*

† ‡ §

Abstract

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§

Abstract

I Introduction

(i)

(ii)

Q

M

Ψ

most

likely

Q M Ψ

(i)

(ii)

(iii)

II Entropy and the Pricing Kernel

M_{t+1}

P_{it} i

X_{it+1}

$$P_{it} = \mathbb{E}_t \left[M_{t+1} X_{it+1} \right].$$

$\mathbb{E}_t$

t

$$M_t = m(\theta, t) \times \psi_t$$

$m(\theta, t)$ t

$\theta \in \mathbb{R}^k$ ψ_t

$m(\theta, t)$ $m(\theta, t) =$

$m(\Delta c_t, \theta)$ $\Delta c_t \equiv \log \frac{C_t}{C_{t-1}}$ C_t t

$$\mathbf{0} = \mathbb{E}\left[m\left(\theta,t\right)\psi_t \mathbf{R}_t^e\right] \equiv \int m\left(\theta,t\right)\psi_t \mathbf{R}_t^e dP$$

$\mathbb{E}$ $\mathbf{R}_t^e \in \mathbb{R}^N$

P

$$\mathbf{0} = \int m\left(\theta,t\right)\frac{\psi_t}{\bar{\psi}}\mathbf{R}_t^e\, dP = \int m\left(\theta,t\right)\mathbf{R}_t^e\, d\Psi \equiv \mathbb{E}^{\Psi}\left[m\left(\theta,t\right)\mathbf{R}_t^e\right]$$

$\bar{x} \equiv \mathbb{E}\left[x_t\right]$ $\frac{\psi_t}{\bar{\psi}} = \frac{d\Psi}{dP}$ Ψ

P

Ψ P

θ Ψ

$$\hat{\Psi} \equiv \arg\min_{\Psi} D\left(\Psi||P\right) \equiv \arg\min_{\Psi} \int \frac{d\Psi}{dP}\ln\frac{d\Psi}{dP}dP \qquad \mathbf{0} = \int m\left(\Delta c_t,\theta\right)\mathbf{R}_t^e d\Psi.$$

Ψ

$D\left(\Psi||P\right)$

Ψ P

$m\left(\theta,t\right)$ ψ_t

Q

$$\hat{Q} \equiv \arg\min_{Q} D\left(Q||P\right) \equiv \arg\min_{\Psi} \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP \qquad \mathbf{0} = \int \mathbf{R}_t^e dQ \equiv \mathbb{E}^Q \left[\mathbf{R}_t^e\right]$$

Q P

Ψ Q

$$\begin{aligned} \hat{\Psi} &\equiv \arg\min_{\Psi} D\left(P||\Psi\right) \equiv \arg\min_{\Psi} \int \ln \frac{dP}{d\Psi} dP \qquad \mathbf{0} = \int m\left(\theta, t\right) \mathbf{R}_t^e d\Psi \\ \hat{Q} &\equiv \arg\min_{Q} D\left(P||Q\right) \equiv \arg\min_{Q} \int \ln \frac{dP}{dQ} dP \qquad \mathbf{0} = \int \mathbf{R}_t^e dQ \end{aligned}$$

$\{\psi_t\}_{t=1}^T$

Ψ Q

$\{\psi_t\}_{t=1}^T$

$$\hat{\psi}_t = \frac{e^{\lambda(\theta)' m(\theta,t) \mathbf{R}_t^e}}{\sum_{t=1}^{T} e^{\lambda(\theta)' m(\theta,t) \mathbf{R}_t^e}} \qquad \forall t$$

$\lambda(\theta) \in \mathbb{R}^N$

$$\lambda(\theta) \equiv \arg\min_{\lambda} \frac{1}{T} \sum_{t=1}^{T} e^{\lambda' m(\theta,t) \mathbf{R}_t^e},$$

ψ_t

$$\hat{\psi}_t = \frac{1}{T(1+\lambda(\theta)'m\,(\theta,t)\,\mathbf{R}_t^e)} \quad \forall t$$

$\lambda(\theta) \in \mathbb{R}^N$

$$\lambda(\theta) \equiv \arg\min_{\lambda} - \sum_{t=1}^{T} \log(1+\lambda' m\,(\theta,t)\,\mathbf{R}_t^e),$$

$\{\psi\}_{t=1}^T$ λ

$\lambda\,(\theta)$

$\{\psi_t\}_{t=1}^T$

ψ_t

minimum

amount of information

ψ_t

$\mathbf{Z}_{t-1}$

$\mathbf{Z}_{t-1}$

ψ_t

most likely

$\{\psi_t\}_{t=1}^{T}$

Δ^{T-1}

$$\Delta^{T-1} \equiv \left\{(\psi_1 \quad \psi_2 \qquad \psi_T) : \psi_t \geqslant 0 \quad \sum_{t=1}^{T} \psi_t = 1\right\}$$

$$\left\{\hat{\psi}_t\right\}_{t=1}^{T} \equiv \arg\max \frac{1}{T}\sum_{t=1}^{T}\ln\psi_t \qquad \left\{\hat{\psi}_t\right\}_{t=1}^{T} \in \Delta^{T-1},\ \sum_{t=1}^{T} m(\theta,t)\,\mathbf{R}_t^e\psi_t = \mathbf{0}$$

empirical

likelihood

ψ_t

$\{\psi_t\}_{t=1}^{T}$ $N >> 0$

$t = 1,...,T$

$1/N$ N

$\left\{\tilde{\psi}\right\}_{t=1}^{T}$

$$\tilde{\psi}_t \equiv \frac{n_t}{N}$$

n_t $1/N$

$t.$ $\left\{\tilde{\psi}\right\}_{t=1}^{T}$

$\sum_{t=1}^{T} m(\theta,t)\,\mathbf{R}_t^e\tilde{\psi}_t = 0$ $\{\psi_t\}_{t=1}^{T},$

ψ_t $\left\{\tilde{\psi}_t\right\}_{t=1}^{T}$

$\tilde{\psi}_t$

$$\left\{\tilde{\psi}_t\right\}_{t=1}^{T}$$

$$L\left(\left\{\tilde{\psi}_t\right\}_{t=1}^{T}\right) = \frac{N!}{n_1!n_2!...n_T!}\times T^{-N} = \frac{N!}{N\tilde{\psi}_1!N\tilde{\psi}_2!...N\tilde{\psi}_T!}\times T^{-N}$$

$$\left\{\tilde{\psi}_t\right\}_{t=1}^{T}$$

$$\ln L\left(\left\{\tilde{\psi}_t\right\}_{t=1}^{T}\right) \propto \frac{1}{N}\left(\ln N! - \sum_{t=1}^{T} \ln\left(N\tilde{\psi}_t!\right)\right).$$

