1'(p_B - p_2) \\
&\quad + f_1(p_B - p_2)[1 - F_2(p_2)] + p_B f_1'(p_B - p_2)[1 - F_2(p_2)] \\
&= [1 - F_2(p_2)] f_1(p_B - p_2) \left\{ 2 - \frac{p_2 f_2(p_2)}{[1 - F_2(p_2)]} + \frac{(p_B - p_2)\, f_1'(p_B - p_2)}{f_1(p_B - p_2)} \right\}
\end{aligned}$$

which yields the result as $\frac{\partial^2 \pi}{\partial p_2 \partial p_B} < 0$ if and only if (24) holds.

Finally for condition (26)

$$\frac{d}{dx}\left[\frac{x f_1(x)}{F_1(x)}\right]\bigg|_{x = p_B - p_2} < 0 \Leftrightarrow \frac{(p_B - p_2) f_1'(p_B - p_2) + f_1(p_B - p_2)}{F_1(p_B - p_2)} < \frac{(p_B - p_2) f_1^2(p_B - p_2)}{F_1^2(p_B - p_2)}$$

Multiplying by $\frac{F_1(p_B - p_2)}{f_1(p_B - p_2)}$ this is equivalent to

$$\frac{(p_B - p_2) f_1'(p_B - p_2)}{f_1(p_B - p_2)} - \frac{(p_B - p_2) f_1(p_B - p_2)}{F_1(p_B - p_2)} < -1$$

And using (9) this is equivalent to (24). ■

We noted in the Introduction that stochastic bundling can be seen as a by-product of a 'haggling strategy,' following the insight of Riley and Zeckhauser [22]. Interpreted in this way Proposition 3 is a sufficient condition for haggling to be a dominant strategy for the seller of two goods.

In Section 9 we will give some intuition for condition (25) of Proposition 3. To establish this it is first helpful to demonstrate that stochastic bundling is part of optimal pricing for even very small departures from the standard uniform distribution of consumer valuations.

7 The Tilted Uniform Case

Aguilera and Morin ([1], §6.1), show numerically that the seller optimally uses only deterministic bundles when consumers' valuations are uniform on the square $[0,1]^2$. We demonstrate that this result is a knife edge case: inclusion of stochastic bundles is optimal for even the slightest tilt of the uniform distribution of valuations. We demonstrate this for the tilted uniform on general rectangles $[0,c] \times [0,b]$, encompassing the unit square as a special case.

Consider therefore consumers with valuations on support $[0,c] \times [0,b]$ with $f_2 \equiv 1/b$ (uniform) and $f_1(x) = 1/c + \beta - 2\beta x/c$ for $\beta \in [0, 1/c]$ implying $F_1(x) = (1/c + \beta)\, x - \beta x^2/c$. If $\beta = 0$ then we have the standard uniform case. If $\beta > 0$ then we have a tilted uniform in which there is a downwards sloping density for good 1. An example density function for the tilted uniform distribution on the unit square is depicted in Figure 4.

Proposition 4 *Under assumption ABS, stochastic bundling is optimal for any strictly tilted uniform distribution of consumer valuations: $\beta > 0$.*

Proof. The optimal good 2 price p_2 satisfies (9). Substituting into (24) a sufficient condition

for stochastic bundling to be optimal is

$$\frac{(p_B - p_2) f_1(p_B - p_2)}{F_1(p_B - p_2)} - \frac{(p_B - p_2)\, f_1'(p_B - p_2)}{f_1(p_B - p_2)} > 1 \tag{27}$$

Substituting for the tilted uniform this condition is

$$\frac{1 + c\beta - 2\beta\,(p_B - p_2)}{1 + c\beta - \beta\,(p_B - p_2)} + \frac{2\beta\,(p_B - p_2)}{1 + c\beta - 2\beta\,(p_B - p_2)} > 1$$

This condition can be written:

$$2 - \frac{1 + c\beta}{1 + c\beta - \beta\,(p_B - p_2)} - 1 + \frac{1 + c\beta}{1 + c\beta - 2\beta\,(p_B - p_2)} > 1$$

And so the result follows if

$$\frac{1}{1 + c\beta - 2\beta\,(p_B - p_2)} > \frac{1}{1 + c\beta - \beta\,(p_B - p_2)} \tag{28}$$

We continue to assume that all combinations of goods are bought by a positive measure of consumers. In particular at optimal deterministic prices $f_1\,(p_B - p_2) > 0$. This implies $1 + c\beta - 2\beta\,(p_B - p_2) > 0$ and so both denominators in (28) are positive. Hence the result follows if $1 + c\beta - \beta\,(p_B - p_2) > 1 + c\beta - 2\beta\,(p_B - p_2)$ which is true for $\beta > 0$. Hence stochastic bundling is optimal for $\beta > 0$ as required. ■

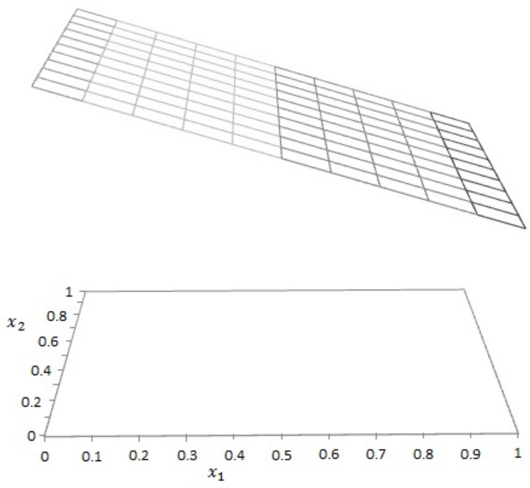


Figure 4: The Tilted Uniform

Notes: The surface plot is of the density function of consumers' valuations when tastes are distributed according to the tilted uniform supported on $[0, 1]^2$. Hence $c = b = 1$ with the tilt parameter $\beta > 0$ capturing the steepness of the tilt.

We will develop an intuition to explain why a downwards sloping density function might be expected to lead to the optimality of stochastic bundling in Section 9.

8 The Shifted Uniform Case

In this section we show that if the support of the standard uniform is moved even marginally out towards higher value consumers, then stochastic bundles again become optimal.

Consider therefore the standard normal distribution on the unit square, with support shifted out to $[b-1,b]^2$ for $b \geq 1$. The standard case is given by $b=1$. As we increase b the support of the consumers' valuations moves up the 45^o line. The density functions are now $f_1 = f_2 = 1$ on the square, and $F_1(x) = F_2(x) = x - (b-1)$. The shifted uniform is displayed graphically in Figure 5.

