Gradualism in dynamic agenda formation

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Abstract

We analyze a dynamic model of agenda formation in which players compete in each period to place their ideal policies on the agenda. In each period, with some probability, a decision maker takes an action from the agenda. We show that in any markov equilibrium of this game, players with extreme ideal policies will always compete to be in the agenda. On the other hand, there is a positive probability that in each round a more moderate policy will arise on the agenda. Therefore, agenda formation is a gradual process which evolves to include better policies for the decision maker but at a relatively slow pace.

1 Introduction

The process of group decision making involves two main (possibly intertwined) sub-processes: that of the formation of a set of alternatives to choose from (which below we refer to as the *agenda*) and that of choosing an option from this set. In some cases, the agenda might be exogenously given. Often however, decision makers are not aware of the feasible options at hand. A newly elected President who has to tackle major issues such as health reform or climate change, is usually not an expert on the subject and must be introduced to the feasible policies. The agenda formation process plays then an important role in bringing the set of feasible options to his or her attention.

Our main assumption is that the agenda formation process is decentralized. Policies are suggested by interested parties whereas the decision maker has no control on

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the content of the agenda. Suggesting a policy may however be costly, as time or attention constraints of the decision maker often induce interested parties to compete -by exerting resources- to place policies on the agenda.[1] In this paper we focus on the dynamic competition between such interested parties.

To fix ideas let us think about the public debate about climate change. In the United States, before the 1980's, the general public was unaware about the evidence or the main parameters of the debate. The intense drought and heat wave in 1988 were the first instances in which scientists publicly claimed that such phenomena were evidence of climate change. Since then, interested parties have used costly media campaigns to bring their favorite policies to the public's attention; among others, bio-fuels producers push for subsidies for producing their own alternative fuel, climate sceptics and polluting industries push towards protecting the status quo, and economists suggest market mechanisms such as taxes on emissions or trading in pollution permits. But how representative are the current policies on the agenda? Are the bio-fuels discussed the best possibilities that are available? Are the new grid feed-in tariffs for solar panels in the UK the type of policy that should be discussed?[2]

More generally, in this paper we will focus on the following questions. Do the best policies arise on the agenda? If they do, how long does it take for these policies to be put forward? What determines the types of policies that are discussed first? How do interests with relatively moderate policy prescriptions fair against more extreme interests?

We propose an infinite horizon model in which a decision maker has to choose an alternative from an agenda. The timing of the decision is stochastic: in each period, with probability ρ, the decision maker will choose his best alternative from the agenda and with probability $1-\rho$ the game continues to the next period. The parameter ρ captures the (stochastic) length of the decision making process.

A finite number of interested players, each with single-peaked preferences on the one-dimensional policy space, try to influence the agenda. In each period, players play an influence game whose winner adds his ideal policy to the agenda. In the

[1] See Cox (2006) for an argument about the importance of time and attention constraints in legislatures. Motivated by such constraints, Copic and Katz (2007) analyze the choice of a set of alternatives that in a later stage would become the agenda for the legislature (see also Barbera and Coelho 2009).

[2] For more on this debate see, "Are we really going to let ourselves be duped into this solar panel rip-off?", *Guardian*, 1st of March 2010.

influence game, players simultaneously take costly actions, and the probability that a player wins is an exogenous function of the vector of costly actions. The family of influence games we consider includes many all-pay mechanisms that have been used in the literature (such as all-pay auctions or the Tullock influence functions).[3] We analyze Markov perfect equilibria of this game where the state is defined as the favorite policy of the decision maker on the current agenda.

The dynamic model brings to the fore the tension between extreme and moderate players. Extreme players are willing to compete harder than others, due to negative externalities. These imply that in short decision making processes (for example when $\rho = 1$) the agenda will be polarized.

Moderate players on the other hand represent better policies for the decision maker. The advantage of moderate players is stronger in long decision making processes: In these cases they need to win fewer contests in order to be chosen by the decision maker compared with extremists, who must repeatedly win contests in order to crowd out better policies. We show that nonetheless, extremists never "give up": Our main result is that for any ρ, including arbitrarily small, at any stage in the process, there is a strictly positive probability that a player different from the most moderate will win. We also show that at any stage there is a substantial probability that a new and more moderate policy will be added to the agenda. Thus the agenda evolves forward with a positive probability, but in a gradual, or relatively slow, manner.

The gradual evolution of the agenda rests on the balancing of the short-run and long-run motivations for placing a policy on the agenda. The short-run motivation relates to the choice of the decision maker today. When the agenda has not fully evolved, many players -even extremists- could potentially influence this choice. On the other hand, players must consider the long-run effects of their actions, if the decision is not taken today. If the process is not gradual, the most moderate player will be on the agenda fairly quickly. Players should then expect their influence on the future to be rather small, and as a result they would tend to act mainly on the basis of the short-run considerations. But these induce relatively extreme players to bid aggressively so that the most moderate player cannot be on the agenda too quickly; thus, any agenda formation process must be gradual and some extreme players are

[3] Becker (1983) laid down the framework of influence functions. See also Skaperdas (1996) and the survey in Konrad (2009).

always active.

We further explore how the parameters of the model - the distribution of ideal policies and the players' preferences - affect the degree of gradualism. We focus on a particular equilibrium, the "fully gradual equilibrium", in which any player becomes active only after all more extreme players have already placed their policy on the agenda. With three players, and simple all-pay-competition functions (in particular, all-pay-auction and a simple Tullock contest function), we show that the existence of such an equilibrium depends on relative rather than absolute polarization in the distribution of ideal policies. In particular, this equilibrium is more likely to exist when an extreme player's ideal policy becomes more extreme. Moreover, this equilibrium does not exist when players care only about winning themselves (office motivation). Thus, negative externalities play an important role in inducing gradualism.

Our model combines two strands in the political economy literature, the one on endogenous agenda formation[4] and the one on influence games[5]. We differ from the first line of literature by focusing on all pay competitions to determine the right to place a policy on the agenda. The second line of research analyzes the direct influence of policy while we focus on indirect influence, via the efforts to place policies on the agenda.[6]

Our model is also related to Osborne *et al* (2004), Osborne and Slivinski (1996) and Besley and Coate (1997) who have analyzed endogenous entry in political models. In contrast to these papers our central focus is on the endogenous cost of entry and on the dynamics of entry.

Other papers have analyzed gradualism in different contexts, albeit stemming from different reasons than the one analyzed in our model. Compte and Jehiel (2004) analyze a bargaining game in which the outside option of the players depends on previous offers. Admati and Perry (1991) show that an agent holds back his payments in contribution games to insure that the other agent contributes his share as well. In a political set up, Polborn and Klumpp (2006) analyze a dynamic competition

[4]Early contributions include Austen-Smith (1989) and Baron and Ferejohn (1989) who consider fixed processes of agents suggesting policies. Recent contributions are Duggan (2006), Penn (2008) and Dutta et al. (2004) who consider endogenous processes.

[5]Becker (1983) and Grossman and Helpman (1994) have mostly used contest functions or auctions to model how players can directly affect political outcomes.

[6]An exception in this literature which is related to our approach is Austen-Smith (1995) who assumes that lobbies need to pay a fixed exogenous "access" cost in order to be heard by a politician.

(primaries) between two candidates, via a contest function, to win different districts. Finally, in a multistage patent race game among two players, Konrad and Kovenock (2009) show that an agent who is losing in the patent race still does not give up, as long as he can win some strictly positive instantaneous prize. A similar result arises in our analysis even when the instantaneous prize converges to zero.

The rest of the paper is organized as follows. In the next section we present the model. In Section 3 we present our main result of gradualism. Section 4 considers the effects of different parameters on gradualism, while Section 5 discusses some possible extensions. All proofs that are not in the text are in an appendix.