N $N \to \infty$

$$\lim_{N\to\infty} \ln L\left(\left\{\tilde{\psi}\right\}_{t=1}^{T}\right) = -\sum_{t=1}^{T} \tilde{\psi}_t \ln \tilde{\psi}_t$$

ψ_t

$$\left\{\hat{\psi}_t\right\}_{t=1}^{T} \equiv \arg\max - \sum_{t=1}^{T} \tilde{\psi}_t \ln \tilde{\psi}_t, \qquad \left\{\widetilde{\psi}_t\right\}_{t=1}^{T} \in \Delta^T, \;\; \sum_{t=1}^{T} m\left(\theta, t\right) \mathbf{R}_t^e \widetilde{\psi}_t = \mathbf{0}.$$

ψ_t

principle of maximum entropy

$$\lim_{N\tilde{\psi}_t\to\infty} \frac{N\tilde{\psi}_t!}{\sqrt{2\pi N\tilde{\psi}_t}\left(\frac{N\tilde{\psi}_t}{e}\right)^{N\tilde{\psi}_t}} = 1$$

II.1 Entropy Bounds

ψ

M

$$\mathbf{0} = \mathbb{E}\left[\mathbf{R}_t^e M_t\right] \equiv \int \mathbf{R}_t^e M_t dP.$$

HJ

Definition 1 (Canonical HJ-bound) *for each $\mathbb{E}\left[M_t\right] = \bar{M}$, the Hansen and Jagannathan (1991) minimum variance SDF is*

$$M_t^*\left(\bar{M}\right) \equiv \underset{\left\{M_t\left(\bar{M}\right)\right\}_{t=1}^{T}}{\arg\min} \sqrt{Var\left(M_t\left(\bar{M}\right)\right)} \text{ s.t. } \mathbf{0} = \mathbb{E}\left[\mathbf{R}_t^e M_t\left(\bar{M}\right)\right]$$

and any candidate stochastic discount factor M_t must satisfy $Var\left(M_t\right) \geq Var\left(M_t^*\left(\bar{M}\right)\right)$.

HJ

$M_t^*\left(\bar{M}\right)$

Q

Definition 2 (Q-bounds) *We define the following probability bounds for any candi-*

$$M_t^*\left(\bar{M}\right) = \bar{M} + \left(\mathbf{R}_t^e - \mathbb{E}\left[\mathbf{R}_t^e\right]\right)' \beta_{\bar{M}}$$

$\beta_{\bar{M}} = Cov\left(\mathbf{R}_t^e\right)^{-1}\left(-\bar{M}\mathbb{E}\left[\mathbf{R}_t^e\right]\right)$

HJ

date stochastic discount factor M_t.

1. *Q1-bound:*
$$D\left(P||\frac{M_t}{\bar{M}}\right) \equiv \int -\ln\frac{M_t}{\bar{M}}dP \geqslant D\left(P||Q^*\right)$$
where Q^* solves Equation (7).

2. *Q2-bound (Stutzer (1995)):*
$$D\left(\frac{M_t}{\bar{M}}||P\right) \equiv \int \frac{M_t}{\bar{M}}\ln\frac{M_t}{\bar{M}}dP \geqslant D\left(Q^*||P\right)$$
where Q^* solves Equation (5).

HJ

HJ Q

Q

Remark 1 *(HJ-bounds as approximated Q-bounds). Let p and q denote the densities of the state x associated, respectively, with the physical, P, and the risk neutral, Q, probability measures. Assuming that there exists a $\mu_p < \infty$ and a $\mu_q < \infty$ such that:*

1. *(Existence of maxima)*

$$\left.\frac{\partial \ln p}{\partial x}\right|_{x=\mu_p} = 0, \quad \left.\frac{\partial \ln q}{\partial x}\right|_{x=\mu_q} = 0;$$

2. *(Finite second moments)*

$$-\left[\left.\frac{\partial^2 \ln p}{\partial x^2}\right|_{x=\mu_p}\right]^{-1} \equiv \sigma_p^2 < \infty, \ -\left[\left.\frac{\partial^2 \ln q}{\partial x^2}\right|_{x=\mu_q}\right]^{-1} \equiv \sigma_q^2 < \infty.$$

We have that, in the limit of the small time interval, a second order approximation of the Q-bounds yields

$$\begin{aligned} D\left(P||\frac{M_t}{\bar{M}}\right) &\propto Var\left(M_t\right), \\ D\left(\frac{M_t}{\bar{M}}||P\right) &\propto Var\left(M_t\right). \end{aligned}$$

Proof.

$a)$

$b)$

$c)$ *diffusion invariance principle*

■

HJ

Q

HJ Q

$Q2$

$Q2$

$$M_t = m(\theta, t) \times \psi_t$$

$m(\theta, t)$

θ ψ_t

Definition 3 (M-bounds) *For any candidate stochastic discount factors of the form in Equation (15), given any choice of the θ parameters of $m(\theta, t)$, we define the following bounds:*

1. *$M1$-bound:*

$$D\left(P||\frac{M_t}{\bar{M}}\right) \equiv \int -\ln\frac{M_t}{\bar{M}}dP \geqslant D\left(P||\frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}}\right) \equiv \int -\ln\frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}}dP$$

where ψ_t^* solves Equation (6) and $\overline{m(\theta,t)\,\psi_t^*} \equiv \mathbb{E}\left[m(\theta,t)\,\psi_t^*\right]$.

2. *M2-bound:*

$$D\left(\frac{M_t}{\bar{M}}||P\right) \equiv \int \frac{M_t}{\bar{M}} \ln \frac{M_t}{\bar{M}} dP \geqslant D\left(\frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}}||P\right)$$
$$\equiv \int \frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}} \ln \frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}} dP$$

where ψ_t^* solves Equation (4).