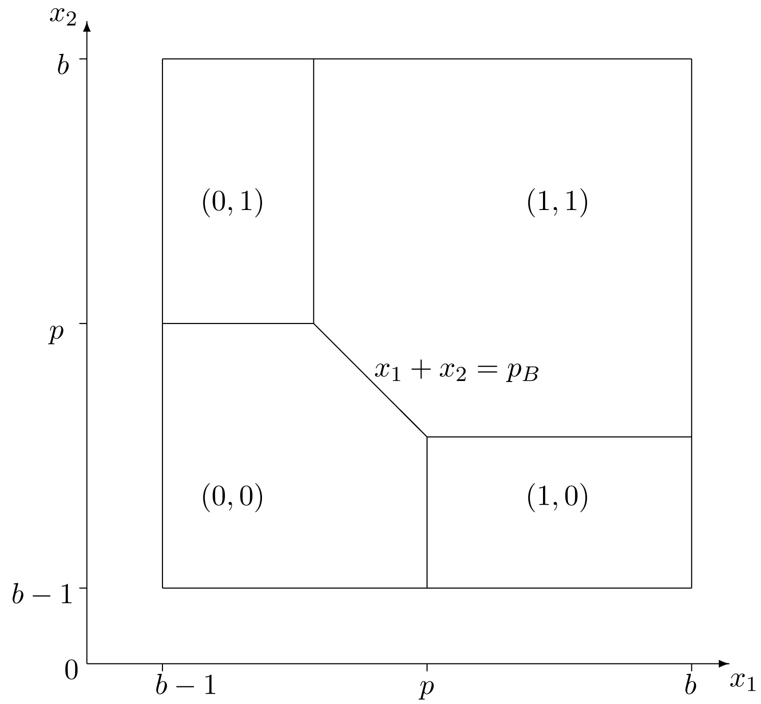


Figure 5: The Shifted Uniform

Notes: The consumers are depicted with support on $[b-1,b]^2 \subset \mathbb{R}_+^2$. The consumer valuations are partitioned according to the purchase decisions under deterministic bundling. The deterministic bundle $(0,1)$ which delivers good 2 with certainty is purchased by consumers in the north-west. The bundles bought in the other three regions of the square are labelled similarly. There is no mass outside the square $[b-1,b]^2$.

Proposition 5 *Under assumption ABS, stochastic bundling is optimal for any strictly shifted uniform distribution of consumer valuations: $b>1$.*

Proof. Following the proof of Proposition 4, a sufficient condition for stochastic bundling to dominate deterministic price schedules is (27). Substituting in for the shifted uniform distribution this can be written

$$\frac{p_B - p_2}{p_B - p_2 - (b-1)} > 1 \tag{29}$$

Under assumption ABS $F_1(p_B - p_2) > 0$ implying $p_B - p_2 - (b-1) > 0$. Hence (29) holds for $b > 1$ giving the result. ■

Taken together Propositions 4 and 5 demonstrate that the optimality of stochastic bundles is robust. They however appear to point to differing intuitions as to when stochastic bundling is likely to be optimal; relying on the slope of the density function in the first case, and the

support of the valuations in the second. Delivering an intuition for the optimality of stochastic bundles is a subject to which we now turn.

9 Intuition and Construction of Profitable Stochastic Bundles

In this section we establish some intuition for the optimality of stochastic bundling. Suppose that a seller has set deterministic bundle prices of $\{p_1, p_2, p_B\}$, and condition (25) of Proposition 3 holds: $\partial^2\pi / \partial p_B \partial p_2 < 0$. This condition implies that for small $\delta > 0$:

$$\frac{\partial \pi}{\partial p_B}(p_1, p_2 - \delta, p_B) - \frac{\partial \pi}{\partial p_B}(p_1, p_2 + \delta, p_B) > 0 \tag{30}$$

The first derivative $\partial\pi/\partial p_B\,(p_1, p_2 + \delta, p_B)$ relates to panel (a) of Figure 6. This first derivative $\partial\pi/\partial p_B$ captures the change in profits if the bundle price were to be raised marginally. Such a bundle price change would result in increased profits from the shaded region in panel (a), coupled with profit losses across the boundary of this region. Consider now the alternative perturbation by which the good 2 price is first lowered to $p_2 - \delta$ so the prices are $\{p_1, p_2 - \delta, p_B\}$, and then the price of the bundle is raised marginally. This would alter the participation regions to the dashed lines shown in panel (a). Condition (30) informs us that the difference between these two thought experiments can be signed. This difference is depicted as panel (b) of Figure 6. As marginal changes in p_B do not alter the boundary between those buying good 2 or nothing, the difference (30) relates only to a band around the boundary between those buying good 2 and the bundle. Using (39) to evaluate (30) and cancelling terms:

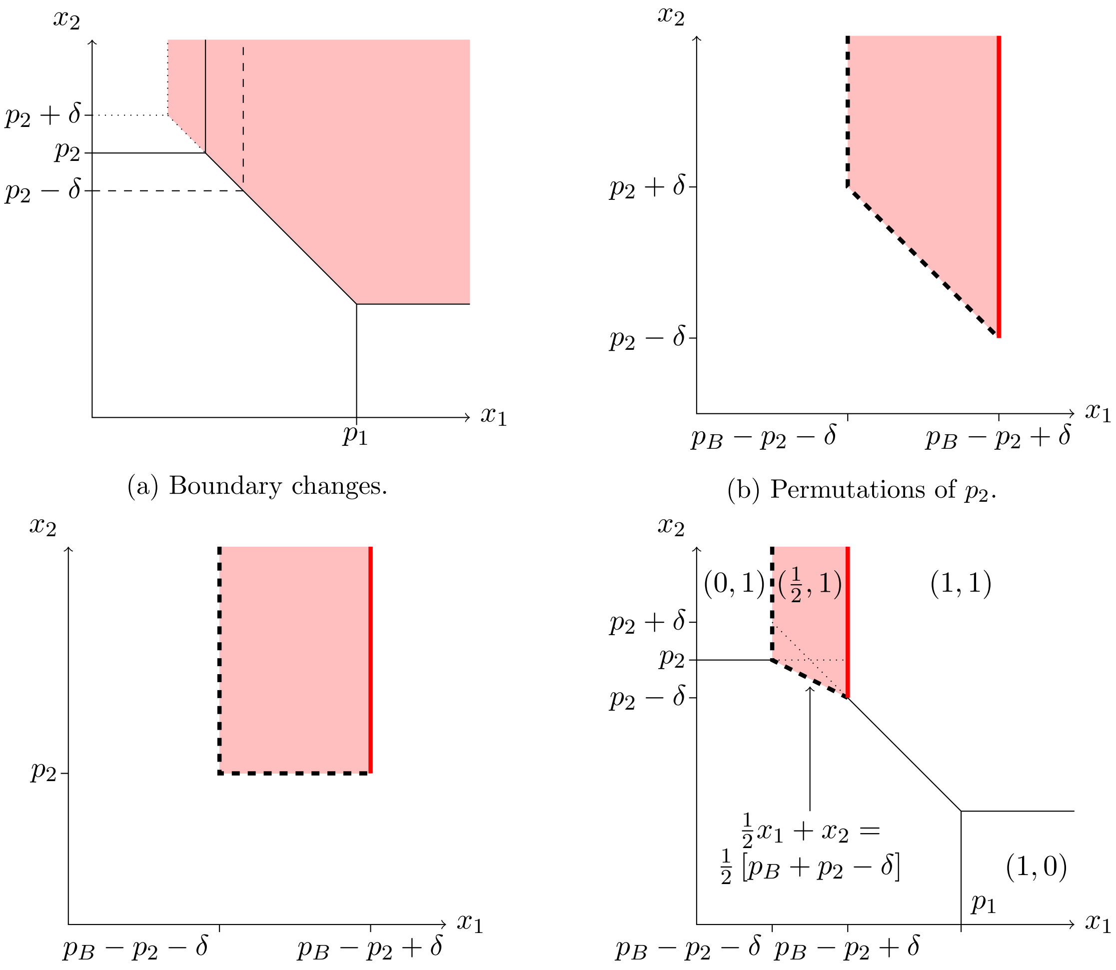
$$\begin{aligned} &(p_B - p_2 - \delta)\left[1 - F_2(p_2 + \delta)\right] f_1(p_B - p_2 - \delta) - (p_B - p_2 + \delta)\left[1 - F_2(p_2 - \delta)\right] f_1(p_B - p_2 + \delta) \\ &\quad + p_B \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z)\, f_2(p_B - z)\, dz - \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z)\left[1 - F_2(p_B - z)\right] dz \;>\; 0 \end{aligned} \tag{31}$$

From condition (31) the line integrals with positive coefficients are represented in panel (b) of Figure 6 by the black dashed lines in the figure; the line integral with the negative coefficient is represented by the solid red line; and the area integral is shaded pink. Hence if the seller could determine an alteration to the selling mechanism which increased profits along the dashed lines, and lost it on the pink area and red line, in the right amounts, then this would be profit enhancing. We will see that stochastic bundles can do essentially this.