2 The model

A set of players, $N = \{1, 2, ..., n\}$, are trying to influence a final policy, y, by placing their ideal policies on an evolving agenda. player i's ideal policy is denoted by $x_i \in [-1, 1]$ and the final policy satisfies $y \in \{x_1.., x_i, .., x_n\}$. The utility of player i from the final policy y is $u_i(y) = -|x_i - y|$.[7] Without loss of generality, let $x_1 = 0$, and $|x_i| < |x_{i+1}|$ for all i.

At any stage t in the (infinite) dynamic game, the players engage in an all-pay competition whose details we specify below. The winner of the competition at stage t places his ideal policy x^t on the agenda. The agenda at time t, $A^t \subseteq \{x_1, .., x_i, .., x_n\}$, evolves in the following way: $A^0 = x_n$, $A^t = A^{t-1} \cup x^t$.[8] Note that all players can compete at any stage even if their policy is already on the agenda. After any stage t, with probability $\rho \in (0, 1)$ the game terminates. At the termination node, a decision maker chooses the policy in A^t that is closest to his ideal policy. Assume without loss of generality that his ideal policy is at zero. With probability $1 - \rho$ the game continues to stage $t + 1$.

We now describe the all-pay-competition that the players play at each stage. In this competition each player i places a bid $b_i \geq 0$ which he must pay regardless of the outcome. Bids are placed simultaneously. The probability with which player i wins the competition at stage t is determined according to a function $H_i(b)$, where b is the vector of bids. We assume that the function $H_i(b)$ satisfies the following properties:

[7]This simple environment is presented for expositional purposes. Theorem 1 below holds for more general utility specifications with or without negative externalities.

[8]The results remain the same if $A^0 = \phi$, provided that utilities are well defined on A^0.

H1. $\sum_{i\in N} H_i(b) = 1$

H2. For any $K > 0$, there exists a $K' > 0$ such that if $b_i = \max_j b_j$ and $\frac{b_i}{b_j} > K'$ then $\frac{H_i(b)}{H_j(b)} > K$.

H3. Monotonicity: $H_i(b)$ (weakly) increases in b_i and (weakly) decreases in b_j for $j \neq i$.

Assumption H1 is made for expositional purposes. Assumption H2 is a weaker version of a requirement that if one player bids infinitely more than another player in relative terms, then he must win with a probability that is infinitely large than that of the other player. H3 is a standard monotonicity requirement implying that it is costly to influence decisions.[9] The above set of assumptions are general enough to include many of the functional forms used in the literature, including the generalized Tullock contests,[10] and the all-pay-auction mechanism. Throughout the paper we will sometimes illustrate our results using the following two H functions:

Example 1 *(All Pay Auction)*

$$H_i(b) = \begin{cases} \frac{1}{|\arg\max b_j|} \ \textit{if } i \in \arg\max b_j \\ 0 \ \textit{if } b_i < \max b_j \end{cases}$$

Example 2 *(A simple contest function)*

$$H_i(b) = \begin{cases} \frac{b_i}{\sum_j b_j} \ \textit{If } \exists b_j > 0 \\ \frac{1}{n} \ \textit{otherwise} \end{cases}$$

[9] We find monotonicity to be a natural assumption for the application at hand, although it is quite strong. Instead of both H1 and H3, we can make a weaker assumption. The important feature for our proof is that players who have a marginal probability of winning a competition can only have marginal effects on others' chances when they withdraw their bids. We discuss this further after we present Theorem 1.

[10] See Skaperdas (1996) for an axiomatic approach that imposes more conditions on the general H and yields the generalized Tullock contest. Skaperdas (1996)'s axiomatization uses an independence axiom. Clark and Riis (1998) also use independence and homogeneity and lose anonymity to get non anonymous Tullock functions.

The all-pay feature is a relevant one in political economy, where agents invest efforts and resources to gain the attention of (or access to) a decision maker. In these circumstances, explicit contracts cannot be legally written or enforced and so these efforts must be taken up-front. As it is not transparent how the decision maker assigns access or attention, we use a general H function for deriving our main results (we solve specific examples in Section 4).

Let $J^t \in \{1, ..., n\}$ be the index of the player whose ideal policy the decision maker would choose from the agenda A^{t-1} (the "most moderate" policy in A^{t-1}). The utility of player i from the game is therefore

$$\sum_{t=1}^{\infty}(1-\rho)^{t-1}(\rho EU_i^t(b^t) - b_i^t)$$

where

$$EU_i^t(b^t) = \sum_{j=1}^{n} H_j(b) u_i(x_{\min\{j,J^t\}})$$

Finally, we say that a player is *active* in period t (in some equilibrium) if the measure of non-zero bids in his support is strictly positive in this equilibrium.

We will focus our analysis on Markov Perfect Equilibria in which players condition their bids, on and off the equilibrium path, only on the state variable J^t and ignore both the time index t and past histories. When H is continuous our model satisfies the sufficient conditions in Escobar (2008) for the existence of a mixed strategy Markov perfect equilibrium.

Proposition 1 (Escobar (2008)) Suppose H *is continuous. A Markov Perfect Equilibrium exists.*

Note that although in the all-pay-auction mechanism (Example 1) H is not continuous, we find an MPE in the examples that we solve and thus we rather specify our characterization result in Theorem 1 without imposing continuity on H.

Our main interest in this paper is to consider the dynamics of the process of agenda formation. But first we consider the benchmark of $\rho = 1$ which highlights the importance of negative externalities. These imply that some degree of polarization always arises in the agenda formation process. For simplicity we focus here on distributions of preferences that include players with the most extreme ideal policies 1 and -1.

Proposition 2: *There exists an $\bar{\varepsilon} > 0$, such that in any equilibrium, for any interval $I \subset [-1, 1]$ of size $\varepsilon < \bar{\varepsilon}$, the probability that the winning policy is in I is strictly smaller than one.*

Proof of Proposition 2: Suppose by way of contradiction that for all $\bar{\varepsilon} > 0$ there exists a distribution of ideal policies, an equilibrium and an interval I of size $\varepsilon < \bar{\varepsilon}$ such that the probability that a policy from within I wins is one. Note that the willingness to pay of players within I is bounded by ε. From H1 and H3 all others must bid zero. Choose the player who is furthest from the interval I. The willingness to pay of this player is at least $(1 - \frac{\varepsilon}{2})$. By submitting a bid $k\varepsilon$, so that $k \to \infty$ and $k\varepsilon \to 0$, by H2, this player wins with a probability converging to one while his bid converges to zero. This implies an expected utility close to zero. Alternatively, in equilibrium his expected payoff is at most $-(1 - \frac{\varepsilon}{2})$ and hence he has a profitable deviation.■

Example 1 *(all-pay-auction):* In this case, $\bar{\varepsilon} = 1$. To see why, note that for any interval of smaller size there will exist a player outside this interval whose expected distance from the final policy is larger than the length of the interval. As the highest bidder wins with probability one, such a player will have a higher willingness to pay than any of the active players and will therefore deviate and submit the bid that allows him to win against all.

Example 2 *(simple contest function):* In this case $\bar{\varepsilon} = 2$; in the unique equilibrium the two most extremists are the only ones to compete, and thus the probability that the winning policy is in any smaller interval is at most a half (we prove this in the Appendix).

