Limited Liability and Mechanism Design in Procurement*

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Abstract

In the presence of cost uncertainty, limited liability introduces the possibility of default in procurement. If financial soundness is not perfectly observable, then financially weaker contractors are selected with higher probability in any incentive compatible mechanism. Informational rents are associated with the probability of default. By selecting the financially weakest contractor, stronger price competition (auctions) may not only increase the probability of default but also the contractors' expected rents. Thus, weak conditions are sufficient for auctions to be suboptimal. In particular, we show that pooling firms with higher assets may reduce the cost of procurement even when default is costless for the sponsor.

KEYWORDS: Procurement, limited liability, bankruptcy.

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1 Introduction

The high frequency of bankrupt bidders in high-stake auctions and procurement, especially in the construction industry[1], has lead researchers to move away from traditional auction theory with deep pocket bidders to analyze the possibility of default.[2] It is now well understood that limited liability makes bidders risk-loving by cutting off the downside losses. The implied tendency of bidders to bid more aggressively increases the probability of contractors' default.[3] Default or bankruptcy may be costly for the sponsor: it implies delays in the completion of the project, litigation costs,[4] the cost of the new procurement process, etc. The literature explains why broke bidders are frequent - the poorest bidder is the most aggressive -, compares the performance of different auction formats,[5] and analyses the possibility of insurance.[6] However, little is known about the feasibility constraints that this

[1]In the US during 1990-1997 more than 80,000 contractors went bankrupt leaving unfinished private and public construction projects with liabilities exceeding \$21 billion (Dun & Bradstreet Business Failure Record).

[2]Examples are Waehrer (1995), Zheng (2001),and Board (2007) in forward auctions and Calveras et al. (2004) and Parlane (2003) and Engel and Wambach (2006) in procurement auctions.

[3]Empirical evidence seems to confirm that budgetary issues are an important factor in explaining company failure: Arditi et al. (2002) show that budgetary issues explain 60.2% of the company failures in the US construction industry.

[4]White (1989) reports direct administrative costs for liquidating a firm in the US for which bankruptcy courts keep record: these make up 7.5%-21% of the liquidation proceeds.

[5]Parlane (2003) and Board (2007) show that if bidders have the same budget but different valuations the second price auction induces higher prices, higher bankruptcy rates and lower utilities than the first price auction. In fact, Parlane shows that the first price auction induces the highest prices among all efficient auctions. Engel and Wambach (2006) obtain comparable results for a procurement setting. They also provide an illustrative comparison between a second-price auction with a multi-source second-price auction without switching costs and a lottery. Both papers show that the sponsor's preferences over different mechanisms crucially depend on the bankruptcy costs.

[6]Calveras et al. (2004) focus on the regulatory practice of surety bonds (financial collateral) and show that these bonds reduce and sometimes eliminate the problem of

possibility of default imposes on trading mechanisms, or about the optimal mechanism in the presence of limited liability.[7] The present paper makes a first step in attacking this problem. We analyze procurement with limited liability. Bidder heterogeneity is modeled in terms of financial assets, and therefore in terms of financial robustness, which is private information.[8] We study a model with a continuum of types (financial assets), and common uncertainty about the cost of carrying out the contract at the time of signing it. That is, in order to concentrate on the effect of limited liability, we abstract from (cost) efficiency differences between contractors.

Under these assumptions, and even excluding the possibility of payments not associated to awarding the contract,[9] incentive compatibility constraints are complex, and in particular do not imply linear relationships between rents and award probabilities. However, it is still true that these probabilities and an initial condition uniquely determine contractors' rents and the expected costs of different mechanisms (revenue equivalence). More importantly, incentive compatibility imposes that these win probabilities are monotonically decreasing and contract prices monotonically increasing in the value of the

abnormally low tenders (low bids with high bankruptcy risk).

[7] Che and Gale (2000) characterize the optimal mechanism for selling to a budget-constrained buyer. This is equivalent to analyzing the one-bidder no-default case. Pai and Vohra (2008) extend the previous analysis to n-bidders in a discrete type space. Parlane (2003) discusses mechanism design when firms differ in their costs but not in their financial situation.

[8] Another strand of the literature (e.g. Waehrer 1995, Engel and Wambach, 2006, Board, 2007) model bidder heterogeneity by assuming equal assets but different valuations/costs. Our approach allows to disentangle efficiency and default effects by concentrating on the latter.

[9] In this environment it seems reasonable to assume that sponsors cannot use trading mechanism with transfers from losers to the winner or from losers to the sponsor. They are difficult to reconcile with private information on financial assets and limited liability.

bidder's financial assets. In other words, in any feasible mechanism and no matter how high bankruptcy costs are, firms in a bad financial situation are at least as likely to be assigned the project as financially solvent firms.

The intuition behind this "perverse" result is in fact simple. Given any contract price, a firm with fewer assets is better protected against bad cost realizations than a more sound firm. Therefore, the former is always willing to trade price for probability of winning if the latter is. In fact, informational rents are linked to the probability of default, in this setting. Thus, firms with less assets expect higher rents in any incentive compatible mechanism.

Under these circumstances, when designing the trading mechanism the sponsor should combine the goal of reducing informational rents with the need for keeping the bankruptcy probability low. Given the incentive compatibility constraints, the second goal in isolation is achieved by a posted price, which randomly assigns the contract and therefore does not select worse firms with (strictly) higher probability. A more striking result is that, contrary to what could be expected, the first goal in isolation is not necessarily achieved by an auction, which maximizes price competition. Indeed, even when bankruptcy costs are negligible (or inexistent), informational rents might not be minimized by a mechanism that minimizes the contracted price by selecting the firm with lowest assets. Lower contracted prices, which necessarily go hand in hand with higher probabilities of selecting less solvent firms and higher probabilities of default, bolster informational rents. We find sufficient conditions that guarantee that, even with no costs associated to default, auctions are suboptimal.

Our results can be put in perspective by relating them to those of Manelli and Vincent (1995). That paper characterizes the optimal procurement mechanism in an environment where the valuations of the uninformed buyer over the potential sellers' goods (quality) are positively correlated with the sellers' opportunity cost of supplying the good. As in the present paper, auctions or price competition lead to awarding the project to the least preferred supplier and mechanisms with some randomization in the winning probabilities may be optimal. In our setting, bidders with worse financial status are those that would result in higher default probability for the sponsor and then, other things (contract price) equal, are less attractive if default is costly. Although the relationship is less direct in our case, we can think of this bankruptcy exposure as a sort of lower quality from the sponsor's viewpoint. However, the parallel breaks when we consider that even with no bankruptcy costs price competition might still be undesirable for the sponsor. Indeed, in our model informational rents are more subtle and are linked to the probability of default, whether default is per se costly or not. This probability is higher when price competition is more intense. Thus, even under regularity assumptions on the inverse hazard rate, it may be in the interest of the sponsor to blunt price competition.

We analyze the implications of these results for the design problem of sponsors by considering a discrete (three type) parameterized example. We show that pooling the two higher types may result in a lower cost for the sponsor even when bankruptcy brings no additional cost to the sponsor. Pooling the lower types, on the contrary, could never reduce the cost for

the sponsor. We also show that these insights extend to a general K type space: pooling at the top may result in a lower cost whereas pooling at the bottom would always increase the cost. Thus, the discrete case lends some justification for blunting price competition in the presence of limited liability even for low costs of default, but raises questions about the right way to do so.

The remainder of the paper is organized as follows. In section 2 we present the model. Section 3 contains our main results for the continuous type model. In section 4 we relate our results to those of Manelli and Vincent (1995), show that auctions maximize price competition, and offer conditions that make auctions suboptimal even in the absence of default costs. We also study the discrete case in this section. Section 5 concludes. All proofs are relegated to a technical appendix.

2 The Model

A risk neutral buyer (sponsor) procures an indivisible contract for which he is willing to pay V, which we assume large enough as to make the possibility of no contracting unattractive. There are N risk-neutral potential contractors all with the same cost, c, unknown at the time of signing the contract. Thus, c is a random shock which for economy of notation we assume to be uniformly distributed on $[0, 1]$. Potential contractors differ in their initial financial status. Let $A_i \geq 0$ denote the value of the assets of potential contractor $i = 1, 2, ..., N$. A_i is contractor i's private information. Each A_i is an independent realization of a random variable with support $\left[\underline{A}, \overline{A}\right]$, with $\underline{A} \geq 0$, density

function $f(\cdot)$, and distribution $F(\cdot)$. We denote by A the vector of firms' types, $A = (A_1, A_2, ..., A_N)$.[10]

Contractors have limited liability, i.e. the losses of firm i cannot be larger than A_i. Therefore, if awarded the project and after c is realized firm i will close down if A_i plus the net profits from undertaking the project, $(P - c)$, fall below 0 .