The same thought experiment can be applied reversing the order in which good 2 and the bundle prices are perturbed. Specifically we compare the first profit derivative with respect to price p_2, $\partial\pi/\partial p_2$, at prices $\{p_1, p_2, p_B - \delta\}$ and $\{p_1, p_2, p_B + \delta\}$. If condition (25) of Proposition 3 holds then

$$\frac{\partial \pi}{\partial p_2}(p_1, p_2, p_B - \delta) - \frac{\partial \pi}{\partial p_2}(p_1, p_2, p_B + \delta) > 0 \tag{32}$$

Panel (c) of Figure 6 depicts this difference which again relates only to a band around the boundary between those buying good 2 and the bundle. Explicitly combine (7) and (8), to yield



(c) Permutations of p_B.

(d) Stochastic bundle.

Figure 6: Second Profit Derivative Condition and Optimal Stochastic Bundling

Notes: Panel (a) plots the participation regions from deterministic prices $\{p_1, p_2 + \delta, p_B\}$. The shaded area denotes those buying the bundle of both goods. The derivative $\partial\pi/\partial p_B\,(p_1, p_2 + \delta, p_B)$ is found as the profit change on the area shaded adjusted by the profit changes along the line integrals at the boundary of this region. Panel (b) depicts graphically condition (30) whose sign is a corollary of condition (25) of Proposition 3: $\partial^2\pi/\partial p_B \partial p_2 < 0$. The difference between $\partial\pi/\partial p_B\,(p_1, p_2 - \delta, p_B)$ and $\partial\pi/\partial p_B\,(p_1, p_2 + \delta, p_B)$ is captured by the band displayed in the panel. Note that the scale has been blown up as compared to panel (a). The line integrals with positive coefficients in condition (31) are represented by the black dashed lines in the panel; the line integral with the negative coefficient is represented by the solid red line; and the area integral is shaded pink. Panel (c) depicts graphically condition (32) whose sign is a corollary of condition (25) of Proposition 3: $\partial^2\pi/\partial p_B \partial p_2 < 0$. The difference between $\partial\pi/\partial p_B\,(p_1, p_2, p_B - \delta)$ and $\partial\pi/\partial p_B\,(p_1, p_2, p_B + \delta)$ is captured by the band displayed in the panel. Note that the scale has again been blown up as compared to panel (a). The line integrals with positive coefficients in condition (33) are represented by the black dashed lines in the panel; the line integral with the negative coefficient is represented by the solid red line; and the area integral is shaded pink. Panel (d) depicts the participation regions in consumer valuation space (x_1, x_2) from the addition of the single stochastic bundle $(1/2, 1)$ at a price of $(1/2) \cdot [p_B + p_2 - \delta]$ to the deterministic bundling prices $\{p_1, p_2, p_B\}$. This stochastic bundle only alters the purchase behaviour of those consumers who were close to indifferent between buying good 2 only or the bundle. The panel is drawn at the same scale as panel (a). Panel (d) depicts the rate of change of profit with respect to discount, δ : line integrals with positive coefficients are represented by the black dashed lines in the panel; the line integral with the negative coefficient is represented by the solid red line; and the area integral is shaded pink. The boundaries from panels (b) and (c) are superimposed as the dotted lines.

$\partial\pi/\partial p_2$ and applying to (32) we have

$$\begin{aligned}(p_B - p_2 - \delta)\,[1 - F_2(p_2)]f_1(p_B - p_2 - \delta) - (p_B - p_2 + \delta)\,[1 - F_2(p_2)]f_1(p_B - p_2 + \delta) \\ +p_2 \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z)\, f_2(p_2)\, dz - \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z)\,[1 - F_2(p_2)]\, dz \quad > \quad 0\end{aligned} \tag{33}$$

Once again the line integrals with positive coefficients are represented in panel (c) of Figure 6 by the black dashed lines; the line integral with the negative coefficient is represented by the solid red line; and the area integral is shaded pink. The areas differ slightly between panels (b) and (c), though we will see not by a significant amount.

For comparison it is now helpful to study the case in which the seller, alongside deterministic bundling prices $\{p_1, p_2, p_B\}$, introduces the stochastic bundle $(1/2, 1)$, in which good 2 is delivered for certain and good 1 with probability $1/2$, at a price of $(1/2) \cdot [p_B + p_2 - \delta]$. The participation regions from this lottery are depicted in panel (d) of Figure 6. Once again this figure is colour coded: if one were to marginally increase δ so that the price of the lottery declined slightly, then the seller would lose profits over the pink shaded region and gain (lose) profits along the boundary marked by a black dashed (solid red) line.

We will now show that under condition (25) of Proposition 3 this lottery is a profitable addition to the deterministic bundle (Proposition 6, below). We will also show that a marginal reduction in the lottery price (panel (d) of Figure 6) generates a change in the sellers' profits which is equivalent to the average of panels (b) and (c) of Figure 6 to first order in δ. Thus the profitability of the lottery could have been inferred from conditions (30) and (32) which are themselves immediate consequences of condition (25) of Proposition 3 (Claim 2, below).

Proposition 6 *If one of the conditions of Proposition 3 holds, then for small δ the introduction of the stochastic bundle $(1/2, 1)$ at a price of $(1/2) \cdot [p_B + p_2 - \delta]$ is profitable for the seller.*

Proof. Consider a deformation to the boundary function in the region $x_1 \in [p_B - p_2 - \delta, p_B - p_2 + \delta]$ for some small δ to a boundary $\varphi(\cdot)$ given by

$$\varphi(x_1) = p_2 - \frac{1}{2}\,[x_1 - (p_B - p_2 - \delta)] \tag{34}$$

Note that $\varphi(p_B - p_2 + \delta) = p_B - (p_B - p_2 + \delta)$ so that the deformation rejoins the deterministic bundle tariff. As $\varphi'(x_1) = -1/2$ this deformation is equivalent to the introduction of the lottery $(1/2, 1)$. The price of this lottery is given by $\varphi(x_1) - x_1\varphi'(x_1) = \frac{1}{2}\,[p_B + p_2 - \delta]$. The new consumer participation regions are therefore plotted in panel (d) Figure 6. That the addition of this stochastic bundle raises profits for δ small is proved as Claim 3 in Appendix A. ■

To complete our intuition we confirm an equivalence between the profit changes introduced by marginal reductions in the price of the lottery $(1/2, 1)$, and the conditions (30) and (32) depicted as panels (b) and (c) of Figure 6. Denote the profit available with the lottery $(1/2, 1)$ priced at a reduction of $\delta/2$ on the actuarially fair level as $\Pi(p_1, p_2, p_B, \delta)$.