3 Gradualism in dynamic agenda formation

We now turn our attention to the dynamic game. Our main result is concerned with the positive features of the dynamic agenda formation process: How does the agenda evolve, and how active are extremists. In short games (large ρ), as illustrated in Proposition 2, extremists have a high intensity to win: with a substantial probability the decision maker might take a decision today and being on the agenda implies instantaneous benefits. On the other hand, when ρ converges to zero, such benefits do not exist. The most moderate player has a stark advantage in this case; he needs to win only once in order to guarantee that the decision maker will choose his ideal

policy. With such high degrees of "patience" he is bound to win at some point. One may conjecture that in this case extremists will stay out of the competition, not to waste fruitless resources. Our main result shows that even when the game is long, extremists never "give up":[11]

Theorem 1 *There exists an $\varepsilon > 0$ such that in any Markov Perfect Equilibrium, for any state $J > 1$, for any ρ, (i) player 1 wins with a probability lower than $1 - \varepsilon$. (ii) Some player $i < J$ wins with a probability larger than ε.*

Theorem 1 implies that the most moderate player will eventually win but that other policies will arise on the agenda with a strictly positive probability. Even if the instantaneous benefits converge to zero, extreme players (who can potentially still be chosen by the decision maker) keep on competing for a place on the agenda. As a result the most moderate player is never guaranteed to win any stage with too large a probability.

The intuition behind this result could be understood through the decomposition of players' incentives into short-run and long-run considerations. In particular, we show that players' willingness to pay at each stage determines their decision of whether to be active or not. A typical willingness to pay of a player i at state J in the game takes the form,

$$w_i^J = \rho \tilde{X}_i^J + (1 - \rho)(V_i^{\min\{i,J\}} - \tilde{V}_i^J).$$

The first term, $\rho \tilde{X}_i^J$, is the short term incentive to win and its magnitude is of order ρ; $\tilde{X}_i^J$ is player *i's static* utility difference between winning and being inactive, i.e., if the decision maker were to pick a policy today, and is thus multiplied by ρ.

The second expression represents the long-run effect of being active. The expression $V_i^{\min\{i,J\}} - \tilde{V}_i^J$ represents the effect of today's action on future continuation values; $V_i^{\min\{i,J\}}$ is the continuation value following player i winning this period's contest while $\tilde{V}_i^J$ represents the expected continuation value if player i remains inactive. The key to the proof of the Theorem is showing that the short-run and the long-run effects are of the same magnitude when $\rho \to 0$.

To understand the magnitude of the long term consideration, $V_i^{\min\{i,J\}} - \tilde{V}_i^J$, we first prove part (ii) of the Theorem which implies that the game will endogenously end (i.e., reach $J = 1$) in finite time in expectations. In turn, as we show, this implies

[11] The proof is provided in the Appendix.

that the difference in continuation values $V_i^{\min\{i,J\}} - \tilde{V}_i^J$ is also of order ρ, as the future payoff is comprised of a finite number (in expectations) of instantaneous benefits.[12]

To see how we prove part (ii), and how our assumptions about H play a role, let us first consider a game with just two players, 1 and 2, and the state $J = 2$. Suppose to the contrary that when $J = 2$ there exists a sequence of equilibria in which player 1 wins with probability ε converging to zero (which must therefore hold for almost each bid in his support). As a result, almost every bid in the support of player 1, b_1^*, must be infinitely small than his willingness to win, i.e., $b_1^* \leq \varepsilon w_1^2$. To see why, note that withdrawing and bidding zero instead of b_1^* implies, by H1 and H3, that player 1 will lose at most ε probability of winning and in turn his relative benefit of winning, his willingness to win. As positive bids have to be weakly better than withdrawing, we get a bound on the bids of player 1 in equilibrium.

But if player 1's bids are converging to zero, player $2's$ best reply must do so as well, as he need not waste too much resources in order to win against very low bids. In particular, such a best reply is a sequence of $b_2^* = \gamma b_1^* \leq \gamma\varepsilon w_1^2$ where $\gamma \to \infty$ and $\gamma\varepsilon \to 0$, so that $b_2^* \to 0$. By H2, this guarantees winning with probability converging to one and the highest possible utility. We can therefore also derive a limit on the bids of player 2 in equilibrium.

We have concluded that a player who loses almost for sure, player 1, has a willingness to win which is infinitely higher than all equilibrium bids. This however cannot arise in equilibrium. player 1 could then place winning bids which cost infinitely less than his gain of winning - w_1^2- a profitable deviation.[13] We have therefore reached a contradiction to the proposed sequence of equilibria.

Thus, from state $J = 2$, we must move forward with a strictly positive probability so that the game will end in finite time; the long term incentives of the players must therefore be of order ρ, as this is what they can aspire to gain and the most they will be willing to pay for.[14]

EXTENDING THE PROOF OF PART (II) TO MANY PLAYERS: Lemma A1 in the appendix extends the argument above for N players and all states J. To do so we use an induction on J which relies on the structure of the game which renders it is

[12]That is, $0 < \lim_{\rho\to 0} |\frac{V_i^{\min\{i,J\}}(\rho) - \tilde{V}_i^J(\rho)}{\rho}| < \infty$.

[13]In particular, by H2, as above, he can deviate and place a bid b_1' such that $\frac{b_1'}{b_2^*} \to \infty$ and $\frac{b_1'}{w_1^2} \to 0$.

[14]Of course one has to take into account the future bids of the players. In the Appendix we show that these are of order ρ or lower.

impossible to move from some state J to any state $J' > J$.[15] Extending the above argument to more than two players involves considering several issues.

First, we have relied above on the fact that a marginal (one with a very small chance of winning) player's bid must be infinitely smaller than his willingness to win. With more than two players, when a marginal player withdraws his bid, he may substantially change the balance of power between the other players who compete. Assumptions H1 and H3 guarantee that this is not the case; alternatively, we can make a weaker (and more direct) assumption stating that when a marginal player withdraws, he does not substantially affect the ratio of the winning probabilities of other players.[16]

The presence of negative externalities complicates the proof for many players for another reason; when a losing player withdraws and shifts even a small probability to others, he may shift it to the player he fears most. His bids are therefore infinitely smaller than his willingness to win against his worst (remaining) enemy. This implies that we need to consider the worst case scenario for each player, or more generally all bilateral comparisons between players. To do this we use the inductive structure of the game.

Finally, note that when there are many players, at state J, some players with $i > J$ may still fight in order to defend J as the current state. This implies that many players might submit bids to support the same policy (as well as multiplicity of equilibria). The proof involves deriving bounds on bids for all these players.[17]

We have so far concluded that both the short-run and the long-run considerations composing the willingness to win of players are comparable and of order ρ. The implication of this is that player 1 cannot win any stage with probability converging to one (as stated in part (i)); as players' short-run and long-run incentives are comparable in equilibrium, no player has an absolute endogenous advantage which implies a quick resolution of the game.[18]

More generally, in equilibrium, the magnitude of the two effects -the short-run and the long-run- must be balanced. Intuitively, if it is expected that the most

[15] As we have done above, we also rely on the structure of the MPE.

[16] Note that this holds for the Tullock contest functions also for non-marginal players (the independence assumption).

[17] We discuss this issue more in Section 5.

[18] To show this formally, we again use the assumptions on H and repeat a similar argument to the one illustrated above to show that Player 1 cannot win with a probability converging to zero.

moderate player will be on the agenda very quickly, this will imply that $V_i^{\min\{i,J\}}$ is fairly close to $\tilde{V}_i^J$ and so the short-run incentives will play an important role. But then other more extreme players will decide to be active - as they are the ones who are most motivated to win today and realise the short term gains. But as extreme players are active, our initial supposition that the most moderate player wins quickly cannot be sustained. As a result, equilibria will always involve some short-run considerations, and relatively extreme players will always be active.

Theorem 1 establishes gradualism for a general environment, but does not tell us how slow (or quick) is the rate of gradualism. For example, although we know that at no stage player 1 wins almost for sure, it is not clear at which stage he actually becomes active. We now explore in more detail the dynamics of gradualism, the role played by negative externalities and how the distribution of players' ideal policies affects gradualism.