The sponsor chooses a procurement mechanism to award the contract. After the contract is awarded at a price P, the cost c is realized and publicly observed and the selected contractor either finishes the project or declares bankruptcy. Bankruptcy may imply extra costs for the sponsor. We summarize these costs by a constant C_B, with $C_B \geq 0$. Thus, when the selected contractor declares bankruptcy the sponsor has to bear the realized cost c plus this bankruptcy cost, but can liquidate and seize the assets of the firm at that moment, $A_i + P$. Thus, if the sponsor signs a contract with firm i (with assets A_i) at a price P and the realized cost is c, then the utility of the sponsor is

$$U^S = \begin{cases} V - P & \text{if } P - c + A_i \geq 0 \\ V - c + A_i - C_B & \text{otherwise.} \end{cases} \tag{1}$$

These payoffs can be considered as the reduced form payoffs of a continuation game where the sponsor asks for bids from the rest of contractors once the realized cost c has been revealed.

Note that we are implicitly assuming that the sponsor cannot use trading mechanisms with transfers from losers to the winner and/or the sponsor.

[10] This way of modeling bidder-heterogeneity is due to Che and Gale (1996) who model bidder-heterogeneity in terms of wealth instead of value in a forward auction. It is also used in Zheng (2001).

Besides being unrealistic, such (meaningful) trading mechanisms would be difficult to reconcile with limited liability and private information on financial assets.

We now summarize the timing of the model:

1. Nature chooses the financial value A_i of each firm. Each firm privately learns its financial assets.

2. The sponsor announces the procurement process.

3. Firms submit their "bids" or messages, and the project is awarded according to the rules announced by the sponsor. The price P is set according to these rules.

4. The cost parameter c is realized. If the assets of the selected firm i are such that $A_i + P - c \geq 0$, then the firm finishes the project. Otherwise it declares bankruptcy.

5. The sponsor and the firms realize their payoffs. The selected firm retains a financial value of $\max\{0, A_i + P - c\}$, all other firms retain their assets, and the contractor obtains a payoff defined in (1).

3 Implementable mechanisms

We restrict attention to trading mechanisms where prices are deterministic after conditioning on all types, A, and the identity of the winner. Thus, we allow for mechanisms such as the second price auction, but we do not consider mechanisms where the sponsor uses random devices that are not

related to the primitives of the problem to (partially) determine the price. The reader can trivially extend the proof of Lemma 1 below to check that there is nothing to be gained by considering a larger class of mechanisms. Thus, a mechanism is a pair (σ, P), where $P : \left[\underline{A}, \overline{A}\right]^N \to R^N$, and $\sigma : \left[\underline{A}, \overline{A}\right]^N \to \Delta^N$. We interpret $\sigma_i(A)$ as the probability that supplier i is assigned the project when A is the vector of assets. $P_i(A)$ represents the price of the contract if the vector of assets (types) is A *and* supplier i is assigned the project. Thus, $P_i(A)$ is not the unconditional expected payment to contractor i. In a conventional auction design problem, that would be all that would matter both to the contractor and to the sponsor. In our problem it is the joint distribution of contract allocation and payments that matters.

The sponsor faces participation or individual rationality constraints, IR: for all i and for all A_i,

$$U_i(A_i; \sigma, P) \equiv E_{A_{-i}} \sigma_i(A) \left[\int_0^{\min\{1, P_i(A)+A_i\}} (P_i(A) + A_i - c) dc - A_i \right] \geq 0. \tag{2}$$

Also, the sponsor faces incentive compatibility constraints, IC, which in this setting means that for all A_i and $\widehat{A}_i$, and all i,[11]

$$\begin{aligned} &U_i(A_i; \sigma, P) \geq \\ &E_{A_{-i}} \sigma_i(A_{-i}, \widehat{A}_i) \left[\int_0^{\min\{1, P_i(A_{-i}, \widehat{A}_i)+A_i\}} (P_i(A_{-i}, \widehat{A}_i) + A_i - c) dc - A_i \right]. \end{aligned}$$

The sponsor's goal is to minimize the cost of the project. That is, to minimize

[11]Note that we are implicitly assuming that bankruptcy cannot be claimed when funds are sufficient to cover the cost. That is, that assets cannot be hidden once the supplier has applied for bankruptcy.

$$E_A \sum_i \sigma_i(A) \left[P_i(A) + \int_{\min\{1, P_i(A)+A_i\}}^{1} (c + C_B - P_i(A) - A_i) dc \right].$$

First we show that the sponsor needs to consider only mechanisms where the price depends on the type and identity of the winner, but not on the types of other bidders.

Lemma 1 *For any IC, IR mechanism* (σ, P)*, there exists a mechanism* $(\sigma, \overline{P})$*, where* $\overline{P}_i(A)$ *is constant on* A_{-i}*,* $(\sigma, \overline{P})$ *is also IC and IR, results in the same expected rents for each firm* i *and each value of* A_i*, and results in expected (weakly) lower cost for the sponsor.*

Proof. See appendix. ∎

Intuitively, randomness in the price is not a useful instrument for the sponsor. Indeed, higher dispersion in the price for the winner induces higher probability of default, which at best represents a cost for the sponsor that the contractor does not appropriate.

Given Lemma 1 we can consider only mechanisms where the payments are independent of types other than that of the winner. Thus, let us define $\Psi_i(A_i) = E_{A_{-i}} \sigma_i(A)$. $\Psi_i(A_i)$ is the expected probability that bidder i obtains the contract when conditioning on his information, A_i. Incentive compatibility implies a crucial monotonicity property of trading mechanisms. Indeed,

Lemma 2 *If* (σ, P) *is IC, then* $U_i(A_i; \sigma, P)$ *is continuous and monotone decreasing in* A_i*. Monotonicity is strict if* $1 > P_i(A_i) + A_i$*.*

Proof. See appendix. ■

Lemma 2 shows that informational rents are linked to low asset holdings, not to solvency. This monotonicity result also implies monotonicity of allocation probabilities and prices. This is our most important result.

Lemma 3 *In any IC mechanism* $\Psi_i(A_i)$ *is monotonically decreasing and* $P(A_i)$ *monotonically increasing.*

Proof. See appendix. ■

Lemma 3 unveils the "perverse effects" of limited liability. By cutting off the downside losses, limited liability makes firms risk-loving. The fewer assets a firm has, the stronger is this effect. Firms with many assets will be more conservative since they have more to lose. Therefore, they are only willing to procure the project at higher prices. Firms with fewer assets can imitate firms with higher assets and will only be stopped from doing so and accept a lower price if this increases their win probabilities. In other words, even when choosing the procurement mechanism optimally firms in a weak financial situation are at least as likely to win the project as more solvent firms. We further examine $U_i(A_i; \sigma, P)$ to better understand the source of informational rents.

From continuity and monotonicity, $U_i(A_i; \sigma, P)$ is differentiable almost everywhere if it is bounded. It is bounded from below by IR and can be bounded above in a search for an optimal mechanism. Now, let U_i be differentiable at A_i. Applying the envelope theorem, when it exists the derivative of U_i at A_i is

$$-\left[1 - (P_i(A_i) + A_i)\right] \Psi_i(A_i). \tag{3}$$

when $P_i(A_i) + A_i < 1$, and zero otherwise. Recall that $[1 - (P_i(A_i) + A_i)]$ is the probability that firm i defaults if assigned the contract. Thus, the source of informational rents is the probability of default.

Note that (3) is not linear in Ψ_i taking into account the relationship of Ψ_i and A_i. In fact, $[1 - (P_i(A_i) + A_i)]$ depends on the mechanism itself. However, it is still true that IC eliminates one degree of freedom in the choice of mechanism. That is, we can still prove a "revenue equivalence" result in this setting.