Claim 2

$$\begin{aligned}\frac{\partial}{\partial\delta}\Pi\left(p_1,p_2,p_B,\delta\right) &= \frac{1}{4}\left[\frac{\partial\pi}{\partial p_B}\left(p_1,p_2-\delta,p_B\right)-\frac{\partial\pi}{\partial p_B}\left(p_1,p_2+\delta,p_B\right)\right] \quad \textit{(cf. (30))}\\ &+\frac{1}{4}\left[\frac{\partial\pi}{\partial p_2}\left(p_1,p_2,p_B-\delta\right)-\frac{\partial\pi}{\partial p_2}\left(p_1,p_2,p_B+\delta\right)\right] \quad \textit{(cf. (32))}\\ &+\mathcal{O}\left(\delta^2\right)\end{aligned}$$

When at deterministic prices the cross-partial derivative of profits $\partial^2\pi/\partial p_2\partial p_B < 0$, then Proposition 3 guarantees that there exist stochastic bundles which can be profitably added to the optimal deterministic prices. Proposition 6 explicitly constructs such a profitable addition to the optimal deterministic prices. The seller can always increase her profits by offering a stochastic bundle of good 2 for sure and the possible addition of good 1 with positive probability. Consumers close to indifferent between buying both goods or just good 2 are induced to purchase the stochastic bundle by a small price reduction on the actuarially fair retail price. These consumers were depicted in the shaded region of panel (d) of Figure 6. This stochastic bundle allows the seller to screen out those consumers who value good 2 a great deal, and value good 1 at just below the level at which they would have purchased it.

Claim 2 demonstrates that the change in profits created by the introduction of this lottery is, to first order in the discount for buying it, captured by comparing the gain in profits from increasing the bundle price at deterministic prices $\{p_1, p_2-\delta, p_B\}$ and at $\{p_1, p_2+\delta, p_B\}$ (panels (a) and (b) of Figure 6). The difference in the participation regions includes the consumers who would buy the lottery $(1/2, 1)$; the marginal increase in p_B then picks out just those consumers in this set who would buy the lottery. Under the conditions of Proposition 3, profits can be increased by altering the probability of receiving good 1 across this group of consumers.

Condition (25) of Proposition 3 is the condition which guarantees that it is profitable to exploit the boundary between those who buy the bundle and those who buy just good 2. Inspection of panels (b) and (c) of Figure 6 shows that there are essentially two ways in which the inequalities (30) and (32) can hold:

1. If we have $f_1(p_B-p_2-\delta) >> f_1(p_B-p_2+\delta)$ then it is possible for the profit gain on the vertical dashed lines of panels (b) and (c) in Figure 6 to outweigh the loss on the solid vertical red line, as well as the loss over the area integral. This requires that the density function $f_1(\cdot)$ be downwards sloping. This is the rationale for the optimality of stochastic bundling under a tilted uniform distribution (Section 7).

2. Alternatively if $\int_{p_B-p_2-\delta}^{p_B-p_2+\delta} f_1(z)\left[p_2 f_2(p_2)-\left[1-F_2(p_2)\right]\right] >> 0$ then it is again possible for the profit gain on the horizontal or slanted dashed lines of panels (b) and (c) in Figure 6 to outweigh the loss over the area integral, as well as the profit contributions across the vertical boundary lines. This condition is satisfied if the revenue secured from extra business is significant and there is a substantial mass of consumers at the three way boundary between not buying, buying good 2 or buying the bundle. This is so in the example of the shifted uniform (Section 8).

A competition authority may be concerned at the practice of using randomness, as profitable

stochastic bundles can involve extracting greater consumer surplus by reducing the probability of serving those with a relatively higher value for the stochastically supplied product, and increasing the probability of serving those with a relatively lower valuation. Thus it would seem that profitable stochastic bundling should lower consumer surplus. We will demonstrate that this logic is incomplete. To encourage the consumers to purchase the profitable stochastic bundle of Proposition 6 a price reduction is required, and this transfer to consumers can increase overall consumer welfare. Thus the introduction of the stochastic bundle can be a welfare improvement.

Proposition 6 (ii) *If one of the conditions of Proposition 3 holds, then the introduction of the stochastic bundle in Proposition 6 increases consumer surplus.*

It is not however a general result that consumer surplus rises with stochastic bundling. As noted the stochastic bundle allows the seller to screen out from the bundle consumers those who value good 2 highly, but good 1 much less. The remaining bundle consumers now have a higher average valuation for the bundle. The seller can therefore increase her profits further by altering the deterministic prices in the presence of the stochastic bundle. This effect can lower consumer surplus. Nevertheless, Proposition 6(ii) demonstrates that one cannot condemn stochastic bundling as *per se* detrimental to consumer surplus.

10 Reversal Of The No Time Discrimination Result

In this section we establish a correspondence between stochastic bundling and temporal price discrimination by a monopolist. Stokey [26] demonstrated that a single-good monopolist who can commit to a price path through time would optimally not price discriminate over time. Any reduction in the price to attract lower value consumers later in time would encourage too many higher demand consumers to delay their consumption also. This insight parallels that of Riley and Zeckhauser [22] who demonstrated that a single-good monopolist would not wish to price discriminate over the probability of delivery, that is to 'haggle,' as this would cannibalise demand from too many of the higher value consumers.

Little is known about the temporal multiple-good monopolist problem due to the complexity of price discriminating over multiple dimensions. Here however we develop the parallel between stochastic sales and delayed sales and demonstrate that price discriminating through time can be optimal for the multiple-good monopolist.

To develop the intuition for our analogy recall that Proposition 3 has identified conditions under which there exists an assignment of utilities of the form $u(x) = x_2 - \varphi(x_1)$ which generates more profit for the seller than the best deterministic bundle prices. In this assignment consumer x receives the stochastic bundle with delivery probabilities $(-\varphi'(x_1), 1)$, and must pay a price of $\varphi(x_1) - x_1\varphi'(x_1)$. Now note that any stochastic bundle of the form $(q_1, 1)$ for a price of p can be interpreted as a dynamic-delivery offer under which the consumer pays p, receives good 2 immediately, and receives good 1 after time τ such that $e^{-r\tau} = q_1$, where r is the discount rate. Under both interpretations a consumer x derives utility $q_1x_1 + x_2 - p$. We can formalise this correspondence:

Proposition 7 *Given any utility permutation $u(x) = x_2 - \varphi(x_1)$ with $x_1 \in [0, \tilde{x}]$ which is allowable (so satisfies (16) through (18)), the same consumer utilities can be implemented by adding to the deterministic bundle prices a menu of dynamic prices $\{q_1, p\}$ such that for total price p, good 2 is delivered immediately, and good 1 is delivered at time τ such that $e^{-r\tau} = q_1$.*