4 Full gradualism

In a fully gradual equilibrium, all policies are placed on the agenda before player 1 becomes active. More generally, player $i-1$ becomes active only at state i; thus, the decision maker, if the game is long enough, becomes aware of the full set of feasible options. To achieve this, at each stage, only two players can win with a strictly positive probability and the agenda evolves starting from the most extreme policies and adding more and more moderate policies:

Definition 1: In a *fully gradual equilibrium,* at any state $J>1$, only players J and $J-1$ are active.[19]

To see whether such an equilibrium exists, for simplicity, we focus on $N = \{1,2,3\}$ (results can be generalized to any set N). We first consider a distribution of policies in which $x_2 > 0$ and $x_3 < 0$ (where $|x_3| > x_2$), or a "two-sided" influence game, in which players 2 and 3 are on different sides of the most moderate player. We show that full gradualism exists when *relative* extremism ($\frac{|x_3|}{x_2}$) is large enough. In Section 4.2 we consider the case in which $x_3 > x_2 > 0$; we show that in

[19] An extension of a fully gradual equilibrium is that at some state J, player $J-1$ competes against some player $i > J$ where player i "defends" the policy of J, as when he wins a contest, the state remains J. Our results are robust to a more general definition of full gradualism and we discuss such "defending" behaviour in Section 5.

such "one-sided" influence games, full gradualism is harder to sustain. In Section 4.3 we focus on the role of negative externalities and show that full gradualism does not arise in their absence for any distribution of policies. Throughout, to obtain more specific comparative static results, we consider the H functions defined in Examples 1 and 2, the all-pay-auction and the simple contest function.

4.1 Existence of a fully gradual equilibrium

At $J = 2$, for any $|x_3| > x_2$, it is an equilibrium for only players 1 and 2 to compete, and they will keep on competing until player 1 wins. To construct a fully gradual equilibrium we therefore have to check if, at $J = 3$, it is an equilibrium for only players 2 and 3 to be active. The binding condition that we need to check is whether or not player 1 will prefer to enter the competition as well.

Consider first the case when $x_3 < 0$. When ρ is large and close to 1, it is easy to sustain full gradualism. Intuitively, players 2 and 3's willingness to win vis a vis each other is of the order of $|x_3| + |x_2|$ whereas that of player 1 is at most $|x_3|$ and he is therefore priced out of the competition. For some parameters (for example when $|x_3| \approx x_2$), in the all-pay-auction, full gradualism is the unique equilibrium when ρ is large enough.

It is not surprising that when negative externalities play a large role then large polarization in the form of full gradualism takes place in equilibrium; we show however that this is also the case when $\rho \to 0$, when player 1's dynamic advantage is stark, and can potentially outweigh negative externalities:

Proposition 4: *Suppose that $x_3 < 0$ and consider the all-pay-auction and simple contest functions. (i) The existence of a full gradual equilibrium depends only on relative distances: In particular, for all ρ, full gradualism arises when $\frac{|x_3|}{x_2}$ is large enough. (ii) Full gradualism is an equilibrium for a larger set of parameters under the all-pay-auction.*

Somewhat counter-intuitively, player 1 stays out if player 3 - which represents his worst case scenario, is located far enough from him. The reason is that the distance between these players affects mainly the willingness to win of player 3, who knows that in expected finite time the agenda will include the ideal policy of player 1 (by Theorem 1). On the other hand, when considering whether to deviate, player 1 is aware that player 2 wins the game between 2 and 3 more often (due to his advantage

of being more moderate), and thus his willingness to win in $J = 3$ is guided mainly by his distance from player 2.

It is therefore the *relative values of* x_2 *and* x_3 that are important in characterizing the equilibria above. This prediction is different from models in which the payments in equilibria are exogenously fixed (for example, the citizen-candidate or town meeting models) and so the absolute values of ideal policies matter.[20] In our model, when we polarize society for example by "stretching" both the ideal policies of 2 and 3 to maintain $\frac{|x_3|}{x_2}$ fixed, the identities of the winners of the agenda game do not change (although the magnitude of the payments changes).

When only players 2 and 3 compete, in the all-pay-auction (and more generally for highly competitive H functions), the optimal deviation -if such exists- for player 1 is to place the highest equilibrium bid. He will therefore become active if his willingness to win is higher than the highest bid, in which case he will win for sure and extract some rent. The highest bid in the game between players 2 and 3 is their minimum willingness to win. Insuring that the willingness to win of 1 is lower than this minimum provides us with the condition that polarization ensures full gradualism.

In the simple contest function, player 3 has to be even further away for player 1 to stay out. Under such function, competition is less aggressive. As a result, in the equilibrium between players 2 and 3, player 3 wins more often compared with the all-pay-auction, which encourages player 1 to become active to avoid his worst-case scenario. Moreover, as competition is less aggressive, an optimal deviation for player 1 will actually consist of a small bid which can affect the results with little effort (as opposed to the more competitive all-pay auction). This makes a deviation easier and in particular, player 1 may deviate even his willingness to win is lower than that of the other two players.

4.2 Gradualism in one-sided versus two-sided in‡uence games

We now compare between the case above in which $x_3 < 0$ -a "two sided" influence game- and $x_3 > 0$ -a "one sided" influence game. To make this comparison meaningful we fix $|x_3|$, so that the distances of players 2 and 3 from player 1 are fixed.

Proposition 5: *Fix* $|x_3|$*. For both the all-pay-auction and the simple contest function, when* $x_3 > 0$ *full gradualism is less likely to be sustained than when* $x_3 < 0$.

[20] See Besley and Coate (1997) and Osborne et al (2004).

Although $|x_3|$ is fixed, player 1 is more motivated to deviate when the distribution is one-sided. The reason is that in the supposed equilibrium between player 2 and 3, player 3 is not as disadvantaged when he is on the same side of player 2. Thus, he wins more often. This increases the willingness to win of player 1 and gives a higher incentive for player 1 to deviate and become active.

Although harder to sustain, full gradualism still exists in this case as well: in such an equilibrium, polarization in the present builds on less polarization in the future. As the future unfolds, players affect outcomes less as polarization decreases, and thus they focus on the present. In the present though negative externalities are rather important and imply large polarization. We will now examine the role of negative externalities in sustaining full gradualism.

4.3 Gradualism and players' motives

In many set ups, it is reasonable to consider the pure winning incentives of politicians or interest groups.[21] We now consider the extreme case in which agents care only about being chosen by the decision maker but not about the policy that will be enacted if they are not chosen. This implies that there are no negative externalities in agents' utilities.

In the previous sections we have shown how negative externalities can persist throughout the dynamic process to yield a fully gradual equilibrium, in which player 1 is not active unless all other policies have already been placed on the agenda. We now show that when there are no negative externalities, full gradualism is impossible to sustain.

Assume that player $i's$ utility from $x_i = y$ is some v, and 0 otherwise.[22] We still consider three players, whereas x_3 can be either positive or negative. Note that again, when $J = 2$, only players 1 and 2 can compete. We can then show:

Proposition 6: *For both the all-pay auction and the simple contest function, for all ρ, there is no full gradualism: In equilibrium, player 1 is active in every stage and wins every stage with a strictly positive probability.*

[21] Our results are robust to the inclusion of small office or winning motivations.

[22] The case of equal valuations is the natural case to consider as an extension of our political economy model. If valuations differ, we can construct valuation vectors in which extreme players have relatively large valuations, to somewhat mimic the negative externalities that exist in our model.