Lemma 4 *Two mechanisms that share $\Psi_i(A_i)$ and give the same rents to bidders of the highest type also share $P_i(A_i)$.*

Proof. See appendix ■

Thus, we only need to consider $\Psi_i(A_i)$, i.e., σ, when we compare rents and expected costs of two different mechanisms. We can represent the expected payment for the sponsor as the sum of the expected cost of the project, contractors' expected utility, and expected bankruptcy costs:

$$c + \sum_i E_A U_i(A_i; \sigma, P) + E_A \sum_i \sigma_i(A) \left[1 - (P_i(A_i) + A_i)\right] C_B \tag{4}$$

On the other hand, integrating equation (3) we get[12]

$$U_i(A_i; \sigma, P) = U_i(\overline{A}; \sigma, P) + \int_{A_i}^{\overline{A}} \left[1 - (P_i(x) + x)\right] \Psi_i(x) dx.$$

Using this equation we can write

$$E_{A_i} U_i(A_i; \sigma, P) = U_i(\overline{A}; \sigma, P) + E_A \int_{A_i}^{\overline{A}} \sigma_i(A_{-i}, x) \left[1 - (P_i(x) + x)\right] dx.$$

[12] In the proof of Lemma 4 we show that U_i is Lipschitz continuous and therefore the Fundamental Theorem of Calculus holds for U_i.

Hence we can rewrite (4) as:

$$c + \sum_i U_i(\overline{A}; \sigma, P) +$$
$$\sum_i E_{A_i}\left[\int_{A_i}^{\overline{A}} \Psi_i(x)\left[1 - (P_i(x) + x)\right] dx + \Psi_i(x)\left[1 - (P_i(A_i) + A_i)\right] C_B\right].$$

Notice that the third term of this expression captures the informational rents and the bankruptcy costs incurred. This third term can be written as

$$\sum_i \int_0^{\overline{A}} \left[\int_{A_i}^{\overline{A}} \Psi_i(x)\left[1 - (P_i(x) + x)\right] dx + \Psi_i(A_i)\left[1 - (P_i(A_i) + A_i)\right] C_B\right] dF(A_i)$$

or, changing the order of integration,

$$\begin{aligned} & \sum_i \int_0^{\overline{A}} \Psi_i(A_i)\left[1 - (P_i(A_i) + A_i)\right]\left(\frac{F(A_i)}{f(A_i)} + C_B\right) f(A_i) dA_i \qquad (5) \\ = \; & \sum_i \int_0^{\overline{A}} E_{A_{-i}} \sigma_i(A)\left[1 - (P_i(A_i) - c + A_i)\right]\left(\frac{F(A_i)}{f(A_i)} + c_B\right) f(A_i) dA_i \\ = \; & E_A \sum_i \sigma_i(A)\left[1 - (P_i(A_i) + A_i)\right]\left(\frac{F(A_i)}{f(A_i)} + C_B\right). \end{aligned}$$

We know from Lemma 3 that $P_i(A_i)$ is monotonically increasing, hence $P_i(A_i) + A_i$ also increases in A_i. Therefore, the probability of default, namely $\max\left[0, 1 - (P_i(A_i) + A_i)\right]$ decreases in A_i. So lower types are always associated with higher bankruptcy costs. However, this monotonicity may be lost in the term that represents informational rents since $\left[1 - (P_i(A_i) + A_i)\right] \frac{F(A_i)}{f(A_i)}$ may not be monotone even if the inverse hazard rate is monotonically increasing. This makes the design of an optimal mechanism complex even in the absence of default costs.

Also, by the individual rationality constraint $U_i(\overline{A}; \sigma, P) \geq 0$. Since the highest type has no incentive problems to reveal its type, one may expect that the optimal mechanism assigns zero rents to this type. This is indeed

the case if $C_B = 0$. However, $U_i(\overline{A};\sigma,P)$ also determines the probability of default, and hence it affects bankruptcy costs. Thus, it may be in the sponsor's interest to leave some rents to the highest type if this reduces the probability of default sufficiently and bankruptcy costs are high. To see this assume that $C_B \to \infty$. In this case it is optimal to avoid bankruptcy altogether, which can be achieved only if $P_i(\underline{A})+\underline{A} \geq 1$, which in turn implies that $P_i(\overline{A}) + \overline{A} > 1$, so that, by IC, $U_i(\overline{A}) > 0$.

4 Auctions and the design of optimal mechanisms

From now on we can dispense with the subscript that refers to a particular bidder. The problem we have been analyzing shares some important features with the problem studied by Manelli and Vincent (1995). In their paper, suppliers with cost (type) c supply a good of unobservable quality $v(c)$. If the buyer trades with a supplier with cost c at a price P, then the buyer's surplus is $v(c) - c - \pi$, where $\pi = P - c$. Given π, and assuming that $v(c) - c$ is increasing in c, the buyer prefers high-type c suppliers. However, IC also imposes that the probability of trade is non increasing in c. In our problem, we could denote the probability of non-default by α, and define a transformed willingness to pay for the project as $\widehat{V} = V - (1-\alpha)C_B$. If the buyer trades with a type A supplier, the buyer's surplus is $\widehat{V} - \pi - Ec$. Since $\alpha = \max\{(P + A), 1\}$ is increasing in A, we have that when $C_B > 0$, $\widehat{V}$ is increasing in the type A just as in Manelli and Vincent (1995). Moreover, IC also implies that the probability of trade is non increasing in A. Thus, our

results can be read in the light of Manelli and Vincent: price competition selects bad types, and then it may be in the interest of the buyer to avoid it. Limited liability and private information about the financial state introduce some sort of quality differential correlated to the bidders' privately known willingness to supply.

However, this is where the parallel stops. Indeed, the interaction between price competition and the buyer's surplus is complicated by the dependency of α on P, and therefore on the mechanism itself. While in Manelli and Vincent (1995) informational rents can be represented by the inverse hazard rate, in our problem this term is

$$(1-\alpha)\frac{F(A)}{f(A)} = [1-(P(A)+A)]\,\frac{F(A)}{f(A)}.$$

Put in other terms, $U_i' = -(1-\alpha)\Psi(A)$. Thus, apart from the direct effect that a change in $\Psi(A)$ (and so in P) has on the rents of suppliers, it also has an indirect effect through its effect on the probability of default $(1-\alpha)$. As a result, even if the inverse hazard rate is monotone, it may well be that $(1-\alpha)\frac{F(A)}{f(A)}$ (which depends on the mechanism itself) is not. In this case, even if the default cost C_B is zero, so that $\widehat{V}$ is independent of the supplier's type, the optimal mechanism does not need to be one that selects the lowest type.

An important consequence of this is that, even when $C_B = 0$, auctions need not be optimal. In order to prove this we first show that under mild conditions, auctions maximize price competition, i.e., select the contractor with lowest value of asset.

Lemma 5 *Assume* $\overline{A} > \frac{1}{2}$. *A first price auction has a symmetric equilibrium,* $b(A)$*, where* $b(A)$ *is strictly increasing in* A *for* $A < \frac{1}{2}$*, and* $b(A) = \frac{1}{2}$ *for* $A \geq \frac{1}{2}$.

Proof. See appendix. ■

The condition that $\overline{A} > \frac{1}{2}$ simplifies the proof of existence but is not crucial to the result.[13] Firms with assets in excess of $\frac{1}{2}$ will never default in any IR mechanism, and then will always be pooled (i.e., be selected with in the same probability at the same price) in any IC mechanism. Thus, an auction indeed maximizes price competition and therefore selects the contractor with the lowest asset. But - as is shown in the next lemma - under weak conditions such a mechanism will be suboptimal even for $C_B = 0$.

Lemma 6 *For any* $C_B \geq 0$*, there exist non strictly monotone, IC mechanisms that result in higher surplus for the buyer than mechanisms that assign the contract to the bidder with lowest* A *(auctions) if* $f'(A)$ *exists and is negative for* A *close to* $\overline{A}$.

Proof. See appendix. ■

We can gain some further intuition for this result by looking at the difference between contract price and realized price, i.e., the price paid to the contractor that finalizes the contract. When increasing the probability that higher types win the contract by, say, pooling a certain interval of types, the mechanism does not only result in a higher contract price: it also results in a higher expected value of the assets of the winner. Thus, it increases

[13] $\overline{A}$ can be arbitrarily close to $\frac{1}{2}$. Also, the assumption $\overline{A} > \frac{1}{2}$ is virtually equivalent to an atom at $\frac{1}{2}$.

the probability that the contract price will coincide with the realized cost for the sponsor. In contrast, the auction selects the contractor with lowest assets, so that although it minimizes the contract price it also maximizes the probability that the realized cost for the sponsor exceeds that price.

We can further investigate the type of simple mechanisms that may be attractive from the point of view of the sponsor by substituting a discrete type space for the continuous type space that we have been studying up to this point. In particular, assume that A can take on three values, A_k, $k = 0, 1, 2$, and for simplicity assume that $A_0 = 0$, $A_1 \in \left(0, \frac{1}{2}\right)$ and $A_2 = \frac{1}{2}$. We denote the probability of the three types, respectively as α_0, α_1, and $\alpha_2 = 1 - (\alpha_0 + \alpha_1)$. Also for simplicity, we will assume that there are only two bidders. It will become clear below that none of these assumptions, not even the three-type assumption, is restrictive.