Proof. Given any allowable utility permutation function $\varphi(\cdot)$, then there exists a suite of stochastic bundles which generates consumer utilities $\{u(x)\}$ by Lemma 2. Further, under the utility permutation function $\varphi(\cdot)$, consumer x receives the stochastic bundle with delivery probabilities $(-\varphi'(x_1), 1)$, and must pay a price of $\varphi(x_1) - x_1\varphi'(x_1)$. Note therefore that consumer optimisation over the set of stochastic bundles $\{(-\varphi'(z), 1)$ at price $\varphi(z) - z\varphi'(z) : z \in [0, \tilde{x}]\}$ must generate utility $\{u(x)\}$. If it did not then the utility permutation $u(x) = x_2 - \varphi(x_1)$ would not have been incentive compatible and therefore not allowable. The analogue between stochastic bundles and delayed delivery is now immediate: the package of good 2 delivered immediately and good 1 delivered after time τ is offered if there exists $z \in [0, \tilde{x}]$ such that $-\varphi'(z) = \exp(-r\tau)$, and the total price is $p = \varphi(z) - z\varphi'(z)$. ■

Proposition 7 allows us to recast the results of this study in terms of optimal dynamic price discrimination:

Proposition 8 (Optimality of Temporal Price Discrimination) *A multiple-good monopolist will benefit from introducing price discrimination through time for good 1 conditional on the purchase of good 2, if one of the conditions of Proposition 3 is satisfied.*

Proof. Under the conditions of Proposition 3 there exists an allowable permutation $\varphi(x_1)$, such that for each consumer who participates the seller receives revenue $\varphi(x_1) - x_1\varphi'(x_1)$, and the consumer receives utility $x_2 - \varphi(x_1)$, and this is more profitable for the seller than the optimal deterministic, non-time-varying, prices. By Proposition 7 there exists a menu of dynamic prices for good 1 conditional on the purchase of good 2 which implements this allocation of revenues to the seller and utilities to the buyers. Hence there exists a menu of dynamic prices which delivers a higher profit than the optimal fixed prices. ■

Proposition 8 establishes conditions under which a two-good monopolist increases her profits by price discriminating over time. Therefore we have established that the 'no temporal price discrimination' result for the monopolist does not extend past the single product case. For example, the analysis of Sections 7 and 8 demonstrate that the slightest tilt or shift of the uniform distribution results in the optimality of dynamic price discrimination.

The dynamic price discrimination we have studied can be interpreted in a manner which is closer to marketing practice: the seller offers the goods at fixed prices $\{p_1, p_2, p_B\}$, and conditional on the purchase of good 2, offers good 1 to the consumer for a further payment which is time-varying. Such a strategy is known in marketing as *cross-selling*. Hence one might refer to the strategy we have identified as the optimality of dynamic cross-selling. The dynamic prices available only conditional on the purchase of good 2 allows the seller to focus the distortion in purchasing behaviour on just one of the boundaries: that between the purchasers of good 2 and the bundle. Thus the economics underlying the optimality of temporal price discrimination parallel those discussed in Figure 6 .

Finally note that we can construct profitable dynamic cross-sell strategies by using this

isomorphism between stochastic bundling and temporal price discrimination and reinterpreting the analysis of Section 9:

Proposition 9 *If one of the conditions of Proposition 3 is satisfied then for sufficiently small* $\delta > 0$, *the introduction of the cross-sell such that good* 1 *is offered to the purchasers of good* 2 *at time* $t = -\frac{1}{r}\ln\left(\frac{1}{2}\right)$ *for price* $p_{1X} = p_B - p_2 - \delta$ *is profit increasing for the seller.*

Proof. If the consumer buys good 2 and then the cross-sell to good 1 he/she derives utility

$$x_2 - p_2 + e^{-rt}\left(x_1 - p_{1X}\right) = x_2 - \left[p_2 - \frac{1}{2}\left[x_1 - \left(p_B - p_2 - \delta\right)\right]\right].$$

Hence the introduction of the cross-sell option alters the consumers' utility to match the perturbation (34). The seller receives net present value of revenue from each consumer who participates of:

$$p_2 + e^{-rt}p_{1X} = \frac{1}{2}\left(p_B + p_2 - \delta\right)$$

This is identical to that generated by the stochastic bundle given in Proposition 6. Hence the result follows from that Proposition. ■

11 Conclusion

It was the objective of this paper to study when and how stochastic bundling is part of the multiproduct monopolist's optimal selling mechanism. We showed that stochastic bundling allows the seller to target individual consumer participation boundaries. When the slope of consumer valuations is consistently such that there is a large mass of consumers buying a subset of the bundle near to the boundary, then stochastic bundles can be used to draw those consumers profitably into a greater volume of purchase. This is why the slightest tilt of the uniform distribution of valuations results in the optimality of stochastic bundling. For analogous reasoning, when there is a mass of consumers who are not purchasing, but who value the products sufficiently above cost, then stochastic bundles can again be used to draw these consumers profitably into a greater volume of purchase. This is why the slightest shift of the uniform distribution of valuations results in the optimality of stochastic bundling. Further, this targeting of the consumer boundary can be mapped to an appropriate intertemporal price discrimination across multiple goods. We have described the profitable deviation as dynamic cross-selling.

We hope that our work will contribute to several avenues of scholarly work. Most directly the search for optimal multiproduct price schedules is yet to be solved, despite the many contributions we have reviewed in Section 2. We have identified a class of stochastic bundles which increase profits beyond the optimal deterministic bundle prices: the stochastic bundles include one good with certainty and the second with some probability. In the two type case of Section 3 this family was sufficient to deliver the fully optimal mechanism. The outstanding question is whether the same is true in the general setting we study, so that the calculus of variations conditions of Section 5 construct the fully optimal tariff. Pavlov [20] holds out the hope that this might indeed be the case under the regularity condition of McAfee and McMillan [18] as the optimal tariff must deliver some product for certain.

Secondly our work opens up a parallel between stochastic bundling and optimal monopoly temporal price discrimination. The literature here, as noted in Section 2, has predominantly focused on the setting in which the monopolist cannot commit to prices over time and so is liable to the Coase conjecture. Our work suggests that the setting of monopoly commitment is not so straight forward: the optimality of no price discrimination through time does not extend beyond the single good setting. Optimal targeting of consumers is a fruitful study for many scholars in marketing and related disciplines. This literature has often taken an empirical approach in which trial and error lead to identification of which consumer types should receive cross-sell promotions after a period of time. We hope that our work will contribute theoretical insights to these approaches.

A Omitted Proofs

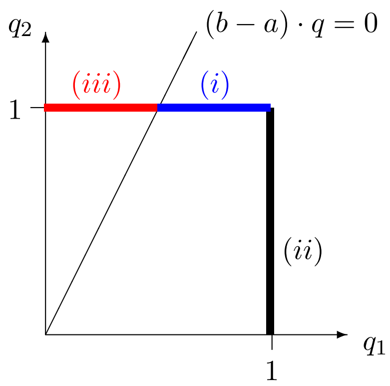


Figure 7: Stochastic Bundle Space for the Motivating Example (Section 3, Proposition 1).
Notes: The figure shows the feasible space of stochastic bundles: $\{q^a, q^b\} \in [0,1]^2$. The line defined by $(b-a)\cdot q = 0$ lies anticlockwise of the point $(1,1)$ as condition (1) that the consumers are not strongly ordered implies $(b-a)\cdot(0,1) < 0$; and condition (2) implies $(b-a)\cdot(1,1) \geq 0$. The boundary of the feasible region is subdivided into distinct labels ((i) shown in blue, (ii) shown in black, and (iii) shown in red) which are referred to in the proof of Proposition 1.