Q

$$D\left(P||\frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}}\right) \geq D(P||Q^*) \qquad D\left(\frac{m(\theta,t)\,\psi_t^*}{\overline{m(\theta,t)\,\psi_t^*}}||P\right) \geq D(Q^*||P)$$

$m(\theta,t)$

ψ_t

$m(\theta,t)$

ψ_t M_t

Definition 4 (Ψ-bounds) *For any candidate stochastic discount factors of the form in Equation (15), given any choice of the θ parameters of $m(\theta,t)$, two lower bounds for the relative entropy of ψ_t are defined as follows:*

1. *Ψ1-bound:*

$$D\left(P||\frac{\psi_t}{\bar{\psi}}\right) \equiv -\int \ln \frac{\psi_t}{\bar{\psi}} dP \geqslant D\left(P||\frac{\psi_t^*}{\bar{\psi}}\right)$$

where ψ_t^* solves Equation (6);

2. Ψ *2-bound*

$$D\left(\frac{\psi_t}{\bar{\psi}}||P\right) \equiv \int \frac{\psi_t}{\bar{\psi}} \ln \frac{\psi_t}{\bar{\psi}} dP \geqslant D\left(\frac{\psi_t^*}{\bar{\psi}}||P\right)$$

where ψ_t^* *solves Equation (4).*

Ψ

$D\left(\frac{\psi_t^*}{\bar{\psi}}||P\right)$ $D\left(\frac{m(\theta,t)}{\overline{m(\theta,t)}}||P\right)$ $D\left(Q^*||P\right)$

$D\left(\frac{\psi_t^*}{\bar{\psi}}||P\right)$ $D\left(Q^*||P\right)$ $D\left(\frac{m(\theta,t)}{\overline{m(\theta,t)}}||P\right)$

ψ_t

$M_t = m\left(\theta, t\right)$

ψ_t Ψ

$D\left(\frac{\psi_t^*}{\bar{\psi}}||P\right) = D\left(P||\frac{\psi_t^*}{\bar{\psi}}\right) = 0$ HJ Q M

Q HJ

$m\left(\theta, t\right)$

θ Q

HJ

ψ_t

Definition 5 (Volatility bound for ψ_t) *For each $\mathbb{E}[\psi_t] = \bar{\psi}$, the minimum variance ψ_t is*

$$\psi_t^*(\bar{\psi}) \equiv \underset{\{\psi(\bar{\psi})\}_{t=1}^T}{\arg\min} \sqrt{Var(\psi_t(\bar{\psi}))} \; s.t. \; \mathbf{0} = \mathbb{E}\left[\mathbf{R}_t^e m(\theta,t)\,\psi_t(\bar{\psi})\right]$$

and any candidate SDF must satisfy the condition $Var(\psi_t) \geq Var(\psi_t^*(\bar{\psi}))$.

θ

$$\psi_t^*(\bar{\psi}) = \bar{\psi} + \left(\mathbf{R}_t^e m(\theta,t) - \mathbb{E}[\mathbf{R}_t^e m(\theta,t)]\right)' \beta_{\bar{\psi}}$$

$$\beta_{\bar{\psi}} = Var(\mathbf{R}_t^e m(\theta,t))^{-1} \left(-\bar{\psi}\mathbb{E}[\mathbf{R}_t^e m(\theta,t)]\right)$$

$$\sigma_{\psi^*} = \bar{\psi}\sqrt{\mathbb{E}[\mathbf{R}_t^e m(\theta,t)]' Var(\mathbf{R}_t^e m(\theta,t))^{-1} \mathbb{E}[\mathbf{R}_t^e m(\theta,t)]}$$

$$\sigma_{\psi^*} \equiv \sqrt{Var(\psi_t^*(\bar{\psi}))}$$

Ψ

ψ_t M_t

Ψ

Ψ

$M_t =$

$m(\theta,t)\,\psi_t$ M_t

$M_t = m(\theta,t)$

$\psi_t = 1$ t θ_0

$$Var\left(M_t\left(\theta_0\right)\right) \equiv Var\left(m\left(\theta_0, t\right)\right) \geq Var\left(M_t^*\left(\bar{M}\right)\right)$$

$Var\left(M_t^*\left(\bar{M}\right)\right)$

θ_0 HJ

θ_0

ψ_t

ψ_t

$$\mathbb{E}\left[\mathbf{R}_t^e m\left(\theta_0, t\right)\right] \equiv \mathbb{E}\left[\mathbf{R}_t^e M_t\left(\theta_0\right)\right] = \mathbf{0}$$

III Data Description

$1929 - 2009$

$1947 : Q1 - 2009 : Q4$

Q M Ψ

$i)$

$ii)$ $iii)$ $iv)$

$v)$ $vi)$

t

excluding

IV An Illustrative Example: the Consumption CAPM with Power Utility

$$M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

δ γ

$\frac{C_{t+1}}{C_t}$

$i)$

$ii)$

$$M_{t+1}^{S} = \delta^{1+S}\left(\frac{C_{t+1+S}}{C_t}\right)^{-\gamma} R_{t+1,t+1+S}^{f}$$

S

$R_{t+1,t+1+S}^{f}$ $t+1$ $t+1+S$

$S = 0$

$\theta = \gamma$ $M_t = m(\theta, t) = \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$

$\psi_t = \delta$ t $\theta =$

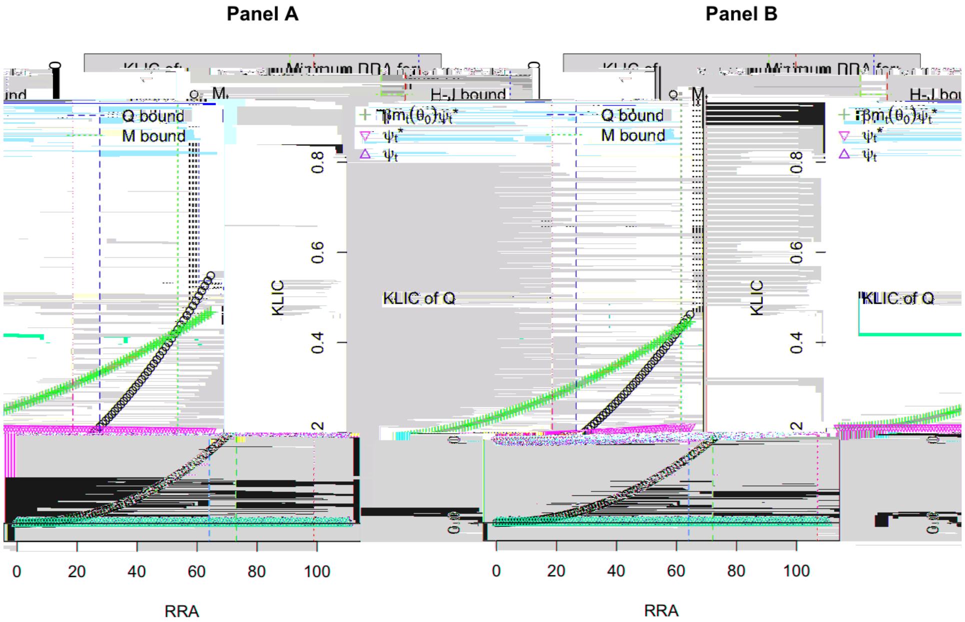
γ $m(\theta, t) = \left(\frac{C_{t+1+S}}{C_t}\right)^{-\gamma}$ $\psi_t = \delta^{1+S} R_{t+1,t+1+S}^{f}$