The absence of negative externalities sharpens the advantage of player 1. In particular, in the game between 2 and 3, player 2 cares only about winning while if he loses to player 3 he is not concerned with perhaps a very far away implemented policy. This implies that the willingness to win of player 1 is higher than that of player 2, when player 1 is assumed to be non-active at $J = 3$: player 1 knows that winning will result in the highest prize whereas player 2 knows that winning against player 3 will just yield this prize with some probability (as if the game doesn't end, player 1 will challenge him):

Lemma 1: *Suppose that player 1 is not active at $J = 3$, then for all H, $w_1^3 > w_2^3$.*

Proof of Lemma 1: To see this, let w_{ij}^k denote the willingness to win of player i against player j in state k and let V_i^k denote the continuation value of player i in state k. Note that $w_2^3 = w_{23}^3$ and that $w_1^3 = H_2 w_{12}^3 + H_3 w_{13}^2 = H_2 w_1^2 + H_3 w_{13}^2$. We now show:

(i) $w_1^2 \geq w_2^2$:

$$w_1^2 = \rho v + (1-\rho)(v - V_1^2) \geq \rho v + (1-\rho)(V_2^2) = w_2^2$$

as $V_1^2 + V_2^2 \leq v$, which is the largest feasible prize in the game. Intuitively, player 1 is more keen to win against player 2 than the opposite as by winning player 1 terminates the game and gets the prize.

(ii) $w_2^2 > w_{23}^3$:

$$w_2^2 = \rho v + (1-\rho)V_2^2 \geq \rho v + (1-\rho)(V_2^2 - V_2^3) = w_{23}^3.$$

Intuitively, player 2 is more keen to win against player 1 than in the game vs. player 3, as losing to player 1 ends his hopes of winning.

(iii) $w_{13}^3 > w_1^2$:

$$w_{13}^3 = \rho v + (1-\rho)(v - V_1^3) > \rho v + (1-\rho)(v - V_1^2) = w_1^2$$

as $V_1^2 > V_1^3$: in state $J = 3$ player 1 does not participate and gains nothing, and can get some payoff only when $J = 2$ arises, thus V_1^3 is a discounted payoff of V_1^2.

These three results imply that $H_2 w_1^2 + H_3 w_{13}^2 > w_{23}^3$ or that $w_1^3 > w_2^3$.$\square$

As explained in Section 4.1, the higher willingness to win of player 1 than that of an active player is sufficient to insure that player 1 will deviate and enter the game both in the all-pay-auction and in the simple contest function.[23]

[23] We show that this is a necessary condition for a deviation in an all-pay-auction in Lemma A2

5 Discussion

We now consider several of our main assumptions and some possible extensions.

Strategic choice of policies. In the model we analyze, players can only propose their ideal policies. Alternatively one can analyze a model in which players strategically choose policies from a set of feasible policies. The considerations of players in this case would be similar to the ones in our model. In particular, the short-run incentives will lead players to choose policies close to their ideal ones. As we show that the short-run and long-run considerations have to be balanced in equilibrium, Theorem 1 can be generalized to this alternative model, when the set of feasible policies is finite.

"Cost-sharing" to overcome the advantage of moderates. When $N > 2$, players may keep on fighting even when they have no chance of winning, in order to "defend" the most moderate position on the current agenda. This implies "free riding" as well as multiplicity of equilibria. In addition, the possibility that players may fight for each other raises the following question: What if many agents coordinate their efforts in fighting for a particular policy? Could they overcome the advantage of player 1 by for example having a different player fighting player 1 at each stage? Suppose that there are infinitely many players with ideal policy x_2, and that at each period, a different player fights against player 1. We find that in this case, the advantage of player 1 is still maintained. For example, under the all-pay-auction, there exists such a "cost-sharing" equilibrium but the probability that player 1 wins each stage is bounded away from zero (we show this in the Appendix).

Uncertainty about the decision maker's preferences. In our model players know the preferences of the decision maker. If these are unknown, or more interestingly, if these change with time, the distinction between moderates and extremists becomes blurred. Consider the example with three players we analyze in sections 4.1 and 4.2. Instead of switching the ideal policy of player 3 from being left of the median to right of the median, we could have kept the position of 3 but changed the preferences of the decision maker and switch his ideal policy towards that of player 3. Player 3 would have then the advantage of being a moderate while having the intensity of preferences of an extremist.

---

in the appendix, and a sufficient condition for a deviation in the simple contest function in Lemma A3.

Strategic decision makers and unawareness. Our model abstracts from an important and interesting strategic consideration of decision makers- when to stop the process and take a decision. Note that when we allow the decision maker to be strategic we need to model the decision maker's beliefs about what policies could potentially arise on the agenda in the future. One way to do this is to assume that the decision maker knows the ideal policies of all players but is bound to choose from a formal agenda. Indeed, as long as the probability that the game ends is always bounded from below, one can generalize the results of Theorem 1 to the case of a strategic decision maker.

Alternatively, a generalization of our model to a strategic but unaware decision maker -who is not aware of feasible policies- is more problematic. To analyze such a model one would have to formalize the expectations of the decision maker about future policies that might arise on the agenda. To this end, one would need a model of unawareness that could be applied to this context.

Incomplete information. In the analysis above, players have complete information about other players' ideal policies or more generally about the available set of policies. A generalization of the model to incomplete information is complicated by the fact that the identity of those who actually place their policies on the agenda reveals information about their types. This creates an additional strategic aspect in which players might be more reluctant to compete as they would lose information rents. Levy and Razin (2009) analyzes this new insight in an endogenous agenda formation model with two periods, two players, and private information on ideal policies. A possible generalization of this analysis to the framework of the current model is left for future research.

6 Appendix

6.1 Proofs for Section 2

Further to Proposition 1, we prove here the specific result for Example 2.

Claim 1 *Assume the simple contest function* and $\rho = 1$. *The unique equilibrium is that the two most extreme players with positions at* -1 *and* 1 *are the only ones who are active.*

Proof of Claim 1 : Suppose that at least three players are active in equilibrium.

Let M be the set of active players. Rename the set of active players with player 1 being the player with the minimum x_i and player m being the player with the highest x_i among the active players. Each $i \in M$ will satisfy the f.o.c:

$$\begin{aligned} b_1(x_i - x_1) + ... + b_{i-1}(x_i - x_{i-1}) + b_{i+1}(x_{i+1} - x_i) + ... + b_m(x_m - x_i) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i \neq 1, m \\ b_2(x_2 - x_1) + ... + b_m(x_m - x_1) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i = 1 \\ b_1(x_m - x_1) + ... + b_{m-1}(x_m - x_{m-1}) &= (\sum_{j=1}^{m} b_j)^2 \text{ if } i = m \end{aligned}$$

Take the difference between the f.o.c of two consecutive individuals in M, i and j and assume w.l.o.g that $i < j$:

$$b_1(x_i - x_j) + ... + b_{i-1}(x_i - x_j) + b_i(x_i - x_j) + b_j(x_j - x_i) + ... + b_m(x_j - x_i) = 0$$

This implies that

$$\sum_{l=1}^{i} b_l = \sum_{l=j}^{m} b_l$$

But if we take $i = 1$ and $j = 2$ and then $i' = 2$ and $j' = 3$ we get:

$$\begin{aligned} b_1 &= b_2 + \sum_{l=3}^{m} b_l \text{ and} \\ b_1 + b_2 &= \sum_{l=3}^{m} b_l, \end{aligned}$$

implying that $b_2 = 0$, a contradiction. Therefore any equilibrium has at most two active players.