A mechanism in this setting can be represented by a set of six values, (Ψ_0, Ψ_1, Ψ_2) and (P_0, P_1, P_2) that denote the probability of winning and the price for each of the three types. Notice that we are only considering symmetric mechanisms. Also, notice that Ψ_k is positive for all k since in case both bidders draw the same type each is assigned the contract with probability 1/2 (the contract is always allocated). That is, $\Psi_k \geq \frac{\alpha_k}{2}$. Individual rationality for type 2 amounts to $P_2 \geq 1/2$. The fact that the probability of allocation is one also eliminates one degree of freedom in the choice of Ψ_k. So, the sponsor's problem is to choose four values (say Ψ_0, Ψ_1, P_0, P_1) to minimize the cost of the contract. The (other) constraints that the sponsor faces are the IC constraints for types 0 and 1, and monotonicity of Ψ_k together

with the conformity of Ψ with some probability σ.

These are the elements that define the problem for the sponsor whose solution is the optimal mechanism. For now, let us consider a simpler question: would a mechanism that selects the lowest type with probability one and then guarantees the lowest expected contract price be always optimal for the sponsor if $C_B = 0$? Or else, would the sponsor prefer to pool some of the types even when there are no costs associated to bankruptcy?

If we assume away bankruptcy costs, $C_B = 0$, then the goal for the sponsor is to minimize $\sum_{k=1}^{3} \alpha_k U(A_k)$. We begin by comparing a mechanism that assigns the contract to the lowest type to one where the two higher types are pooled. That is, a mechanism with $\Psi_0 = (1 - \frac{\alpha_0}{2})$ and $\Psi_1 = (1 - \alpha_0 - \frac{\alpha_1}{2})$ with one with $\Psi_2^P = \Psi_1^P = \frac{1-\alpha_0}{2}$. Note that in this latter case $P_2^P = P_1^P = 1/2$, and that only the incentive compatibility for type 0 will bind. That constraint can be written as

$$(1 - \frac{\alpha_0}{2}) \int_0^{P_0} (P_0 - c) dc \geq \frac{1 - \alpha_0}{2} \int_0^{1/2} (\frac{1}{2} - c) dc$$

or

$$U^P(A_0) = (1 - \frac{\alpha_0}{2}) \frac{{P_0}^2}{2} \geq \frac{1 - \alpha_0}{2} \frac{1}{8}, \tag{6}$$

which at the optimum (among such mechanisms) is satiated. Therefore (6) with equality determines the rents of a type 0 bidder in such mechanism. On the other hand, the rents for type 1 are

$$U^P(A_1) = \frac{1 - \alpha_0}{2} (\int_0^{\frac{1}{2}+A_1} (\frac{1}{2} + A_1 - c) dc - A_1) = \frac{1 - \alpha_0}{2} (\frac{(\frac{1}{2} + A_1)^2}{2} - A_1). \tag{7}$$

Since (6) is satiated and type 2 expects no rents, the expected rents for a

bidder are

$$\sum_{k=1}^{3}\alpha_k U^P(A_k)=\alpha_0\frac{1-\alpha_0}{2}\frac{1}{8}+\alpha_1\frac{1-\alpha_0}{2}(\frac{(\frac{1}{2}+A_1)^2}{2}-A_1).$$

Let us turn to a mechanism that assigns the contract to the lowest type. It will have to guarantee incentive compatibility for the two lower types. That is

$$U(A_0)=(1-\frac{\alpha_0}{2})\frac{P_0^2}{2}\geq(1-\alpha_0-\frac{\alpha_1}{2})\frac{P_1^2}{2}, \tag{8}$$

for type 0, and

$$U(A_1)=(1-\alpha_0-\frac{\alpha_1}{2})[\frac{(P_1+A_1)^2}{2}-A_1]\geq\frac{1-\alpha_0-\alpha_1}{2}(\frac{(\frac{1}{2}+A_1)^2}{2}-A_1), \tag{9}$$

for type 1. Again, notice that there is no reason to leave any slack, so that both constraints will be satiated at the optimal (in this restricted set) mechanism. Therefore, (9) with equality determines P_1, and then (8) with equality determines P_0. As a function of P_1 defined implicitly by (9) with equality, the expected rents in this mechanism are

$$\sum_{k=1}^{3}\alpha_k U(A_k)=\alpha_0(1-\alpha_0-\frac{\alpha_1}{2})\frac{P_1^2}{2}+\alpha_1\frac{1-\alpha_0-\alpha_1}{2}(\frac{(\frac{1}{2}+A_1)^2}{2}-A_1).$$

Notice that $U^P(A_1)>U(A_1)$. Indeed, the probability that type 2 obtains the contract is larger in the pooled mechanism and the price in both mechanisms is the same for this type. Since type 1 obtains its rents from the possibility of imitating the type 2 bidder, its rents must be larger in the pooled mechanism. This argument is independent on the number of types and in particular of the existence of type 0. This immediately implies that when $C_B=0$ the optimal mechanism with only two possible types must

minimize the probability that the high type obtains the contract, i.e., assign the contract to the lowest type.

However, pooling the two highest types also has an effect on the rents of type 0. Indeed, it reduces the probability that type 1 obtains the contract but increases the price type 1 receives. Therefore, the probability that type 0 still obtains the contract if it decides to imitate type 1 is also reduced although the price it gets in that case is higher. This possibility of imitation is the source of rents for type 0 (i.e., (8) holds with equality). We know that this change in probability and price is profitable for type 1 itself, but type 0 is relatively more interested in probability of winning than in price. So, depending on the underlying parameters pooling at the top may well reduce bidder 0's rent. And in that case the increase in the rents of type 1 may be outweighed by the decrease in the rents of type 0. We now provide examples for both possibilities. In the first example pooling increase the rents of type 0 and type 1. In the second example pooling decreases the rents of type 0 and increases the rents of type 1, but the first effect dominates.

Example 1 *Assume that* $\alpha_0 = .8$, $\alpha_1 = .1$, *and* $A_1 = .1$. *In this case, the price* P_1 *defined implicitly by (9) is* 0.403322 *in the mechanism that assigns the contract to the lowest type. The rents for type* 0 *are* 9.76×10^{-3}, *and the rents for type* 1 *are* 4×10^{-4}. *Total expected rents in this mechanism are* 1.016×10^{-2}. *In the pooled mechanism,the rents for type* 0 *are* 0.01 *and for type* 1 *are* 8×10^{-4} *so that total expected rents are* 1.08×10^{-2}.

Example 2 *Assume that* $\alpha_0 = .1$, $\alpha_1 = .1$, *and* $A_1 = .3$. *In this case, the price* P_1 *defined implicitly by (9) is* $0.486\,65$ *in the mechanism that assigns the*

contract to the lowest type. The rents for type 0 *are* $0.100\,65$*, and the rents for type* 1 *are* $7.997\,7\times10^{-3}$*. Thus total expected rents in this mechanism are* $1.086\,5\times10^{-2}$*. In the pooled mechanism, the rents for type* 0 *are* $0.056\,25$ *and for type* 1 *are* 0.009 *so that the expected rents are* 6.525×10^{-3}*.*

Thus, pooling of the two highest types may reduce informational rents and the resulting expected price for the sponsor. Could this be attained by pooling the two lower types instead? The answer is in the negative. If the two low types are pooled, then type 1 would still expect the same rents, whose source would be the possibility of imitating the unaffected high type 2 bidder.[14] Moreover, these rents would now come from an increased probability of obtaining the contract (namely $\Psi_1 = 1 - \frac{\alpha_0}{2} - \frac{\alpha_1}{2}$) and a lower price. If that trade off leaves type 1 indifferent, then an imitating type 0 will prefer the higher probability. Thus, the rents of type 0 would certainly

[14] These rents are still given by the right hand side of equation (9).

be higher in the mechanism that pools the two low types.[15] Moreover, this effect of pooling the two low types is independent of whether type 1 is itself pooled to type 2 or not. That is, total pooling (a posted price $P = \frac{1}{2}$) always leaves more rents to contractors than a mechanism that only pools the two high types.[16]