$1947:1 - 2009:4$ HJ

$\gamma = 10$ $S = 11$

1 A

γ HJ $Q1$ $M1$ $\Psi1$



ψ_t

γ

$+$

HJ

$\gamma \geqslant 64$

$Q1$ $\gamma \geqslant 72$

γ $M1$

γ

$\gamma = 107$ $\Psi 1$

γ

γ $\Psi 1$ γ

B $Q2$ $M2$ $\Psi 2$

$Q2$ $M2$ γ 73

99 $\Psi 2$ γ

Q HJ

M Q Ψ

M

2 1

A HJ $Q1$ $M1$

$\gamma \geqslant 22$ 23 46

1 A

$\Psi 1$ γ B $Q2$

$M2$ $\gamma \geqslant 24$ 47 $\Psi 2$

γ

10 $\hat{\gamma} = 1.5$

60%

Ψ

3 A

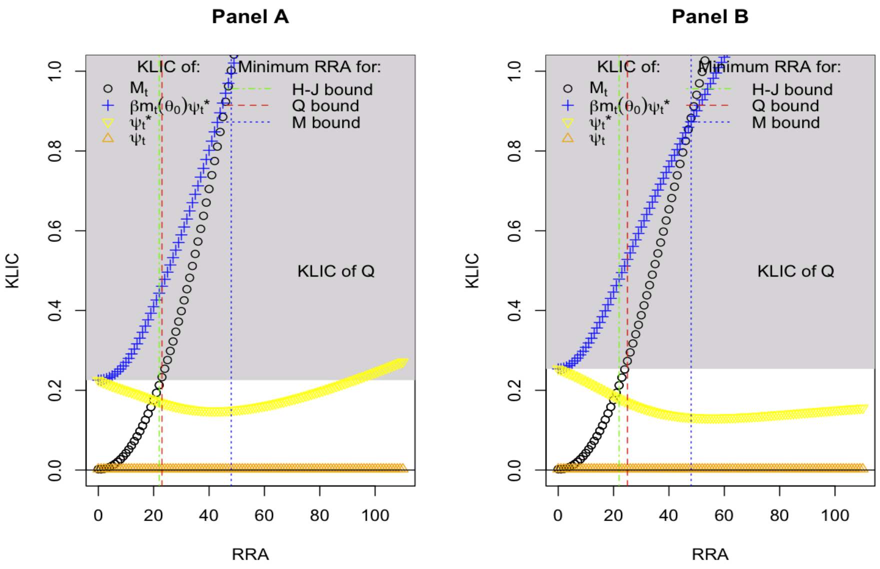
$$\gamma = 10$$

$$m(\theta, t) = \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}$$

ψ_t^*

$$M_t = \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \psi_t^*$$

0.98ron7n96365558299()275(a)11()rorano23902240()373()8()11(m)35(6)10())8()5n



21.7% 2.5

0.37 0.06

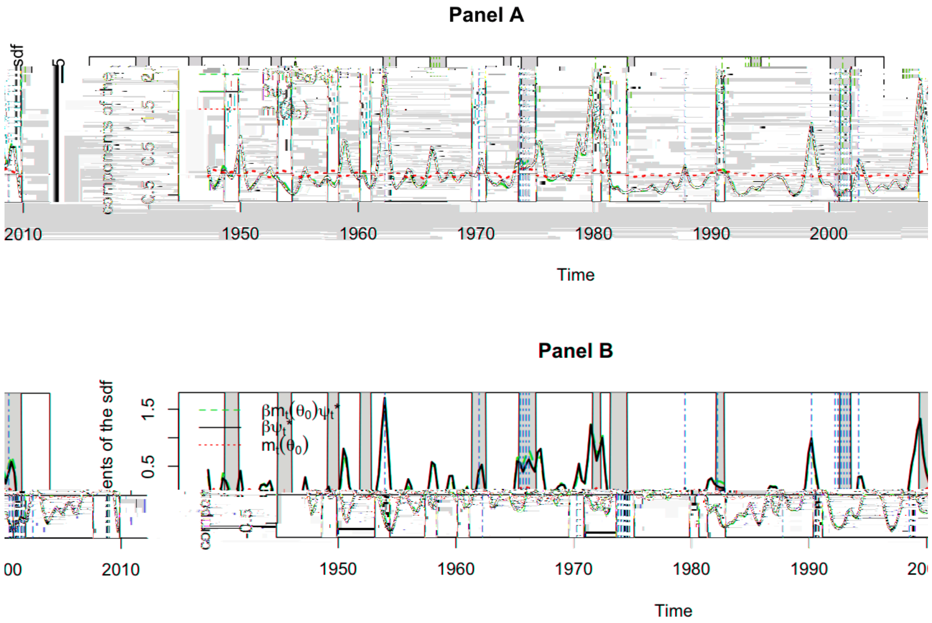
R^2 54.1% $\gamma = 10$

5.2%

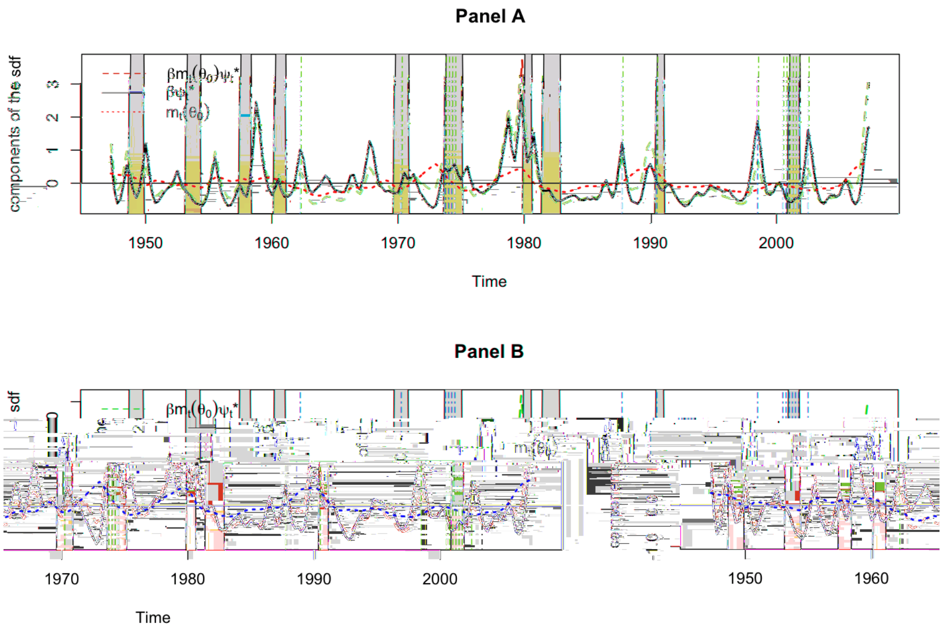
0.92

B

$\gamma = 10$



γ



γ

V Application to More General Models of Dynamic Economies

M_t

C_t

ψ_t

$$M_t = \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \psi_t$$

γ HJ Q M

Ψ

γ

Q HJ M

HJ Q

V.1 The Models Considered

V.1.1 External Habit Formation Model: Campbell and Cochrane (1999)