Suppose two players i and j are active and it is not the case that they are the players at 1 and -1. Suppose without loss of generality that $x_i, x_j \neq 1$. Note that in equilibrium they both submit a symmetric bid b^*. Let ε be the distance between the policies of i and j. The solution to the first order conditions implies that,

$$\begin{aligned} b\varepsilon &= (2b)^2 \Leftrightarrow \\ b &= \frac{\varepsilon}{4} \end{aligned}$$

Now write down the first order condition for player at $x_i = 1$ at zero,

$$\frac{(1 - x_i)}{\varepsilon} + \frac{(1 - x_j)}{\varepsilon} - 1,$$

and note that as $(1 - x_i) > \varepsilon$ or $(1 - x_j) > \varepsilon$ this implies that the expression above is positive, and hence this cannot be an equilibrium.

Finally suppose that only the players with ideal policies 1 and -1 are active, and they each bid $\frac{1}{2}$. For any other player i, the first order condition evaluated at zero is,

$$\frac{1}{2}(x_i + 1) + \frac{1}{2}(1 - x_i) - 1 = 0$$

As the first order condition is positive for any $b > 0$ only if it is strictly positive for $b = 0$, this corresponds to an equilibrium. $\square$

6.2 Proof of Theorem 1

Let u_{ij}^l be the utility of player i when player j wins at state l, abstracting from the possible payments made by player i at state l. That is, $u_{ij}^l = -\rho|x_i - x_k| + (1-\rho)V_i^k$ where $k = \min\{l, j\}$. Let $w_{ij}^l = u_{ii}^l - u_{ij}^l$ denote the willingness to win of player i against player j in state l.

We will often look at sequences of equilibria in which $\rho \to 0$ and then the relevant willingness to win is $w_{ij|_\rho}^l$, i.e., corresponding to a sequence of equilibria computed for a sequence $\rho \to 0$. We will often suppress the notation for ρ in these expressions, writing w_{ij}^l for $w_{ij|_\rho}^l$. In what follows we use the following terms to refer to the magnitudes of sequences converging to zero. We say that w_{ij}^l is of *order* ρ if $0 < \lim_{\rho\to 0}\left|\frac{w_{ij|_\rho}^l}{\rho}\right| < \infty$ for any sequence of equilibria computed for a sequence $\rho \to 0$. Similarly we say that w_{ij}^l is of *order* ρ *or lower* if $0 \leq \lim_{\rho\to 0}\left|\frac{w_{ij|_\rho}^l}{\rho}\right| < \infty$ and of an order ρ *or higher* if $0 < \lim_{\rho\to 0}\left|\frac{w_{ij|_\rho}^l}{\rho}\right| \leq \infty$. We will also sometimes write $x(\rho) \approx y(\rho)$ for two functions $x(\rho)$ and $y(\rho)$ to imply that $\lim_{\rho\to 0}\frac{x(\rho)}{y(\rho)} = 1$.

We first prove the following Lemma.

Lemma A1: *(i) There exists an* $\bar{\varepsilon} > 0$, *such that for all* ρ, *for all states* l, *the probability that some* $i < l$ *wins is larger than* $\bar{\varepsilon}$. *(ii) For any state* l *and for any* $j \neq 1$, w_{ji}^l *is of order* ρ *or lower, and* w_{21}^l *is strictly positive and of order* ρ.

Proof of Lemma A1:

First note that w_{1i}^l for all i is of an order ρ or higher for any state, as $w_{1i}^l = \rho|x_{\min\{i,l\}}| + (1-\rho)(-V_1^{\min\{i,l\}})$ where $V_j^J \leq 0$ for all j, J.

We will prove the Lemma by induction.

Consider $J = 2$. We will first show that the probability that player 1 wins is bounded away from zero. Suppose to the contrary, that there exists a sequence of equilibria, as $\rho \to 0$, in which player 1 wins with probability ε_n converging to zero. In what follows we assume mixed strategies for the players denoted by f_i and suppress the notation for the element of the sequence and for state 2.

As $\Pr(1 \text{ wins}) = \int_{b_1} f_1(b_1) \Pr(1 \text{ wins}|b_1) db_1 < \varepsilon$, then $\Pr(1 \text{ wins}|b_1) < k\varepsilon$ for a measure of at least $1 - \frac{1}{k}$ of bids in the support of player 1 for all $k > 2$. Choosing a sequence of $k \to \infty$ and $k\varepsilon \to 0$ this implies that for almost any bid b_1^* in the support of player 1, $\Pr(1 \text{ wins}|b_1^*) < k\varepsilon$.

We now compare the utility of each bid in the support of player 1 with a bid of zero. Given the strategies of all other players, player 1 is better off using b_1^* rather than zero only if:

$$\sum_{i \neq 1} (\Pr(i \text{ wins}|b_1 = 0) - \Pr(i \text{ wins}|b_1^*))(w_{12}) \geqslant b_1^*$$

For almost all b_1^*, by $H1$ and $H3$, $\Pr(i \text{ wins}|b_1 = 0) - \Pr(i \text{ wins}|b_1^*) < k\varepsilon$, implying that $b_1^* < k\varepsilon w_{12}$.

Consider other active players. A possible strategy for each such player j is to bid a sequence of $b_j = \gamma b_1^*$ where $\gamma \to \infty$ and $\gamma k\varepsilon \to 0$ so that $b_j \to 0$. By $H2$, such bid guarantees winning (and thus maintaining state 2, as in the equilibrium) with probability converging to 1 and a bid converging to zero. Thus the equilibrium strategy of all other active players must involve bids b_j^* with $\frac{b_j^*}{w_{12}} \to 0$.

Now we reach a contradiction. player 1 can deviate from his equilibrium strategy and place a bid b_1' such that $\frac{b_1'}{b_j^*} \to \infty$ and $\frac{b_1'}{w_{12}} \to 0$. His (relative) gain is w_{12} while his (relative) cost is at most infinitely smaller than w_{12}, yielding a strictly positive benefit.

Now let the probability that 1 wins in $J = 2$ denoted by z, which is bounded from zero. Let us focus on an individual j, for which $u_j(x_1) < u_j(x_2)$. Note that

$$V_j^2 = \frac{-z|x_j| - (1-z)(\rho|x_j - x_2|) - \tilde{b}_j^2}{1 - (1-\rho)(1-z)}$$

where $\tilde{b}_j^J$ refers to the expected payments of player j in state J. We have:

$$\begin{aligned} w_{j1}^2 &= \rho|x_2| + (1-\rho)(V_j^2 + |x_j|) \\ &= \rho|x_2| + (1-\rho)\left(\frac{\rho(1-z)|x_j| - (1-z)(\rho|x_j - x_2|) - \tilde{b}_j^2}{1 - (1-\rho)(1-z)}\right) \end{aligned}$$

By $H1$ and $H3$, $0 \leq \tilde{b}_j^2 < (1-z)w_{j1}^2$, implying that

$$\rho|x_2|(1-(1-\rho)(1-z)) + (1-\rho)(\rho(1-z)|x_j| - (1-z)(\rho|x_j - x_2|)) < w_{j1}^2 \text{ and}$$

$$w_{j1}^2 < \rho|x_2| + (1-\rho)\frac{\rho(1-z)|x_j| - (1-z)(\rho|x_j - x_2|)}{1-(1-\rho)(1-z)}$$

and therefore w_{j1}^2 is of order ρ or lower.

Note that for $j = 2$ we have

$$\rho|x_2|(1-(1-\rho)(1-z)) + (1-\rho)\rho(1-z)|x_j| < w_{j1}^2 \text{ and}$$

$$w_{j1}^2 < \rho|x_2| + (1-\rho)\frac{\rho(1-z)|x_j|}{1-(1-\rho)(1-z)}$$

implying that $w_{21}^2 > 0$ and that w_{21}^2 is of order ρ.

Note further that for all $j > 2$, $w_{j2}^2 = 0$.