Do these insights extend to a general, K-type case? The answer is in

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[15] The rents for type 1 are given by

$$\begin{aligned} U(A_1) &= (1-\alpha_0-\frac{\alpha_1}{2})[\frac{(P_1+A_1)^2}{2}-A_1] = (1-\frac{\alpha_0}{2}-\frac{\alpha_1}{2})[\frac{(P_1^L+A_1)^2}{2}-A_1] \qquad (10) \\ &= \frac{1-\alpha_0-\alpha_1}{2}(\frac{(\frac{1}{2}+A_1)^2}{2}-A_1), \end{aligned}$$

where P_1 is the price for type 1 in the mechanism assigning the project to the lowest bidder and P_1^L the price for type 1 (and also type 0) when pooling at the bottom. Recall $P_1^L < P_1$. We can reformulate equation (10) as:

$$\frac{2-\alpha_0-\alpha_1}{2-2\alpha_0-\alpha_1} = \frac{(P_1+A_1)^2-2A_1}{(P_1^L+A_1)^2-2A_1} \tag{11}$$

Under pooling at the bottom type 0 gets rents

$$U^L(A_0) = (1-\frac{\alpha_0}{2}-\frac{\alpha_1}{2})\frac{P_1^{L2}}{2} \tag{12}$$

while his rents when the lowest bid wins are

$$U(A_0) = (1-\alpha_0-\frac{\alpha_1}{2})\frac{P_1^2}{2} \tag{13}$$

We now have to compare $U^L(A_0)$ with $U(A_0)$. Using (11)this is equivalent to comparing

$$\frac{(P_1+A_1)^2-2A_1}{(P_1^L+A_1)^2-2A_1} \text{ with } \frac{P_1^2}{P_1^{L2}}$$

which is equivalent to comparing $(2-A_1)\left(P_1+P_1^L\right)$ with $2P_1P_1^L$. Given $A_1 \leq \frac{1}{2}$ and $P_1^L < P_1 \leq \frac{1}{2}$, the former is clearly bigger than the latter, which implies that the rents of type 0 are higher under pooling at the bottom.

[16] Indeed, in the former the rents of type 1 would be higher: they would still be given by the right hand side of (7) with only substituting $\frac{1}{2}$ for $\frac{1-\alpha_0}{2}$. But the rents of type 0 would also be higher. They would still be given by the right hand side of (6) also substituting $\frac{1}{2}$ for $\frac{1-\alpha_0}{2}$. That is, not only the win probability of type 2 would be higher, but also the win probability of type 1, the one that could be imitated by type 0.

the positive. To see this, assume that there are additional types $A_3 ..., A_K$, with $A_k > A_{k-1}$. We will first discuss pooling at the bottom. Since the win probabilities of the other types $k > 1$ are unaffected, pooling of types 0 and 1 (and consequently, changing both P_0 and P_1 so that the IC constraints for these types still hold with equality), would result in the same effects discussed in the three type case. Thus, for arbitrary discrete type spaces we conclude that pooling the lowest types will never be optimal when $C_B = 0$. A trivial corollary is that total pooling will never be optimal when $C_B = 0$.

Similarly, for arbitrary discrete type spaces, pooling the two highest types will always increase the rents of the second highest type: the value of Ψ_K goes up and P_K remains unchanged, determined by the IR constraint for type K, so that by imitating type K type $K - 1$ can guarantee higher rents. However, every other type below $K - 1$ might see its rents reduced. On the one hand, the values of Ψ_k for $k < K - 1$ are unaffected. On the other hand, using exactly the same argument as in the three type case for type 0, the rents for type $K - 2$ might be lower, which means that P_{K-2} would also be lower. Iterating this argument, we conclude that in such case P_k would be lower for all types $k < K - 1$. That is, the possibility of reducing rents by pooling the higher types is not particular to the three types case.

Example 3 *Assume $K = 7$, $A_k = \frac{k-1}{12}$, and $\alpha_k = \frac{1}{7}$ for all k. That is, there are seven types distributed uniformly (in value and probability) on the interval $[0, \frac{1}{2}]$. It is optimal to pool type 3 and above, and separate types below 3.*

This discussion tells us that, in a procurement setting with limited liabil-

ity but *small default costs* blunting price competition may be desirable at the top, but not at the bottom. Therefore, price floors, which are often used by public administrations, would tend to blunt competition at the wrong side.[17] In practice, pooling at the top could be obtained by requiring a minimum discount in the auction. Consider the continuous case again: assume that the sponsor announces (i) a maximum price of $\frac{1}{2}$ and (ii) that all bids that are above some price $\widehat{P} < \frac{1}{2}$ will be treated as the same bid. It is simple to extend the proof of Lemma 5 to show that to each such $\widehat{P}$ corresponds a cut-off value $\widehat{A}$ such that types above $\widehat{A}$ bid $\frac{1}{2}$ and types below $\widehat{A}$ use a strictly monotone bidding function. The effect of this "distortion" is to reduce the probability of bankruptcy for these top types. In a standard auction problem distorting the allocation for low valuation types by reducing their probability of winning causes a reduction in the rents of all higher valuation types. In a similar way, here "distorting" the allocation of high A types by reducing their probability of default causes a reduction in the rents of lower A types.

Not surprisingly, as the cost of bankruptcy C_B increases, pooling of types becomes even more attractive. Also, the relative impact of pooling at the top and at the bottom becomes less clear cut. Our discussion above has shown that pooling at the top increases the price (and thereby decreases the probability of default) for the pooled types but might reduce all prices (and hence increase the probability of bankruptcy) for lower types. This latter

[17] Once we have pooled the two lowest types, we could consider pooling the third lowest type as well. The argument used to show that pooling the two lowest types increases the rents of contractors shows that this could only further increase the contractors' expected rents. Thus, a price floor, whether it only pools the two lowest types or some larger set of types at the bottom, is never optimal.

effect is absent when pooling at the bottom. However, the prices of the pooled types move in different directions. On the one hand, pooling at the top increases the probability of obtaining the contract for higher types, and therefore these types see their price reduced (and hence their probability of default increased). On the other hand, when pooling at the bottom the price for the lowest type, who now has a lower probability of obtaining the contract but has to be guaranteed higher rents, must increase (and consequently its probability of default must decrease). It is not difficult to construct examples where one or the other partial pooling is more effective in reducing the probability of default.[18] Of course, complete pooling unambiguously reduces the probability of default. Thus, when C_B is large enough, complete pooling, i.e. posted prices, are optimal mechanisms for assigning the contract.

5 Conclusions

We have shown that limited liability results in a perverse selection in procurement when the cost of the project is common but uncertain, and firms differ in their financial strength. Indeed, in this case incentive compatibility implies that selecting the more sound firm is not feasible. In fact, the stronger the price competition the more likely it is to select the financially weakest firm. This is an unfortunate aspect of price competition when the costs of default are high. Perhaps more surprisingly, even if these costs are inexistent, fiercer price competition, and so a higher likelihood of selecting fi-

[18] For instance, in our example 2 above, pooling at the bottom reduces this probability of default more than pooling at the top. But if we change the value of A_0 to .1 then the opposite is true.

nancially weaker firms at lower contract prices, may be against the interest of the sponsor. This happens because informational rents, and not only default costs, are linked to the probability of default. Thus, mechanisms that give rise to higher probability of default may also result in higher informational rents. We have provided sufficient conditions under which the sponsor will always prefer to curtail price competition somehow when the space of types is a continuum. These are far from necessary. By considering a finite type space, we have argued that limited liability may give some foundations to the usual practice in public procurement of blunting price competition even if default costs are low. However just what sort of limits to price competition are appropriated is a delicate issue. For instance, our discussion shows that price floors, which are commonly used by some public procurers and tend to pool firms at the lower levels of financial strength, may be counterproductive.[19]

Our analysis has abstracted from efficiency differentials. Allocation efficiency is usually appropriately managed by price competition. Thus, our results should be handled with care. When efficiency differentials are suspected to be important, one needs to search for the optimal balance between the two goals of selecting the most efficient firm and the financially fittest. This is an issue left for further research.

[19] Another mechanism that is commonly used to limit price competition is the average bid auction in which the bidder closest to the average bid wins. Decarolis (2008) provides empirical evidence using data on the road construction Industry in Italy that the average bid auctions have generated both significant inefficiencies in contract's allocation and high costs of procurement.

6 Appendix

Lemma 1 For any IC, IR mechanism (σ, P), there exists a mechanism $\left(\sigma, \overline{P}\right)$, where $\overline{P}_i(A)$ is constant on A_{-i}, $\left(\sigma, \overline{P}\right)$ is also IC and IR, results in the same expected profits for each firm i and each value of A_i, and results in (weakly) expected lower cost for the sponsor.