Proof of Proposition 1. As noted, the seller's revenue maximisation problem is equivalent to maximising $\sum_{k\in\{a,b\}} \lambda^k(k\cdot q^k - U^k)$ subject to the incentive compatibility constraints (3) and (4). The constraints (3) and (4) are plotted in panel (b) of Figure 1. There are three possible cases depending on the signs of $(b-a)\cdot q^a$ and $(b-a)\cdot q^b$. We can solve for the optimal bundle within each case:

1. *The feasible band in $\left(U^a, U^b\right)$ lies strictly above the origin:* $0 < (b-a)\cdot q^a \leq (b-a)\cdot q^b$.

   This is the case depicted in panel (b) of Figure 1. In this case, the minimal surplus which can be left to the consumers satisfies $U^a = 0$, and $U^b = (b-a)\cdot q^a$. The seller's expected revenue is

$$R = \lambda^a\left(a\cdot q^a - 0\right) + \lambda^b[b\cdot q^b - (b-a)\cdot q^a] = \lambda^b b\cdot q^b + \left(a - \lambda^b b\right)\cdot q^a. \tag{35}$$

   The feasible set of bundles, in this case requires

$$q^a, q^b \in [0,1]^2 \text{ and } 0 < (b-a)\cdot q^a \leq (b-a)\cdot q^b \tag{36}$$

   As revenue (35) is linear it is maximised at an extreme point of the feasible set of bundles $\left\{q^a, q^b\right\}$. Fix q^a and consider Figure 7. Constraints (36) and (1) imply that q^b must lie weakly clockwise of q^a on either (i) or (ii). Suppose first that q^a is fixed on the boundary (ii). Revenue is increasing in q^b, and so optimality requires the seller to set $q^b = q^a$. Substituting into revenue (35) yields $R = a\cdot q^a$ yielding optimal bundle $q^a = q^b = (1,1)$ in this case. The second possibility is that q^a is on the boundary (i). In this case optimality requires $q^b = (1,1)$. Substituting into revenue (35) yields $R = \lambda^b b\cdot(1,1) + \left(a - \lambda^b b\right)\cdot q^a$. Note that $\left(a - \lambda^b b\right)$ has positive second coordinate and may have a negative first coordinate. If $a_1 - \lambda^b b_1 \geq 0$ then revenue is increasing in q^a and so optimality would require $q^a = (1,1)$. If however $a_1 - \lambda^b b_1 < 0$ then revenue is declining in q^a and so there is no maximum point in the feasible set (36).

   The optimal bundle in this case is therefore $q^a = q^b = (1,1)$ yielding revenue $R^{\text{case 1}} =$

$a_1 + a_2$. The full bundle is sold to everyone at a low price $a_1 + a_2$. The social welfare is maximised, and b types keep a rent.

2. *The feasible band in* $\left(U^a, U^b\right)$ *encompasses the origin:* $(b-a) \cdot q^a \leq 0 \leq (b-a) \cdot q^b$

   In this case, the minimal surplus which can be left to the consumers satisfies $U^b = U^a = 0$. The seller's expected revenue is $\sum_{k \in \{a,b\}} \lambda^k k \cdot q^k$. The feasible set of bundles, in this case requires

$$q^a, q^b \in [0,1]^2 \text{ and } (b-a) \cdot q^a \leq 0 \leq (b-a) \cdot q^b \tag{37}$$

   As revenue is linear it can be separately maximized in q^a and q^b. Optimality requires that q^b must lie on boundary (i) or (ii) of Figure 7. The unique maximal point is $q^b = (1,1)$. The optimal q^a must lie on (iii), and so optimality requires $(b-a) \cdot q^a = 0$ yielding $q^a = \left(\frac{a_2 - b_2}{b_1 - a_1}, 1\right)$.

   The optimal seller strategy in this case is therefore to sell the full bundle to b types at a high price $b_1 + b_2$, and a types buy a stochastic bundle. Social welfare is not maximised in this case but all rents are extracted. Expected revenue is $R^{\text{case 2}} = \lambda^b(b_1 + b_2) + \lambda^a \left(a_1 \frac{a_2 - b_2}{b_1 - a_1} + a_2\right)$.

3. *The feasible band in* $\left(U^a, U^b\right)$ *lies strictly below the origin:* $(b-a) \cdot q^a \leq (b-a) \cdot q^b < 0$

   In this case, the minimal surplus which can be left to the consumers satisfies $U^b = 0, U^a = -(b-a) \cdot q^b$. The seller's expected revenue is

$$R = \lambda^a[a \cdot q^a + (b-a) \cdot q^b] + \lambda^b b \cdot q^b = \lambda^a a \cdot q^a + (b - \lambda^a a) \cdot q^b.$$

   The feasible set of bundles, in this case requires

$$q^a, q^b \in [0,1]^2 \text{ and } (b-a) \cdot q^a \leq (b-a) \cdot q^b < 0 \tag{38}$$

   Optimality requires that the bundles lie on boundary (iii) of Figure 7. Given q^b, revenue is increasing in q^a. Condition (1) implies q^a must lie anticlockwise of q^b, so optimality requires $q^a = q^b$. The seller revenue is therefore $b \cdot q^b$. This is increasing in q^b and so there is no maximum point within the feasible set (38). This case cannot therefore obtain at the seller's solution.

The only feasible cases are therefore cases 1 and 2. Comparing revenues to determine optimality we have:

$$\begin{aligned}
R^{\text{case 2}} - R^{\text{case 1}} &= \lambda^b(b_1 + b_2) + \lambda^a(a_1 \frac{a_2 - b_2}{b_1 - a_1} + a_2) - (a_1 + a_2) \\
&= \lambda^b(b_1 + b_2 - a_1 - a_2) + \lambda^a a_1 \left(\frac{a_2 - b_2}{b_1 - a_1} - 1\right) \\
&= \frac{b_1 + b_2 - a_1 - a_2}{b_1 - a_1} \left(\lambda^b b_1 - a_1\right).
\end{aligned}$$

So case 2 (stochastic bundling) is optimal when $a_1 - \lambda^b b_1 < 0$. If this inequality is not satisfied then we noted in case 1 above that for these parameters case 1 (deterministic bundling) is feasible

and so optimal. This completes the proof. ■

Proof of Claim 1. Assume we are at an interior solution so that a positive measure of consumers buys all combinations of the goods. Using (6) we have

$$\begin{aligned}\frac{\partial\pi}{\partial p_B} &= p_1\left[1-F_1\left(p_1\right)\right]f_2\left(p_B-p_1\right)+p_2[1-F_2(p_2)]f_1(p_B-p_2) \\ &+\int_{p_B-p_2}^{p_1} f_1\left(z\right)\left[1-F_2\left(p_B-z\right)\right]dz+\left[1-F_2\left(p_B-p_1\right)\right]\left[1-F_1\left(p_1\right)\right] \\ &-p_B\left\{f_1\left(p_B-p_2\right)\left[1-F_2\left(p_2\right)\right]+\int_{p_B-p_2}^{p_1} f_1\left(z\right)f_2\left(p_B-z\right)dz+f_2\left(p_B-p_1\right)\left[1-F_1\left(p_1\right)\right]\right\}\end{aligned} \tag{39}$$