$$M_t = \delta \left(\frac{C_t}{C_{t-1}} \frac{S_t}{S_{t-1}} \right)^{-\gamma}$$

δ γ

$S_t = \frac{C_t - X_t}{C_t}$

$$\ln M_t = \ln \delta - \gamma \Delta c_t - \gamma \Delta s_t$$

$\ln(\psi_t)$

$$\ln \psi_t = \ln \delta - \gamma \Delta s_t$$

ψ S

ψ

$i.i.d.$

$$\Delta c_t = g + \upsilon_t \qquad \upsilon_t \sim i.i.d.N\left(0, \sigma^2\right)$$

$AR(1)$

$$s_t = (1 - \phi)\, \overline{s} + \phi s_{t-1} + \lambda\left(s_{t-1}\right) \upsilon_t$$

$\overline{s}$

$$\begin{aligned} \lambda\left(s_t\right) &= \begin{cases} \frac{1}{\overline{S}} \sqrt{1 - 2\left(s_t - \overline{s}\right)} & s_t \leq s_{max} \\ 0, & s_t > s_{max} \end{cases} \\ s_{max} &= \overline{s} + \frac{1}{2}\left(1 - \overline{S}^2\right) \quad \overline{S} = \sigma \sqrt{\frac{\gamma}{1 - \phi}}. \end{aligned}$$

γ

$(\delta \quad \phi)$ g σ

$\widehat{\upsilon}_t = \Delta c_t - g$

V.1.2 External Habit Formation Model: Menzly, Santos, and Veronesi (2004)

i.i.d.

$$dc_t = \mu_c dt + \sigma_c dB_t$$

μ_c $\sigma_c > 0$ B_t

inverse

$Y_t \equiv \frac{1}{S_t}$

$$dY_t = k\left(\overline{Y} - Y_t\right) dt - \alpha\left(Y_t - \lambda\right)\left[dc_t - E\left(dc_t\right)\right]$$

$\overline{Y}$ k

γ

$\left(\delta \quad k \quad \overline{Y} \quad \alpha \quad \lambda\right)$

μ_c σ_c $\widehat{dB_t} = \frac{[dc_t - E(dc_t)]}{\sigma_c}$

V.1.3 Long-Run Risks Model: Bansal and Yaron (2004)

$$\ln M_{t+1} = \theta \log \delta - \frac{\theta}{\rho} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}$$

$r_{c,t+1}$

δ ρ

$\theta = \frac{1-\gamma}{1-1/\rho}$ γ

Δc_{t+1} Δd_{t+1}

x_t σ_t

$$\begin{aligned} x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t z_{x,t+1} \\ \sigma^2_{t+1} &= (1-\upsilon)\sigma^2 + \upsilon \sigma^2_t + \sigma_w z_{\sigma,t+1} \\ \Delta c_{t+1} &= \mu_c + x_t + \sigma_t z_{c,t+1} \\ \Delta d_{t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t z_{d,t+1} \end{aligned}$$

$z_{x,t+1}$ $z_{\sigma,t+1}$ $z_{c,t+1}$ $z_{d,t+1}$ $i.i.d.\ N(0,1)$

z_t

$z_{m,t}$

x_t σ_t^2

$$
\begin{aligned}
z_t &= A_0 + A_1 x_t + A_2 \sigma_t^2 \\
z_{m,t} &= A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma_t^2 \\
r_{f,t} &= A_{0,f} + A_{1,f} x_t + A_{2,f} \sigma_t^2
\end{aligned}
$$

$z_{m,t}$ $r_{f,t}$ x_t

σ_t^2

x_t σ_t^2 $z_{m,t}$ $r_{f,t}$

$r_{c,t+1} = \kappa_0 + \kappa_1 z_{m,t+1} - z_{m,t} +$

Δc_{t+1} z_t

$$
\begin{aligned}
\ln M_{t+1} &= \left(\theta \log \delta + (\theta - 1)\left[\kappa_0 + (\kappa_1 - 1) A_0\right]\right) + \left(-\frac{\theta}{\psi} + \theta - 1\right) \Delta c_{t+1} \\
&\quad + (\theta - 1)\kappa_1 A_1 x_{t+1} + (\theta - 1)\kappa_1 A_2 \sigma_{t+1}^2 - (\theta - 1) A_1 x_t - (\theta - 1) A_2 \sigma_t^2
\end{aligned}
$$

x_t σ_t^2 x_t σ_t^2

$z_{m,t}$ $r_{f,t}$

$$
\ln M_{t+1} = c_1 - \gamma \Delta c_{t+1} + c_3 \left(r_{f,t+1} - \frac{1}{\kappa_1} r_{f,t}\right) + c_4 \left(z_{m,t+1} - \frac{1}{\kappa_1} z_{m,t}\right)
$$

$$c = (c_1, c_3, c_4)'$$

$$\begin{aligned}
\ln M^q_{t+1} &= \sum_{i=1}^{3} \ln M_{t+i} \\
&= 3c_1 - \gamma \Delta c^q_{t+1} + c_3 \left(r_{f,t,t+3} - \frac{1}{\kappa_1} r_{f,t,t+2} \right) \\
&\quad + c_4 \left[z_{m,t+1} + z_{m,t+2} + z_{m,t+3} - \frac{1}{\kappa_1} \left(z_{m,t} + z_{m,t+1} + z_{m,t+2} \right) \right]
\end{aligned}$$

$\ln \psi_t$

$$\begin{aligned}
\ln \psi_t &= 3c_1 + c_3 \left(r_{f,t,t+3} - \frac{1}{\kappa_1} r_{f,t,t+2} \right) \\
&\quad + c_4 \left[z_{m,t+1} + z_{m,t+2} + z_{m,t+3} - \frac{1}{\kappa_1} \left(z_{m,t} + z_{m,t+1} + z_{m,t+2} \right) \right]
\end{aligned}$$