Proof. For any given $A_i > 0$ such that $E_{A_{-i}}\sigma_i(A) > 0$, and for all A_{-i}, and slightly abusing notation, define $\overline{P}_i(A)$ as the solution to the following equation in $\overline{P}$

$$E_{A_{-i}}\sigma_i(A)\int_0^{\min\{1,\overline{P}+A_i\}}(\overline{P}+A_i-c)dc = \tag{14}$$

$$E_{A_{-i}}\sigma_i(A)\int_0^{\min\{1,P_i(A)+A_i\}}(P_i(A)+A_i-c)dc. \tag{15}$$

(Note that the right hand side equals $U_i(A_i;\sigma,P)+E_{A_{-i}}\sigma_i(A)A_i$.) We first show that this number exists. Define $f(\overline{P}) = \int_0^{\min\{1,\overline{P}+A_i\}}(\overline{P}+A_i-c)dc$. It is easy to see that $f(\overline{P})$ is strictly increasing in $\overline{P}$ as long as $\overline{P} \geq -A_i$, since $E_{A_{-i}}\sigma_i(A) > 0$. Also, trivially, $f(\overline{P})$ is continuous for $\overline{P} \neq 1-A_i$. For $\overline{P} = 1-A_i$, $f(\overline{P})$ has a kink but the right and left limits (exist and) coincide: $\frac{1}{2}$. Also, for $\overline{P} = -A_i$ the left hand side takes a value of 0, whereas for $\overline{P} = \sup_{A_{-i}} P_i(A)$ it takes a value (weakly) larger than the right hand side. Since by IR the right hand side is at least as large as $E_{A_{-i}}\sigma_i(A)A_i > 0$, we conclude that indeed $\overline{P}_i(A_i)$ exists and is unique. Finally, for any given A_i so that $E_{A_{-i}}\sigma_i(A) = 0$ we can define $\overline{P}_i(A)$ as any number. And if $\underline{A} = 0$ and $E_{A_{-i}}\sigma_i(A)\int_0^{\min\{1,P_i(A)+A_i\}}(P_i(A)+A_i-c)dc > 0$, for $A_i = \underline{A}$ we define $\overline{P}_i(A)$ as we defined it for the case $A_i > 0$.

Note that IR and IC of the mechanism $\left(\sigma, \overline{P}\right)$ follow trivially. Now we

prove the last claim, namely that the expected cost with $\sigma, \overline{P}$ is lower. Let $C(\sigma', P')$ represent this expected cost for an arbitrary mechanism (σ', P'). Then, substituting the definition of $\overline{P}$, note that

$$C(\sigma, P) - C(\sigma, \overline{P}) = C_B \times$$
$$\textstyle\sum_i E_{A_i} \left\{ E_{A_{-i}} \sigma_i(A) \left[(1 - \min\{1, P_i(A) + A_i\}) - (1 - \min\{(1, \overline{P}_i(A_i) + A_i\}) \right] \right\}$$

Let us consider any i, and any A_i. If $\overline{P}(A_i) + A_i \geq 1$, then

$$\begin{aligned} & E_{A_{-i}} \sigma_i(A) \left[(1 - \min\{1, P_i(A) + A_i\}) - (1 - \min\{(1, \overline{P}_i(A_i) + A_i\}) \right] \\ = \;& E_{A_{-i}} \sigma_i(A) \left[(1 - \min\{1, P_i(A) + A_i\}) \right] \geq 0. \end{aligned}$$

Assume that $\overline{P}(A_i) + A_i \leq 1$. Then

$$\begin{aligned} & E_{A_{-i}} \sigma_i(A) \int_0^{\min\{1, P_i(A) + A_i\}} (P_i(A) + A_i - c) dc \\ \geq \;& E_{A_{-i}} \sigma_i(A) \int_0^{\min\{1, P_i(A) + A_i\}} (\min\{1, P_i(A) + A_i\} - c) dc \end{aligned}$$

Thus,

$$\begin{aligned} & E_{A_{-i}} \sigma_i(A) \int_0^{\overline{P} + A_i} (\overline{P} + A_i - c) dc \\ \geq \;& E_{A_{-i}} \sigma_i(A) \int_0^{\min\{1, P_i(A) + A_i\}} (\min\{1, P_i(A) + A_i\} - c) dc. \end{aligned}$$

and integrating by parts in both sides, we have

$$E_{A_{-i}} \sigma_i(A) \int_0^{\overline{P} + A_i} c dc \geq E_{A_{-i}} \sigma_i(A) \int_0^{\min\{1, P_i(A) + A_i\}} c dc.$$

Let $\Phi \subset \left[\underline{A}, \overline{A}\right]^{N-1}$ be the set of A_{-i} vectors where $\overline{P} + A_i \geq \min\{1, P_i(A) +$

$A_i\}$, and Φ^C be its complement to $\left[\underline{A}, \overline{A}\right]^{N-1}$. Then we can write

$$\begin{aligned}
& E_{A_{-i}}\sigma_i(A)\int_0^{\overline{P}+A_i} cdc - E_{A_{-i}}\sigma_i(A)\int_0^{\min\{1,P_i(A)+A_i\}} cdc \\
= & \int_{\Phi}\left(\sigma_i(A)\int_{\min\{1,P_i(A)+A_i\}}^{\overline{P}+A_i} cdc\right) dF_{-i}(A_{-i}) \\
& -\int_{\Phi^C}\left(\sigma_i(A)\int_{\overline{P}+A_i}^{\min\{1,P_i(A)+A_i\}} cdc\right) dF_{-i}(A_{-i}) \\
\leq & \int_{\Phi}\left(\sigma_i(A)\left(\overline{P}+A_i-\min\{1,P_i(A)+A_i\}\right)\left(\overline{P}+A_i\right)\right) dF_{-i}(A_{-i}) \\
& -\int_{\Phi^C}\left(\sigma_i(A)\left(\min\{1,P_i(A)+A_i\}-\left(\overline{P}+A_i\right)\right)\left(\overline{P}+A_i\right)\right) dF_{-i}(A_{-i}) \\
= & E_{A_{-i}}\left[\sigma_i(A)\left(-\min\{1,P_i(A)+A_i\}+\left(\overline{P}+A_i\right)\right)\left(\overline{P}+A_i\right)\right] \\
= & \left(\overline{P}+A_i\right)\cdot E_{A_{-i}}\sigma_i(A)\left(-\min\{1,P_i(A)+A_i\}+\left(\overline{P}+A_i\right)\right),
\end{aligned}$$

where $F_{-i}(A_{-i})$ represents the marginal of F on the $-i$ dimensions. The inequality is obtained by substituting in the first integral the sup of c in the interval for c, and in the second integral the inf of c in the interval for c. The last equality follows from the fact that $\left(\overline{P}+A_i\right)$ is independent of A_{-i}.Thus,

$$\begin{aligned}
0 & \leq E_{A_{-i}}\sigma_i(A)\left(-\min\{1,P_i(A)+A_i\}+\left(\overline{P}+A_i\right)\right) \\
& = E_{A_{-i}}\sigma_i(A)\left(1-\min\{1,P_i(A)+A_i\}-\left(1-\left(\overline{P}+A_i\right)\right)\right).
\end{aligned}$$

Now, taking expected value with respect to A_i and adding for all i, we have

$$\begin{aligned}
& 0 \leq \\
& \textstyle\sum_i E_{A_i}\left\{E_{A_{-i}}\sigma_i(A)\left[(1-\min\{1,P_i(A)+A_i\})-(1-\min\{(1,\overline{P}_i(A_i)+A_i\})\right]\right\}
\end{aligned}$$

and the result follows. ■

Lemma 2 If (σ, P) is IC, then $U_i(A_i;\sigma,P)$

Proof. Continuity is trivial. Now let $A_i > A_i^{'}$. From IC,

$$\begin{aligned} & U_i(A_i;\sigma,P) - U_i(A_i^{'};\sigma,P) \leq \\ & \Psi_i(A_i)\left[\int_0^{\min\{1,P_i(A_i)+A_i\}}(P_i(A_i)+A_i-c)dc - A_i\right] - \\ & \Psi_i(A_i)\left[\int_0^{\min\{1,P_i(A_i)+A_i^{'}\}}(P_i(A_i)+A_i^{'}-c)dc - A_i^{'}\right] \end{aligned} \tag{16}$$

If $\min\{1, P_i(A_i)+A_i^{'}\} = 1$ then $\min\{1, P_i(A_i)+A_i\} = 1$ as well, since $A_i > A_i^{'}$. Then

$$U_i(A_i;\sigma,P) - U_i(A_i^{'};\sigma,P) \leq \Psi_i(A_i)\left[\int_0^1 (A_i - A_i^{'})dc - A_i + A_i^{'}\right] = 0.$$