Now substituting in for the density functions and setting to zero gives the optimality condition

$$\begin{aligned}0 &= \frac{p_1}{b}\left[1-\frac{p_1}{c}\right]+\frac{p_2}{c}\left[1-\frac{p_2}{b}\right] \\ &+\int_{p_B-p_2}^{p_1}\frac{1}{c}\left[1-\frac{p_B-z}{b}\right]dz+\left[1-\frac{p_B-p_1}{b}\right]\left[1-\frac{p_1}{c}\right] \\ &-p_B\left\{\frac{1}{c}\left[1-\frac{p_2}{b}\right]+\int_{p_B-p_2}^{p_1}\frac{1}{cb}dz+\frac{1}{b}\left[1-\frac{p_1}{c}\right]\right\}\end{aligned}$$

The integral terms yield $\frac{1}{cb}\left(b-2p_B\right)\left(p_1+p_2-p_B\right)+\frac{1}{2cb}\left[p_1^2-\left(p_B-p_2\right)^2\right]$. Substituting in the optimal values for the prices and collecting terms in multiples of p_B gives the quadratic equation

$$0=\frac{2c}{3b}+\frac{2b}{3c}+1-2p_B\left(\frac{1}{c}+\frac{1}{b}\right)+p_B^2\frac{3}{2cb}$$

This can be solved to yield the optimal bundle price given in (10). Finally under the parameter restriction $2c>b>c/2$ one can observe that $p_B>p_1,p_2$ and so all product combinations are bought by a positive measure of consumers, thus confirming that we are indeed at an interior solution as assumed. ■

Proof of Proposition 2. Existence follows from Hellwig ([14], Theorem 3.1) who demonstrates that the Lagrangian approach is valid even if the function $\varphi'(\cdot)$ is not piecewise continuous. To characterise the solution, compute the directional derivative of the Lagrangian (20) at φ in the direction of a variation h:

$$\begin{aligned}DL(\varphi)h = &\int_0^{\tilde{x}}\left\{\begin{array}{c}[-f_2(\varphi)[\varphi-x_1\varphi']+(1-F_2(\varphi))]\,h(x_1)f_1(x_1)-[1-F_2(\varphi)]x_1h'(x_1)f_1(x_1) \\ -\lambda(x_1)h''(x_1)\end{array}\right\}dx_1 \\ &-\mu_1h'(0)-\mu_2h(\tilde{x})+\mu_3h'\left(\tilde{x}\right).\end{aligned}$$

Now integration by parts yields

$$\int_0^{\tilde{x}}\lambda(x_1)h''(x_1)dx_1=\left[\lambda(x_1)h'(x_1)\right]_0^{\tilde{x}}-\int_0^{\tilde{x}}\lambda'(x_1)h'(x_1)dx_1$$

Thus

$$DL(\varphi)h = \int_0^{\tilde{x}} \left\{ \begin{array}{c} \{-f_2(\varphi)[\varphi - x_1\varphi'] + (1 - F_2(\varphi))\}\, h(x_1)f_1(x_1) \\ + \{\lambda'(x_1) - [1 - F_2(\varphi)]x_1 f_1(x_1)\}\, h'(x_1) \end{array} \right\} dx_1$$
$$- \mu_1 h'(0) - \mu_2 h(\tilde{x}) - \left[\lambda(x_1)h'(x_1)\right]_0^{\tilde{x}} + \mu_3 h'(\tilde{x}).$$

Integration by parts again yields:

$$DL(\varphi)h = \int_0^{\tilde{x}} \left\{ \begin{array}{c} \{-f_2(\varphi)[\varphi - x_1\varphi'] + (1 - F_2(\varphi))\}\, f_1(x_1) \\ - \left\{\lambda''(x_1) - \frac{d}{dx_1}\{[1 - F_2(\varphi)]x_1 f_1(x_1)\}\right\} \end{array} \right\} h(x_1)dx_1 \tag{40}$$
$$- \mu_1 h'(0) - \mu_2 h(\tilde{x}) - \left[\lambda(x_1)h'(x_1)\right]_0^{\tilde{x}} + \left[\left\{\lambda'(x_1) - [1 - F_2(\varphi)]x_1 f_1(x_1)\right\} h(x_1)\right]_0^{\tilde{x}} + \mu_3 h'(\tilde{x}).$$

At an optimal perturbation φ the directional derivative must vanish for all variations h, and so the term in h in the integral should be zero, which gives the Euler equation:

$$\begin{aligned} -f_2(\varphi)[\varphi - x_1\varphi']f_1(x_1) + [1 - F_2(\varphi)]\, f_1(x_1) &= \lambda''(x_1) - \frac{d}{dx_1}\{[1 - F_2(\varphi)]x_1 f_1(x_1)\} \\ -f_2(\varphi)[\varphi - x_1\varphi']f_1(x_1) + [1 - F_2(\varphi)]f_1(x_1) &= -[x_1 f_1(x_1)]'[1 - F_2(\varphi)] + x_1 f_1(x_1) f_2(\varphi)\varphi' + \lambda''(x_1) \end{aligned}$$

This simplifies to yield (22).

The directional derivative (40) vanishes generating further transversality conditions. The term in $h(0)$ yields (21). Next suppose that $\varphi'(\tilde{x}) > -1$, by complementary slackness $\mu_3 = 0$. The transversality condition in $h'(\tilde{x})$ yields $\lambda(\tilde{x}) = 0$. The transversality condition in $h(\tilde{x})$ yields, (using that $\varphi(\tilde{x}) = p_B - \tilde{x}$):

$$\lambda'(\tilde{x}) = \mu_2 + \tilde{x} f_1(\tilde{x})[1 - F_2(p_B - \tilde{x})] \tag{41}$$

To complete the derivation of (23) we optimise over the starting point of the perturbation $\tilde{x}$. The total profit of the seller is given by

$$\pi = L(\tilde{x}) + \int_{x_1=\tilde{x}}^{p_1} p_B\,[1 - F_2(p_B - x_1)]\, f_1(x_1)\, dx_1 + [\text{terms independent of } \tilde{x}] \tag{42}$$

Differentiating with respect to $\tilde{x}$ yields the first order condition

$$\mu_2\left[\varphi'(\tilde{x}) + 1\right] - \mu_3\varphi''(\tilde{x}) = [1 - F_2(p_B - \tilde{x})][p_B - \tilde{x} - \tilde{x}\varphi'(\tilde{x})]f_1(\tilde{x}) - \lambda(\tilde{x})\varphi''(\tilde{x}) - p_B\,[1 - F_2(p_B - \tilde{x})]\, f_1(\tilde{x})$$

Using complementary slackness this yields

$$\left[\varphi'(\tilde{x}) + 1\right]\{\mu_2 + \tilde{x}[1 - F_2(p_B - \tilde{x})]f_1(\tilde{x})\} = \mu_3\varphi''(\tilde{x}) \tag{43}$$