γ

ψ

V.1.4 Housing: Piazzesi, Schneider, and Tuzel (2007)

$$M_t = \delta \left(\frac{C_t}{C_{t-1}} \right)^{-1/\sigma} \left(\frac{A_t}{A_{t-1}} \right)^{\frac{(\rho-\sigma)}{\sigma(\rho-1)}}$$

$$A_t = \frac{P_t^c C_t}{P_t^c C_t + P_t^s S_t}$$

A_t P_t^s P_t^c

S_t C_t

σ ρ

$$\ln M_t = \ln \delta - 1/\sigma \Delta c_t + \frac{\rho - \sigma}{\sigma(\rho - 1)} \Delta a_t$$

$\ln \psi_t$

$$\ln \psi_t = \ln \delta + \frac{\rho - \sigma}{\sigma(\rho - 1)} \Delta a_t$$

$\gamma = \frac{1}{\sigma}$ $(\delta \quad \rho)$

V.2 Empirical Results

γ HJ Q M Ψ

A B C D

E F

25 10 10

10 10

HJ $Q1$ $M1$ $\Psi 1$

CC A

γ

HJ $Q1$ $M1$ $\Psi 1$ 1.4

B HJ $Q1$ $M1$ $\Psi 1$

$\gamma = 7.3$ 9.8 9.9 13.9

Q HJ M

Q

$\frac{\gamma}{S_t}$ S_t $\gamma = 2$

$[20, \infty)$ B Q

$\gamma \geqslant 9.8$ $[43.9, \infty)$ M

$\gamma \geqslant 9.9$ $[44.2, \infty)$ Ψ

$\gamma \geqslant 13.9$ $[52.5, \infty)$

10

10 10 10

C D E F

$Q2$ $M2$ $\Psi 2$

[Table I about here]

MSV

HJ $Q1$ $M1$

$\Psi1$ $\gamma = 11.4$ 11.2 12.4 15.7

27.8

31.7 33.9 53.3

$Q2$ $M2$ $\Psi2$

BY A

γ HJ $Q1$ $M1$ $\Psi1$

3.0

B HJ

$\gamma = 4.0$ $Q1$ $M1$ $\Psi1$

$\gamma = 5.0$

$Q2$ $M2$ $\Psi2$ γ

$A - F$

HJ Q M Ψ

PST

HJ $Q1\,(Q2)$ $M1\,(M2)$

$\Psi1\,(\Psi2)$ $\gamma = 19.2$ $19.2\,(19.4)$ $24.4\,(24.3)$

$16.2\,(16.5)$

64.3 75.0 (74.5) 87.2 (83.8) 70.9 (72.5)

Q HJ

M Q

HJ Q M Ψ

γ

Q HJ M

Q

[Table II about here]

filtered SDF

γ

γ

A B C

D E 25 10

10 10 10

$\{\psi_t^*\}_{t=1}^T$

$\{\psi_t^m\}_{t=1}^T$

$\{M_t^* = m_t\psi_t^*\}_{t=1}^T$ $m_t = (C_t/C_{t-1})^{-\gamma}$

$\{M_t^m = m_t\psi_t^m\}_{t=1}^T$

[Table III about here]

$\gamma = 2$ A 1

ψ^*

ψ 0.02 ψ^* 2

0.05

ψ^* 0.06

0.08 $B - E$

γ 1 2

ψ^*

0.00

0.20

0.02

0.24 ψ^*

$$\begin{aligned}
\ln \psi_{t+1} &= c_1 + c_3 \left(r_{f,t+1} - \frac{1}{\kappa_1} r_{f,t} \right) + c_4 \left(z_{m,t+1} - \frac{1}{\kappa_1} z_{m,t} \right) \\
\ln M_{t+1} &= -\gamma \Delta c_{t+1} + \ln \psi_{t+1}
\end{aligned}$$

$c = (c_1, c_3, c_4)'$

c

c

γ 10 3

A 1

0.10 (0.11) c ψ^*

1 2

c

0.10 (0.11) 0.27 (0.29) 2

0.11 (0.12)

0.30 (0.31)

$B - E$

0.12 (0.11) 0.38 (0.38)

0.44 (0.44)

0.80 (0.82)

γ 16 1

$-0.22\,(-0.20)$

0.13 (0.12) ψ^*

$-0.01\,(-0.02)$

$0.19\,(0.17)$

[Table IV about here]

R^2

R^2

3 R^2

4

R^2

R^2

most likely

1

$\{M_t^m\}_{t=1}^T$

M^m

2 3

[Table V about here]

1

0.18

A 1 2

0.54 0.59 ψ^*

[Table VI about here]

ψ^*

VI Conclusion

$a)$

$b)$

$c)$

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A Appendix

A.1 Additional Proofs and Derivations

Proof of Remark 1. p q

P Q

$Q1$ $Q2$

$$D\left(P||\frac{M_t}{\bar{M}}\right) \equiv \int \ln \frac{dP}{dQ} dP = \int p \ln \frac{p}{q} dx$$

$$D\left(\frac{M_t}{\bar{M}}||P\right) \equiv \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP = \int \ln \frac{dQ}{dP} dQ = \int q \ln \frac{q}{p} dx.$$

q p μ_p μ_q

$$\left.\frac{\partial \ln p}{\partial x}\right|_{x=\mu_p} = 0 \quad \left.\frac{\partial \ln q}{\partial x}\right|_{x=\mu_q} = 0$$

$$-\left[\left.\frac{\partial^2 \ln p}{\partial x^2}\right|_{x=\mu_p}\right]^{-1} \equiv \sigma_p^2 < \infty \quad -\left[\left.\frac{\partial^2 \ln q}{\partial x^2}\right|_{x=\mu_q}\right]^{-1} \equiv \sigma_q^2 < \infty$$

$$\begin{aligned}
\ln q &\propto \frac{1}{2}\left[\frac{\partial^2 \ln q}{\partial x^2}\right]\left(x-\mu_q\right)^2 \equiv -\frac{1}{2}\frac{\left(x-\mu_q\right)^2}{\sigma_q^2} \\
\ln p &\propto \frac{1}{2}\left[\frac{\partial^2 \ln p}{\partial x^2}\right]\left(x-\mu_p\right)^2 \equiv -\frac{1}{2}\frac{\left(x-\mu_p\right)^2}{\sigma_p^2}
\end{aligned}$$

q p

$$q \approx N\left(\mu_q; \sigma_q^2\right)$$
$$p \approx N\left(\mu_p; \sigma_p^2\right)$$