On the other hand, if $\min\{1, P_i(A_i) + A_i^{'}\} = P_i(A_i) + A_i^{'} < 1$, then the right hand side of (16) is equal to

$$\begin{aligned} = \; & \Psi_i(A_i)\int_{P_i(A_i)+A_i^{'}}^{\min\{1,P_i(A_i)+A_i\}}(P_i(A_i)-c)dc \\ & -\Psi_i(A_i)\left[A_i\left[1-\min\{1,P_i(A_i)+A_i\}\right] + A_i^{'}\left[1-\left(P_i(A_i)+A_i^{'}\right)\right]\right] \\ < \; & -\Psi_i(A_i)A_i^{'}\left[\min\{1,P_i(A_i)+A_i\} - \left(P_i(A_i)+A_i^{'}\right)\right] \\ & -\Psi_i(A_i)\left[A_i\left[1-\min\{1,P_i(A_i)+A_i\}\right] + A_i^{'}\left[1-\left(P_i(A_i)+A_i^{'}\right)\right]\right] \\ = \; & -\Psi_i(A_i)\left(A_i - A_i^{'}\right)\left[1-\min\{1,P_i(A_i)+A_i\}\right] \leq 0. \end{aligned}$$

The first inequality follows from the fact that $P_i(A) - c$ is decreasing in c and therefore the value of $P_i(A) - c$ is highest when $c = P_i(A_i) + A_i^{'}$, the lower limit of the integral. Also, if $\min\{1, P_i(A_i) + A_i\} < 1$, the last inequality is strict and therefore the result follows. ∎

Lemma 3 In any IC mechanism $\Psi_i(A_i)$ is monotonically decreasing and $P(A_i)$ monotonically increasing.

Proof. Let $A_i' > A_i$. Define for arbitrary values P, x

$$\Pi_i(P,x) = \int_0^{\min\{1,P+x\}} (P+x-c)dc - x$$

Then by incentive compatibility

$$\Psi(A_i)\Pi_i(P(A_i),A_i) \geq \Psi(A_i')\Pi_i(P(A_i'),A_i) \quad (17)$$

$$\Psi(A_i')\Pi_i(P(A_i'),A_i') \geq \Psi(A_i)\Pi_i(P(A_i),A_i') \quad (18)$$

Combining equations (17) and (18) we get

$$\begin{aligned}&\Psi(A_i)\left[\Pi_i(P(A_i),A_i) - \Pi_i(P(A_i),A_i')\right] \geq \quad (19)\\ &\Psi(A_i')\left[\Pi_i(P(A_i'),A_i) - \Pi_i(P(A_i'),A_i')\right]\end{aligned}$$

Note that $\Pi_i(P,A_i)$ is decreasing in A_i and increasing (non-decreasing) in P. Also, $\Pi_i(P,A_i) - \Pi_i(P,A_i')$ is decreasing in P. Indeed,

$$\Pi_i(P,A_i) - \Pi_i(P,A_i') =$$

$$= \begin{cases} \int_{P+A_i}^{P+A_i'}(c-P-A_i)dc + \int_{P+A_i'}^{1}(A_i'-A_i)dc & \text{if } P < 1-A_i', \\ \int_{P+A_i}^{1}(c-P-A_i)dc & \text{if } 1-A_i' \leq P < 1-A_i, \\ 0 & \text{if } P \geq 1-A_i \end{cases}$$

For $P < 1-A_i$ we can calculate the derivative with respect to P,[20] namely:

$$\frac{d\left[\Pi_i(P,A_i) - \Pi_i(P,A_i')\right]}{dP} = -\left[\max(1,P+A_i') - (P+A_i)\right] < 0$$

Now, assume for contradiction that $P(A_i') < P(A_i)$ which since $\Pi_i(P,A_i) - \Pi_i(P,A_i')$ is decreasing in P implies that

$$\left[\Pi_i(P(A_i),A_i) - \Pi_i(P(A_i),A_i')\right] \leq \left[\Pi_i(P(A_i'),A_i) - \Pi_i(P(A_i'),A_i')\right] \quad (20)$$

[20] The only pontentially problematic point might be at $P = 1 + A'$. The function turns out to be differentiable also at this point since the derivative from the left and the right coincide.

Given (20) the inequality (19) can only be satisfied if $\Psi(A_i') \leq \Psi(A_i)$. But $P(A_i') < P(A_i)$ and $\Psi(A_i') \leq \Psi(A_i)$ violates the incentive compatibility constraint of type A_i' (equation 18). Hence $P(A_i') > P(A_i)$. But then $\Psi(A_i') \leq \Psi(A_i)$. Otherwise the incentive compatibility constraint of type A_i would be violated (equation 17).

■

Lemma 4 Two mechanisms that share $\Psi_i(A_i)$ and give the same rents to bidders of the highest type also share $P_i(A_i)$.

Proof. First we show that in any IC mechanism U_i is Lipschitz continuous. Assume otherwise, so that for any K there exist a and b in $\left[\underline{A}, \overline{A}\right]$ with $b > a$, so that $U_i(a; \cdot) - U_i(b; \cdot) > K(b-a)$. But

$$\begin{aligned}
& U_i(a;\cdot) - U_i(b;\cdot) = \Psi_i(b)b - \Psi_i(a)a + \\
& \Psi_i(a)\left[\int_0^{P_i(a)+a} (P_i(a) - c + a)dc\right] - \Psi_i(b)\left[\int_0^{P_i(b)+b} (P_i(b) - c + b)dc\right]
\end{aligned} \tag{21}$$

Now, consider the IC constraint for type b, and abusing notation slightly, let $\widetilde{U}_i(x, y)$ denote the utility of a type y that behaves as type x. With this notation,

$$\begin{aligned}
& \widetilde{U}_i(a, b) - \widetilde{U}_i(b, b) \\
= \; & \Psi_i(a)\left[\int_0^{P_i(a)+b} (P_i(a) - c + b)dc - b\right] \\
& -\Psi_i(b)\left[\int_0^{P_i(b)+b} (P_i(b) - c + b)dc - b\right]
\end{aligned}$$

$$\begin{aligned}
&> b\left(\Psi_i(b) - \Psi_i(a)\right) + \Psi_i(a)\left[\int_0^{P_i(a)+a} (P_i(a) - c + a)dc\right] \\
&\quad -\Psi_i(b)\left[\int_0^{P_i(b)+b} (P_i(b) - c + b)dc\right] \\
&> K\left(b-a\right) - \left[\Psi_i(b)b - \Psi_i(a)a\right] + b\left(\Psi_i(b) - \Psi_i(a)\right) \\
&= \left(K - \Psi_i(a)\right)\left(b-a\right),
\end{aligned}$$

where the first inequality comes from substituting b for a in the first integral term, and the second inequality comes from using (21) and the assumption $U_i(a;\cdot) - U_i(b;\cdot) > K\left(b-a\right)$. We have established that $\widetilde{U}_i(a,b) - \widetilde{U}_i(b,b) > \left(K - \Psi_i(a)\right)\left(b-a\right)$. This, however, for $K > 1$, contradicts IC. Hence, we have proved by contradiction that in any IC mechanism, $U_i(A_i;\sigma,P)$ is Lipschitz continuous and therefore absolutely continuous. Since $U_i(A_i;\sigma,P)$ is monotone and therefore U_i' exists almost everywhere, by absolute continuity $U_i(A_i;\sigma,P) = U_i(\overline{A};\sigma,P) - \int_{A_i}^{\overline{A}} U_i'(x;\sigma,P)dx$. From the envelope theorem (IC), this also means that $U_i(A_i;\sigma,P) = U_i(\overline{A};\sigma,P) + \int_{A_i}^{\overline{A}} \Psi_i(x)(1-(P_i(x)+x))dx$. Using these results, we now show that $U_i(A_i;\sigma,P)$ is completely determined by $\Psi_i(A_i)$ and $U_i(\overline{A};\sigma,P)$. Assume this is not true, so that there exist two mechanisms (σ,P) and $(\sigma,\widehat{P})$ both of them feasible (i.e., IC), such that $U_i(\overline{A};\sigma,P) = U_i(\overline{A};\sigma,\widehat{P})$ and such that $\widehat{P}_i \neq P_i$ for some i. Since $U_i(\overline{A};\sigma,P) = U_i(\overline{A};\sigma,\widehat{P})$, and U_i is continuous for both mechanisms, that means that there exists an arbitrarily small interval $[a,b]$ where $U_i(b;\sigma,P) = U_i(b;\sigma,\widehat{P})$ and $U_i(A_i;\sigma,P) > U_i(A_i;\sigma,\widehat{P})$ for all $A_i \in [a,b)$, or the same is true when we reverse the positions of $\widehat{P}$ and P. Without loss of

generality, let us assume the former case. That is,

$$\int_a^b \Psi_i(x)(1-(P_i(x)+x))dx > \int_a^b \Psi_i(x)(1-(\widehat{P}_i(x)+x))dx.$$