Note finally that for $j's$ such that $u_j(x_1) > u_j(x_2)$, we have

$$\begin{aligned} w_{j1}^2 &= -\rho|x_2| + (1-\rho)(V_j^2 + |x_j|) \\ V_j^2 + |x_j| &= \frac{-\rho(1-z)(|x_2|) - \tilde{b}_j^2}{1-(1-\rho)(1-z)} < 0 \text{ and therefore } \tilde{b}_j^2 = 0 \text{ and:} \\ w_{j1}^2 &= -\rho|x_2| + (1-\rho)\frac{-\rho(1-z)(|x_2|)}{1-(1-\rho)(1-z)} \end{aligned}$$

Therefore w_{j1}^2 is of order ρ or lower.

Note that as w_{j1}^2 is of order ρ or lower this implies that

$$V_j^2 \approx -|x_j|.$$

Induction hypothesis: Assume that for all states $J \leq l-1$: *(i) There exists an* $\bar{\varepsilon} > 0$, *such that for all* ρ, *the probability that some* $i < J$ *wins is larger than* $\bar{\varepsilon}$. *(ii) For any* $j \neq 1$, w_{ji}^l *is of order* ρ *or lower, and* w_{21}^l *is strictly positive and of order* ρ. *(iii)* For all $j \neq 1$, $V_j^J \approx -|x_j|$.

Consider state l. Let ε be the probability that a player with $j < l$ wins. We will show that it cannot be that ε converges to zero. Suppose it does. By arguments similar to above, almost all bids must be infinitely smaller than $\max_j \max_i w_{ji}^l$ for $j < l$ and $i \leq l$.

Now consider player 1. His utility is at most $(1-\varepsilon)u_{1l} + \varepsilon\tilde{u}_{1i}$ where $\tilde{u}_{1i}$ is the expectations over the utility from players $i < l$ winning. On the other hand, there

exists some sequence of bids b_1' with $\frac{b_1'}{\max_j \max_i w_{ji}^l} \to 0$ that guarantees winning with probability almost 1, and $b_1' \to 0$. Thus from such a deviation his utility is $u_{11} - b_1'$ so his gain is $(1-\varepsilon)w_{1l} + \varepsilon \tilde{w}_{1i} - b_1'$.

Note that w_{1l} is of order ρ or higher. If $\max_j \max_i w_{ji}^l = w_{j'i}^l = w_{j'i}^{\min\{j',i\}}$ for some $i < l$, then by the induction $\frac{\max_j \max_i w_{ji}^l}{w_{1l}}$ is bounded. Assume therefore that $\max_j \max_i w_{ji}^l = w_{j'l}^l$ for some $j' < l$, $j \neq 1$. Then player j' can deviate to b_1' and his gain is $(1-\varepsilon)w_{j'l}^l + \varepsilon \tilde{w}_{ji}^l - b_1'$ which is strictly positive as $\tilde{w}_{ji}^l$ is of order ρ or lower, a contradiction.

Thus $\varepsilon > \bar{\varepsilon} > 0$.

Note that $w_{ji}^l = w_{ji}^{\max\{j,i\}}$ for $j, i < l$ and that $w_{ji}^l = 0$ for $j, i > l$. We now show that for all other cases, w_{ji}^l is of order ρ or lower. Note that $w_{21}^l = w_{21}^2$.

Suppose that in equilibrium the state remains l with probability $1-z$ and that p_k^l is the probability that some player $k < l$ wins in state l.

Consider first w_{ji}^l for $j \geq l > i$. We have:

$$
\begin{aligned}
w_{ji}^l &= \rho(|x_j - x_i| - |x_j - x_l|) + (1-\rho)(V_j^l - V_j^i) \\
&= \rho(|x_j - x_i| - |x_j - x_l|) \\
&\quad + (1-\rho)\Big(\frac{-\rho((1-z)|x_l - x_j| + \sum_{k<l} p_k^l |x_k - x_j|) + (1-\rho)\sum_{k<l} p_k^l V_j^k - b_j^l}{1-(1-\rho)(1-z)} - V_j^i\Big)
\end{aligned}
$$

Note that by induction $V_j^i \approx -|x_j|$ and so

$$
\begin{aligned}
w_{ji}^l &\approx \rho(|x_j - x_{i'}| - |x_j - x_l|) \\
&\quad + (1-\rho)\Big(\frac{-\rho((1-z)|x_l - x_j| + \sum_{k<l} p_k^l |x_k - x_j|) + (1-\rho)\sum_{k<l} p_k^l V_j^K - b_j^l}{1-(1-\rho)(1-z)} + |x_j|\Big)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{-\rho((1-z)|x_l - x_j| + \sum_{k<l} p_k^l |x_k - x_j|) + (1-\rho)\sum_{k<l} p_k^l V_j^K - b_j^l}{1-(1-\rho)(1-z)} + |x_j| \\
= &\frac{\rho(1-z)(|x_j| - |x_l - x_j|) + \sum_{k<l} p_k^l w_{j1}^k - b_j^l}{1-(1-\rho)(1-z)}
\end{aligned}
$$

and so

$$
w_{ji}^l \approx \rho(|x_j - x_{i'}| - |x_j - x_l|) + (1-\rho)\frac{\rho(1-z)(|x_j| - |x_l - x_j|) + \sum_{k<l} p_k^l w_{j1}^k - b_j^l}{1-(1-\rho)(1-z)}
$$

Note that $0 < b_j^l < \max_{i'} w_{ji'}^l$, assume this is $w_{ji^*}^l$ (for $i^* < l$). Then we have,

$$\begin{aligned}
& \rho(|x_j - x_{i'}| - |x_j - x_l|)\frac{1-(1-\rho)(1-z)}{2-(1-\rho)(1-z)} + (1-\rho)\frac{\rho(1-z)(|x_j| - |x_l - x_j|) + \sum_{k<l} p_k^l w_{j1}^k}{2-(1-\rho)(1-z)} \\
\lessapprox \;& w_{ji^*}^l \\
\lessapprox \;& \rho(|x_j - x_{i'}| - |x_j - x_l|) + (1-\rho)\frac{\rho(1-z)(|x_j| - |x_l - x_j|) + \sum_{k<l} p_k^l w_{j1}^k}{1-(1-\rho)(1-z)},
\end{aligned}$$

which implies by the induction that $w_{ji^*}^l$ is of order ρ (or lower) and hence b_j^l is of order ρ or lower. But note that then this applies also to w_{ji}^l and so by induction and by the above, w_{ji}^l is of order ρ or lower.

Consider now w_{jk}^l for $j < l \leq k$: After some manipulation and substituting $V_j^j \to -|x_j|$:

$$\begin{aligned}
w_{jk}^l &= w_{jl}^l = \rho(|x_j - x_l|) + (1-\rho)(V_j^j - V_j^l) \\
&\approx \rho(|x_j - x_l|) \\
&\quad +(1-\rho)\left(\frac{-\rho(1-\sum_{i<j} p_i^l)|x_j| + \rho(1-z)|x_j - x_l| - \sum_{i<j} p_i^l w_{j1}^i + \sum_{j<i<l} p_i^l w_{ji}^i + \tilde{b}_j^l}{1-(1-\rho)(1-z)}\right)
\end{aligned}$$

Note that $\tilde{b}_j^l < \max_i w_{ji}^l$. If $\max_i w_{ji}^l = w_{jk}^{\min\{j,k\}}$ for some $k < l$ then we know that from the induction hypothesis that $\tilde{b}_j^l$ is of order ρ or lower. Suppose then that $\max_i w_{ji}^l = w_{jl}^l$. Plugging this maximal value we get that w_{jl}^l must satisfy,

$$\begin{aligned}
& \rho(|x_j - x_l|) + (1-\rho)\left(\frac{-\rho(1-\sum_{i<j} p_i^l)|x_j| + \rho(1-z)|x_j - x_l| - \sum_{i<j} p_i^l w_{j1}^i + \sum_{j<i<l} p_i^l w_{ji}^i}{1-(1-\rho)(1-z)}\right. \\
< \;& w_{jl}^l \\
< \;& -\frac{1-(1-\rho)(1-z)}{(1-\rho)(1-z)}\rho(|x_j - x_l|) \\
& +(1-\rho)\frac{\rho(1-\sum_{i<j} p_i^l)|x_j| - \rho(1-z)|x_j - x_l| + \sum_{i<j} p_i^l w_{j1}^i - \sum_{j<i<l} p_i^l w_{ji}^i}{(1-\rho)(1-z)}
\end{aligned}$$

which is of order ρ or lower by the induction hypothesis. Therefore, $\tilde{b}_j^l$ is of order ρ or lower implying that w_{jk}^l is of order ρ or lower.