$\sigma_q^2 = \sigma_p^2 = \sigma^2.$

$$\begin{aligned}
\int p \ln \frac{p}{q} dx &\approx \int \left[-\frac{1}{2} \frac{\left(x-\mu_p\right)^2}{\sigma^2} + \frac{1}{2} \frac{\left(x-\mu_q\right)^2}{\sigma^2} \right] p dx \\
&= \frac{1}{2\sigma^2} \left[-\sigma^2 + \int \left(x-\mu_q\right)^2 p dx \right] \\
&= \frac{1}{2\sigma^2} \left\{ -\sigma^2 + \int \left[\left(x-\mu_p\right)^2 + \left(\mu_p-\mu_q\right)^2 + 2\left(\mu_p-\mu_q\right)\left(x-\mu_p\right) \right] p dx \right\} \\
&\approx \frac{1}{2\sigma^2} \left(\mu_p-\mu_q\right)^2 = \frac{1}{2\sigma^2} \sigma^2 \sigma_\xi^2 = \frac{1}{2} \sigma_\xi^2
\end{aligned}$$

ξ $\xi \equiv \frac{dQ}{dP},$

$$\int q \ln \frac{q}{p} dx = \frac{1}{2} \sigma_\xi^2.$$

Q P $M_t \propto \xi_t$

$Var\left(M_t\right) \propto \sigma_\xi^2$

$$D\left(P||\frac{M_t}{\bar{M}}\right) \propto Var\left(M_t\right),$$
$$D\left(\frac{M_t}{\bar{M}}||P\right) \propto Var\left(M_t\right).$$

■

Table I: Bounds for RRA, Quarterly Data 1947:2-2009:4

HJ	Q

Table II: Bounds for RRA, Annual Data 1930-2009

	HJ	$Q1/Q2$	$M1/M2$	$\Psi1/\Psi2$
Panel A: Market Portfolio				
CC	0.1	0.1/0.1	0.1/0.1	0.1/0.1
MSV	0.1	0.1/0.1	0.1/0.1	0.1/0.1
BY	4.0	4.0/4.0	4.0/4.0	4.0/4.0
PST	8.6	9.9/9.5	14.9/15.0	6.8/6.8
Panel B: FF 6 Portfolios				
CC	0.3	1.0/0.6	1.0/0.6	1.2/0.7
MSV	0.2	0.2/0.2	0.2/0.2	0.2/0.2
BY	5.0	5.0/5.0	5.0/5.0	5.0/5.0
PST	12.4	16.8/15.2	20.5/17.7	13.2/12.0
Panel C: 10 Size Portfolios				
CC	0.1	0.5/0.3	0.5/0.3	0.5/0.3
MSV	0.2	0.2/0.2	0.2/0.2	0.2/0.2
BY	4.0	4.0/4.0	4.0/4.0	4.0/4.0
PST	10.4	13.6/12.2	15.7/13.7	11.5/10.3
PanelD:10BMPortfolios				
CC	0.2	0.8/0.5	0.8/0.5	0.9/0.5
MSV	0.2	0.2/0.2	0.2/0.2	0.2/0.2
BY	4.0	5.0/5.0	5.0/5.0	5.0/5.0
PST	11.2	15.8/13.8	17.7/15.8	12.7/11.2
Panel E: 10 Momentum Portfolios				
CC	0.4	1.4/0.9	1.4/0.9	1.5/1.0
MSV	0.2	0.2/0.2	0.2/0.2	0.2/0.2
BY	5.0	5.0/5.0	5.0/5.0	5.0/5.0
PST	14.3	18.3/16.9	21.1/18.5	13.9/12.9
PanelF:10IndustryPortfolios				
CC	0.4	1.7/1.0	1.7/1.0	2.2/1.2
MSV	0.2	0.2/0.2	0.2/0.2	0.2/0.2
BY	5.0	5.0/5.0	5.0/5.0	5.0/5.0
PST	14.1	19.7/17.4	22.0/18.9	16.3/14.2

Ψ

Table III: Correlation Between Filtered and Model SDFs, 1947:Q2-2009:Q4

	$\rho\left(\ln\psi_t^*, \ln\psi_t^m\right)$	$\rho\left(\ln M_t^*, \ln M_t^m\right)$		R^2
Panel A: Fama-French 25 Portfolios				
CC	0.02/0.06	0.05/0.08	−0.17	0.17
MSV	0.00/0.04	0.02/0.06	−0.64	0.0003
$BY^{rest.}$	0.10 0.11	0.11 0.12	−1.01	0.03
$(unrest.)$	(0.27) (0.29)	(0.30) (0.31)	(−0.48)	(0.02)
PST	−0.04/ − 0.07	0.06/0.01	−1.79	0.05
Panel B: 10 Size-Sorted Portfolios				
CC	0.19/0.17	0.25/0.23	−0.13	0.83
MSV	0.20/0.22	0.24/0.25	0.73	0.88
$BY^{rest.}$	0.38 0.38	0.37 0.38	0.71	0.84
$(unrest.)$	(0.80) (0.82)	(0.85) (0.86)	(0.89)	(0.94)
PST	−0.22/ − 0.20	−0.01/ − 0.02	−0.56	0.53
Panel C: 10 BM-Sorted Portfolios				
CC	0.16/0.15	0.20/0.20	0.27	0.60
MSV	0.14/0.16	0.18/0.19	0.06	0.24
$BY^{rest.}$	0.34 0.33	0.34 0.34	−0.50	0.07
$(unrest.)$	(0.58) (0.65)	(0.65) (0.60)	(−0.03)	(0.02)
PST	−0.08/ − 0.09	0.09/0.07	−2.31	0.27
Panel D: 10 Momentum-Sorted Portfolios				
CC	0.08/0.06	0.12/0.10	−0.64	0.07
MSV	0.06/0.08	0.10/0.12	−0.56	0.15
$BY^{rest.}$	0.12 0.11	0.13 0.12	−1.06	0.60
$(unrest.)$	(0.44) (0.44)	(0.50) (0.48)	(−0.40)	(0.29)
PST	−0.12/ − 0.14	0.07/0.02	−0.75	0.36
Panel E: 10 Industry-Sorted Portfolios				
CC	0.05/0.04	0.11/0.09	−11.1	0.003
MSV	0.06/0.11	0.10/0.12	−5.39	0.08
$BY^{rest.}$	0.23 0.27	0.25 0.28	−1.58	0.01
$(unrest.)$	(0.55) (0.58)	(0.61) (0.62)	(−1.17)	(0.11)
PST	0.13/0.12	0.19/0.17	−0.69	0.01