This implies that, for some x in $[a,b)$, $\Psi_i(x)(1-(P_i(x)+x) > \Psi_i(x)(1-(\widehat{P}_i(x)+x))$, i.e., $P_i(x) < \widehat{P}_i(x)$. But, from the definition of U_i,

$$U_i(A_i;\sigma,P) = \Psi_i(A_i)\left[\int_0^{P_i(A_i)+A_i}(P_i(A_i)-c+A_i)dc - A_i\right],$$

this implies that $U_i(x;\sigma,P) < U_i(x;\sigma,\widehat{P})$, a contradiction that proves that $U_i(A_i;\sigma,P)$ is completely determined by $\Psi_i(A_i)$ and $U_i(\overline{A};\sigma,P)$ from which the lemma follows. ■

Lemma 5 Assume $\overline{A} > \frac{1}{2}$. A first price auction has a symmetric equilibrium, $b(A)$, where $b(A)$ is strictly increasing in A for $A < \frac{1}{2}$, and $b(A) = \frac{1}{2}$ for $A \geq \frac{1}{2}$.

Proof. If contractors use the common, differentiable bidding strategy $b(A)$, strictly increasing in A for $A < \frac{1}{2}$, then the expected rents for a contractor with assets $A < \frac{1}{2}$ that bids $b(A')$ are

$$\pi(A',A) \equiv \left[\int_0^{b(A')+A}(b(A')+A-c)dc - A\right](1-F(A'))^{n-1}.$$

The first order condition for $b(A)$ to be best response to itself is then

$$\left.\frac{\partial \pi}{\partial A'}\right|_{A'=A} = 0, \tag{22}$$

where

$$\frac{\partial \pi}{\partial A'} = b'(A')(b(A')+A)(1-F(A'))^{n-1} - \\ f(A')(n-1)(1-F(A'))^{n-2}\left[\int_0^{b(A')+A}(b(A')+A-c)dc - A\right]. \tag{23}$$

The differential equation (22) can be written as

$$b' = \frac{\int_0^{b+A} (b + A - c) dc - A}{\frac{1-F(A)}{f(A)} \frac{b+A}{n-1}}, \tag{24}$$

which together with the condition $b(\frac{1}{2}) = \frac{1}{2}$ defines an initial value problem that satisfies the conditions for existence and uniqueness of a solution. The numerator and denominator of (24) are both positive, and therefore the solution is strictly increasing. Finally, the second order conditions are also satisfied. Indeed, (22) defines an identity in A so that $\frac{d}{dA}\left[\frac{\partial \pi}{\partial A'}\big|_{A'=A}\right] = 0$. Thus, checking that $\frac{\partial}{\partial A'}\frac{\partial \pi}{\partial A'}\big|_{A'=A} \leq 0$ is equivalent to checking that $\frac{\partial}{\partial A}\frac{\partial \pi}{\partial A'}\big|_{A'=A} \geq 0$. And indeed,

$$\begin{aligned} \frac{\partial}{\partial A}\frac{\partial \pi}{\partial A'} &= b'(A')\left(1 - F(A')\right)^{n-1} \\ &\quad - f(A')(n-1)\left(1 - F(A')\right)^{n-2}\left(b(A) + A - 1\right), \end{aligned}$$

which is positive at $A' = A$, since $b(A) + A < 1$ at $A < \frac{1}{2}$. For $A \geq \frac{1}{2}$ it is straightforward to show that the best response is indeed $b(A) = \frac{1}{2}$, which induces zero expected rents conditional on winning the auction: any lower bid induces negative expected rents. This concludes the proof. ■

Lemma 6 For any $C_B \geq 0$, there exist non strictly monotone, IC mechanisms that result in higher surplus for the buyer than mechanisms that assign the contract to the bidder with lowest A (auctions) if $f'(A)$ exists and is negative for A close to $\overline{A}$.

Proof. Given $\Psi(A) = [1 - F(A)]^{n-1}$, and $P(A)$, where we dispense with subscripts by virtue of symmetry, construct a new mechanism $\Psi^*(A), P^*(A)$ where $\Psi^*(A) = \Psi(A)$ for all $A \leq A^*$, and for all $A > A^*$, $\Psi^*(A) = \frac{[1-F(A^*)]^{n-1}}{n}$, for some value A^*. Note that Ψ^* is obtained by assigning the

contract to the firm with lowest type if that type is below A^*, and otherwise randomizing the allocation. That is, σ^* exists. Also, for $A > A^*$, let $P^*(A) = P(\overline{A})$, and $P^*(A^*)$ be such that $U(A)$ is continuous at A^*. This completely determines the mechanism. Moreover, the mechanism satisfies IC and IR. We now show that for A^* sufficiently close to $\overline{A}$ the firms' rents are lower in the new mechanism than in the original. In fact, it suffices to show that, for A^* sufficiently close to $\overline{A}$, the rents of a bidder with type A^* in the auction, $U(A^*)$, are larger than in the new mechanism, $U^*(A^*)$. Indeed, the slope of the rents in both mechanisms is the same for types below A^*, and the difference in rents $U^*(A) - U(A)$ for types above A^* is bounded above by $U^*(A^*)$. For A^* sufficiently close to $\overline{A}$, both $U^*(A^*)$ and $(1-F(A^*))$ approach zero, and therefore its product is of second order. So, let us consider

$$
\begin{aligned}
&U^*(A^*) - U(A^*) = \\
&\int_{A^*}^{\overline{A}} [1 - (P(\overline{A}) + A)] \frac{[1 - F(A^*)]^{n-1}}{n} dA \\
&- \int_{A^*}^{\overline{A}} [1 - (P(A) + A)][1 - F(A)]^{n-1} dA.
\end{aligned}
$$

The derivative of this expression with respect to A^* is

$$
\begin{aligned}
&-[1 - (P(\overline{A}) + A^*)] \frac{[1 - F(A^*)]^{n-1}}{n} + [1 - (P(A^*) + A^*)][1 - F(A^*)]^{n-1} \\
&- \int_{A^*}^{\overline{A}} [1 - (P(\overline{A}) + A)] dA \frac{n-1}{n} f(A^*)[1 - F(A^*)]^{n-2}.
\end{aligned}
$$

This is $\frac{1}{n} f(A^*)[1 - F(A^*)]^{n-2}$ times

$$
\begin{aligned}
&\left(-[1 - (P(\overline{A}) + A^*)] + n[1 - (P(A^*) + A^*)]\right) \frac{1 - F(A^*)}{f(A^*)} \qquad (25) \\
&+(n-1) \int_{A^*}^{\overline{A}} [1 - (P(\overline{A}) + A)] dA.
\end{aligned}
$$

The derivative of (25) with respect to A^* is

$$\begin{aligned}
&(1 - n[P'(A^*) + 1])\frac{1 - F(A^*)}{f(A^*)} + (n-1)[1 - (P(\overline{A}) + A^*)] + \\
&\frac{d}{dA^*}\left[\frac{1 - F(A^*)}{f(A^*)}\right]\left(-[1 - (P(\overline{A}) + A^*)] + n[1 - (P(A^*) + A^*)]\right).
\end{aligned}$$

The first term is negative, since $P^* > 0$. Also, if $f' < 0$ then $\frac{d}{dA^*}\left[\frac{1-F(A^*)}{f(A^*)}\right] < -1$, and so the second and third terms together are smaller than

$$n\left([1 - (P(\overline{A}) + A^*)] - [1 - (P(A^*) + A^*)]\right) < 0.$$

Thus, (25) is decreasing in A^* and attains a value of 0 at $A^* = \overline{A}$. Therefore, for A^* close to $\overline{A}$ it takes a positive value. This proves that $U^*(A^*) - U(A^*)$ is increasing in A^* for values of A^* close to $\overline{A}$. Since $U^*(A^*) - U(A^*) = 0$ for $A^* = \overline{A}$, we conclude that $U^*(A^*) < U(A^*)$ for values of A^* close to $\overline{A}$. This proves the result. ■

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