Under the assumption that $\varphi'(\tilde{x}) > -1$ so that $\mu_3 = 0$ we have

$$\mu_2 + \tilde{x} f_1(\tilde{x})[1 - F_2(p_B - \tilde{x})] = 0 \tag{44}$$

Combining with (41) completes the derivation of (23). ■

Proof of Claim 2. As changes in the price of the lottery only affect those consumers who buy, or are indifferent to buying, the lottery, $\frac{\partial}{\partial \delta}\Pi(p_1, p_2, p_B, \delta)$ is given by (48). After algebraic manipulation and the collection of terms we have:

$$
\begin{aligned}
& 4\frac{\partial}{\partial \delta}\Pi(p_1, p_2, p_B, \delta) - \{[\text{LHS } (33)] + [\text{LHS } (31)]\} \\
= & \left\{ \begin{array}{c} [p_B - p_2 - \delta][F_2(p_2+\delta) - F_2(p_2)] f_1(p_B - p_2 - \delta) \\ -[p_B - p_2 + \delta][F_2(p_2) - F_2(p_2 - \delta)] f_1(p_B - p_2 + \delta) \end{array} \right\} \qquad (45) \\
& - \left\{ \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z) \left[\begin{array}{c} F_2(p_2) - F_2\left(p_2 - \frac{1}{2}[z - (p_B - p_2 - \delta)]\right) \\ +F_2(p_B - z) - F_2\left(p_2 - \frac{1}{2}[z - (p_B - p_2 - \delta)]\right) \end{array} \right] dz \right\} \qquad (46) \\
& + \left\{ \begin{array}{c} [p_B + p_2 - \delta] \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_2\left(p_2 - \frac{1}{2}[z - (p_B - p_2 - \delta)]\right) f_1(z)\, dz \\ -p_B \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z) f_2(p_B - z)\, dz - p_2 \int_{p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_1(z) f_2(p_2)\, dz \end{array} \right\} \qquad (47)
\end{aligned}
$$

We wish to show each of the three braces is $\mathcal{O}(\delta^2)$. Take (47), this vanishes at $\delta = 0$. The result follows if $\lim_{\delta \to 0}(47)/\delta = 0$. Using L'hôpital we differentiate wrt δ and evaluate at $\delta = 0$ to have

$$
\left[\frac{\partial}{\partial \delta}(47)\right]_{\delta = 0} = 2 \cdot \left[\begin{array}{c} [p_B + p_2] f_2(p_2) f_1(p_B - p_2) \\ -p_B f_1(p_B - p_2) f_2(p_2) - p_2 f_1(p_B - p_2) f_2(p_2) \end{array} \right] = 0
$$

confirming the result. Analogous calculations follow for (45) and (46) complete the proof. ■

Claim 3 *If one of the conditions of Proposition 3 hold, then for small δ the boundary function (34) is more profitable for the seller than optimal deterministic prices.*

Proof. The change in profit from optimal deterministic prices following the introduction of the new boundary $\varphi(\cdot)$ is:

$$
\begin{aligned}
\Delta\Pi = & \frac{1}{2}[p_B + p_2 - \delta] \int_{x_1 = p_B - p_2 - \delta}^{p_B - p_2 + \delta} \left[1 - F_2\left(p_2 - \frac{1}{2}[x_1 - (p_B - p_2 - \delta)]\right)\right] f_1(x_1)\, dx_1 \\
& - p_2 \int_{x_1 = p_B - p_2 - \delta}^{p_B - p_2} [1 - F_2(p_2)] f_1(x_1)\, dx_1 - p_B \int_{x_1 = p_B - p_2}^{p_B - p_2 + \delta} [1 - F_2(p_B - x_1)] f_1(x_1)\, dx_1
\end{aligned}
$$

Differentiating with respect to δ yields

$$
\begin{aligned}
\frac{d}{d\delta}[\Delta\Pi] = & -\frac{1}{2} \int_{x_1 = p_B - p_2 - \delta}^{p_B - p_2 + \delta} \left[1 - F_2\left(p_2 - \frac{1}{2}[x_1 - (p_B - p_2 - \delta)]\right)\right] f_1(x_1)\, dx_1 \qquad (48) \\
& + \frac{1}{4}[p_B + p_2 - \delta] \int_{x_1 = p_B - p_2 - \delta}^{p_B - p_2 + \delta} f_2\left(p_2 - \frac{1}{2}[x_1 - (p_B - p_2 - \delta)]\right) f_1(x_1)\, dx_1 \\
& + \frac{1}{2}[p_B + p_2 - \delta][1 - F_2(p_2 - \delta)] f_1(p_B - p_2 + \delta) \\
& + \frac{1}{2}[p_B + p_2 - \delta][1 - F_2(p_2)] f_1(p_B - p_2 - \delta) \\
& - p_2[1 - F_2(p_2)] f_1(p_B - p_2 - \delta) - p_B[1 - F_2(p_2 - \delta)] f_1(p_B - p_2 + \delta)
\end{aligned}
$$

This expression vanishes at $\delta = 0$. To establish that $\Delta\Pi > 0$ for small positive δ it suffices to

demonstrate that $\frac{d^2}{d\delta^2}\left[\Delta\Pi\right]_{\delta=0} > 0$. We have

$$\frac{d^2}{d\delta^2}\left[\Delta\Pi\right]_{\delta=0} = \left[1 - F_2\left(p_2\right)\right] f_1\left(p_B - p_2\right)\left\{\frac{p_2 f_2\left(p_2\right)}{\left[1 - F_2\left(p_2\right)\right]} - \frac{\left[p_B - p_2\right] f_1'\left(p_B - p_2\right)}{f_1\left(p_B - p_2\right)} - 2\right\}$$

Under the conditions of Proposition 3, specifically equation (24), we have that $\frac{d^2}{d\delta^2}\left[\Delta\Pi\right]_{\delta=0} > 0$ yielding the result. ■

Proof of Proposition 6 (ii). The change in consumer surplus following the introduction of the stochastic bundle, from that achieved with optimal deterministic prices, is

$$\begin{aligned}
\Delta CS \quad = \quad & \int_{x_1=p_B-p_2-\delta}^{p_B-p_2+\delta} \int_{x_2=\varphi(x_1)}^{\infty} \left[x_2 - \varphi\left(x_1\right)\right] f_2\left(x_2\right) f_1\left(x_1\right) dx_2 dx_1 && (49) \\
& - \int_{x_1=p_B-p_2-\delta}^{p_B-p_2} \int_{x_2=p_2}^{\infty} \left[x_2 - p_2\right] f_2\left(x_2\right) f_1\left(x_1\right) dx_2 dx_1 && (50) \\
& - \int_{x_1=p_B-p_2}^{p_B-p_2+\delta} \int_{x_2=p_B-x_1}^{\infty} \left[x_1 + x_2 - p_B\right] f_2\left(x_2\right) f_1\left(x_1\right) dx_2 dx_1 && (51)
\end{aligned}$$

Line (49) gives the utility ($u\left(x\right) = x_2 - \varphi\left(x_1\right)$) of the consumers in the region where the stochastic bundle is bought. This region is labelled $\left(\frac{1}{2}, 1\right)$ in panel (d) of Figure 6. Lines (50) and (51) subtract off the utility which was achieved from the deterministic products which would have been bought, absent the stochastic bundle. Differentiating ΔCS with respect to δ, and using that $\partial\varphi/\partial\delta = -1/2$ we have

$$\frac{d}{d\delta}\left[\Delta CS\right] = \frac{1}{2}\int_{x_1=p_B-p_2-\delta}^{p_B-p_2+\delta} \int_{x_2=\varphi(x_1)}^{\infty} f_2\left(x_2\right) f_1\left(x_1\right) dx_2 dx_1 > 0$$

Hence consumer surplus is increasing in δ as required. ■

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