Table IV: Correlations between Filtered and Model SDFs, 1930-2009

	$\rho\left(\ln\psi_t^*, \ln\psi_t^m\right)$	$\rho\left(\ln M_t^*, \ln M_t^m\right)$		R^2
Panel A: Fama-French 6 Portfolios				
CC	0.15/0.16	0.23/0.22	−0.75	0.03
MSV	−0.04/ − 0.07	0.04/ 0.07	−0.98	0.21
$BY^{rest.}$	0.33 0.36	0.24 0.29	0.35	0.35
$(unrest.)$	(0.37) (0.42)	(0.62) (0.56)	(0.50)	(0.66)
PST	−0.04/ − 0.01	0.01/ 0.05	−0.79	0.21
Panel B: 10 Size-Sorted Portfolios				
CC	−0.03/ − 0.03	0.07/0.06	−3.88	0.17
MSV	0.06/ − 0.01	0.06/ − 0.01	0.08	0.85
$BY^{rest.}$	0.47 0.50	0.36 0.40	0.86	0.96
$(unrest.)$	(0.59) (0.61)	(0.68) (0.68)	(0.80)	(0.95)
PST	0.17/0.13	0.01/ 0.08	0.11	0.91
Panel C: 10 BM-Sorted Portfolios				
CC	0.07/0.03	0.16/0.10	−3.12	0.01
MSV	−0.08/ − 0.06	0.07/ 0.06	−2.44	0.05
$BY^{rest.}$	0.52 0.53	0.41 0.47	0.40	0.47
$(unrest.)$	(0.52) (0.54)	(0.61) (0.53)	(0.73)	(0.82)
PST	0.22/0.34	0.08/0.08	0.18	0.57
Panel D: 10 Momentum-Sorted Portfolios				
CC	0.26/0.27	0.34/0.33	0.50	0.78
MSV	0.09/0.07	0.09/0.08	−1.51	0.40
$BY^{rest.}$	0.41 0.50	0.31 0.41	−0.33	0.31
$(unrest.)$	(0.48) (0.53)	(0.68) (0.66)	(−0.14)	(0.08)
PST	−0.07/ − 0.06	0.03/ 0.06	−0.45	0.01
Panel E: 10 Industry-Sorted Portfolios				
CC	−0.04/ − 0.04	0.03/0.003	−4.61	0.22
MSV	−0.04/ − 0.10	0.04/ 0.10	−4.72	0.24
$BY^{rest.}$	0.26 0.39	0.20 0.34	−1.24	0.17
$(unrest.)$	(0.31) (0.41)	(0.37) (0.37)	(−0.30)	(0.25)
PST	0.12/0.12	0.07/ 0.20	−7.15	0.56

Table V: Correlations with FF3, 1947:Q2-2009:Q4

	$\ln M_t^m$	$\ln M_t^*$	$\ln \psi_t^*$
Panel A: Fama-French 25 Portfolios			
CC	0.18	0.54/0.59	0.54/0.59
MSV	0.21	0.54/0.59	0.54/0.59
$BY^{rest.}$	0.45	0.54 0.58	0.52 0.57
$(unrest.)$	(0.87)	(0.54) (0.58)	(0.53) (0.57)
PST	0.07	0.49/0.52	0.45/0.50
Panel B: 10 Size-Sorted Portfolios			
CC	0.18	0.88/0.89	0.87/0.89
MSV	0.21	0.87/0.89	0.87/0.89
$BY^{rest.}$	0.45	0.89 0.90	0.86 0.88
$(unrest.)$	(0.90)	(0.89) (0.90)	(0.86) (0.88)
PST	0.07	0.81/0.82	0.75/0.76
Panel C: 10 BM-Sorted Portfolios			
CC	0.18	0.83/0.86	0.83/0.86
MSV	0.21	0.83/0.86	0.83/0.86
$BY^{rest.}$	0.45	0.84 0.86	0.81 0.85
$(unrest.)$	(0.91)	(0.84) (0.86)	(0.81) (0.85)
PST	0.07	0.87/0.89	0.84/0.87
Panel D: 10 Momentum-Sorted Portfolios			
CC	0.18	0.52/0.52	0.51/0.51
MSV	0.21	0.52/0.52	0.51/0.51
$BY^{rest.}$	0.45	0.55 0.53	0.50 0.50
$(unrest.)$	(0.92)	(0.55) (0.53)	(0.50) (0.50)
PST	0.07	0.53/0.51	0.43/0.43
Panel E: 10 Industry-Sorted Portfolios			
CC	0.18	0.65/0.69	0.64/0.68
MSV	0.21	0.65/0.69	0.65/0.68
$BY^{rest.}$	0.45	0.66 0.69	0.62 0.65
$(unrest.)$	(0.88)	(0.66) (0.68)	(0.62) (0.65)
PST	0.07	0.53/0.55	0.47/0.51

Table VI: Correlations with FF3, 1930-2009

	$\ln M_t^m$	$\ln M_t^*$	$\ln \psi_t^*$
Panel A: Fama-French 6 Portfolios			
CC	0.19	0.73/0.78	0.72/0.77
MSV	0.12	0.73/0.78	0.72/0.77
$BY^{rest.}$	0.73	0.77 0.77	0.68 0.72
$(unrest.)$	(0.81)	(0.78) (0.78)	(0.68) (0.72)
PST	0.35	0.81/0.76	0.65/0.67
Panel B: 10 Size-Sorted Portfolios			
CC	0.19	0.82/0.85	0.82/0.85
MSV	0.12	0.83/0.86	0.83/0.86
$BY^{rest.}$	0.73	0.77 0.77	0.71 0.73
$(unrest.)$	(0.84)	(0.78) (0.79)	(0.71) (0.74)
PST	0.35	0.75/0.72	0.64/0.66
Panel C: 10 BM-Sorted Portfolios			
CC	0.19	0.71/0.75	0.72/0.74
MSV	0.12	0.72/0.76	0.72/0.75
$BY^{rest.}$	0.73	0.67 0.60	0.59 0.58
$(unrest.)$	(0.83)	(0.67) (0.60)	(0.59) (0.58)
PST	0.35	0.64/0.23	0.50/0.33
Panel D: 10 Momentum-Sorted Portfolios			
CC	0.19	0.55/0.63	0.58/0.61
MSV	0.12	0.55/0.62	0.57/0.61
$BY^{rest.}$	0.73	0.69 0.69	0.51 0.57
$(unrest.)$	(0.85)	(0.73) (0.73)	(0.60) (0.64)
PST	0.35	0.73/0.70	0.50/0.55
Panel E: 10 Industry-Sorted Portfolios			
CC	0.19	0.49/0.53	0.49/0.53
MSV	0.12	0.50/0.54	0.50/0.55
$BY^{rest.}$	0.73	0.42 0.39	0.38 0.42
$(unrest.)$	(0.86)	(0.42) (0.38)	(0.36) (0.40)
PST	0.35	0.41/0.27	0.34/0.37