Finally, when $j \geq l > i$:

$$w_{ji}^l = \rho(|x_j - x_i| - |x_j - x_l|) + (1-\rho)(V_j^l - V_j^i)$$

By induction $V_j^i \approx -|x_j|$ and as w_{ji}^l is of order ρ or lower we have that $V_j^l \approx -|x_j|$.

When $j < l \leq k$:

$$w_{jk}^l = \rho(|x_j - x_l|) + (1-\rho)(V_j^j - V_j^l)$$

By the induction $V_j^j \approx -|x_j|$ and as w_{jk}^l is of order ρ or lower we have that $V_j^l \approx -|x_j|$.

This completes the proof of Lemma 1.$\square$

We can now prove the Theorem. Lemma A1 establishes (ii). Suppose by way of contradiction that at some state l player 1 wins with probability $1-\varepsilon$ converging to 1. Similar arguments as in Lemma A1 imply that for all $j \neq 1$, for almost all bids in the support of j, $\frac{b_j^*}{\max_{i\leq l} w_{ji}^l} \to 0$ and thus player 1's bid satisfies $\frac{b_1^*}{\max_j \max_{i\leq l} w_{ji}^l} \to 0$ almost surely. Now consider player 2 for whom $w_{21}^l > 0$ and is of order ρ by Lemma A1. player 2 can deviate to some bid b_2' with $\frac{b_2'}{\max_j \max_{i\leq l} w_{ji}^l} \to 0$ and $\frac{b_2'}{b_1^*} \to \infty$ which will guarantee winning with probability converging to 1, and therefore, relative to his equilibrium strategy, a gain of at least $(1-\varepsilon)w_{21} + \varepsilon\tilde{w}_{2i}^l - b_2'$ which is strictly positive as $\tilde{w}_{2i}$ is of order ρ or lower, a contradiction.$\blacksquare$

6.3 Proofs for Section 4

We start with useful results about equilibria in which only two players bid strictly positive bids.

Lemma A2: *Consider all-pay-auctions. Suppose that in equilibrium, at some state J, only two players, i and j, place strictly positive bids with strictly positive probability. Let $w_{ji}(F_i,\rho)$ be the willingness to win of player j against i given the equilibrium strategy F_i and define analogously $w_{ij}(F_j,\rho)$. Without loss of generality, let $w_{ji}(F_i,\rho) \geq w_{ij}(F_j,\rho)$. Then (i) $w_{ij}(F_j,\rho) > 0$ and F_i and F_j are determined by:*

$$F_j(b) = \frac{b}{w_{ij}(F_j,\rho)};\ F_i(b) = \frac{w_{ji}(F_i,\rho) - w_{ij}(F_j,\rho) + b}{w_{ji}(F_i,\rho)} \text{ for all } b \in [0, w_{ij}(F_j)]$$

(ii) For any other player k, let $w_{kh}(F_i,F_j,\rho)$ be the willingness to win of player k against particular player h. Let $w_k(F_i,F_j,\rho)$ be $k's$ willingness to win given the equilibrium strategies of i and j. *If $w_{ki}(\rho)+w_{kj}(\rho) > 0$, then $w_k(F_i,F_j,\rho) = \bar{H}_j(F_i,F_j)w_{kj} + \bar{H}_i(F_i,F_j)w_{ki} \leq w_{ij}(F_j,\rho)$* (where $\bar{H}$ denotes expectations over H).

Proof of Lemma A2: (i) Consider the first order conditions for player i and j :

$$\begin{aligned} f_j(b)w_i(F_j,\rho) &= 1 \\ f_i(b)w_j(F_i,\rho) &= 1 \end{aligned}$$

This implies the form of the distribution function above, with an atom on zero for F_i.[24] (ii) For some player k, any utility maximizing bid must satisfy the first order condition $f_i F_j w_{ki}(\rho) + f_j F_i w_{kj}(\rho) - 1 = 0$ but second order condition, using (i), is $f_i f_j (w_{ki}(\rho) + w_{kj}(\rho)) > 0$. Hence utility maximizing bids are either 0 or the maximum bid which is $w_{ij}(F_j, \rho)$. So for player k not to enter, we must have that his utility from a bid of zero is higher than the utility from the maximum bid, which implies that $w_k(F_i, F_j, \rho) \leq w_i(F_j, \rho)$.■

Lemma A3: *Consider the simple contest function. Suppose that at some state J, only two players, i and j, place strictly positive bids with strictly positive probability in a pure strategy equilibrium. Then (i) $H_i(b_i, b_j) w^J_{ji} = H_j(b_i, b_j) w^J_{ij} = b_i + b_j$ and (ii) for any other player k, $H_i(b_i, b_j) w^J_{ki} + H_j(b_i, b_j) w^J_{kj} \leq H_j(b_i, b_j) w^J_{ij}(b_i, b_j, \rho)$.*

Proof of Lemma A3: The conditions in (i) are the first order conditions for i and j that must hold in equilibrium (and are also sufficient for the simple contest function). For player k, expected utility from the equilibrium, given some bid b is

$$H_i(b_k, b_i, b_j)(-w^J_{ki}) + H_j(b_k, b_i, b_j)(-w^J_{kj}) - b_k$$

and the first order condition is

$$\frac{H_i(b_k, b_i, b_j)}{(b_k + b_i + b_j)} w^J_{ki} + \frac{H_j(b_k, b_i, b_j)}{(b_k + b_i + b_j)} (w^J_{kj}) - 1$$

Note that if the first order condition is positive at some point, then it also must be positive for $b_k = 0$ given monotonicity. Thus, to check a possible deviation, it is sufficient to check that the condition is positive at $b_k = 0$. This together with the conditions in (i) of players i and j imply the following condition:

$$H_i(b_i, b_j) w^J_{ki} + H_j(b_i, b_j) w^J_{kj} \leq H_j(b_i, b_j) w^J_{ij}(b_i, b_j, \rho).\blacksquare$$

Proof of Proposition 4:

All-pay-auctions: We compute first the equilibrium for $J = 2$. Only players 1 and 2 can be active, as explained in the text. The analysis follows Lemma A1: We conjecture that in equilibrium player 2 has a lower willingness to win[25], and is

[24] Standard arguments from auction theory imply the continuity, non atomness and same support of the distribution functions used in equilibrium.

[25] Indeed, conjecturing the opposite leads to a contradiction.

therefore the one who places an atom on zero of size $1-\frac{w_2^2}{w_1^2}$, where[26]

$$\begin{aligned} w_2^2 &= \rho x_2 + (1-\rho)(V_2^2 - V_2^1) \\ w_1^2 &= \rho x_2 + (1-\rho)(V_1^1 - V_1^2) \end{aligned}$$

(recall that subscript indicates the player, and superscript indicates the state). By (ii) in Lemma A1,

$$V_2^1 = -x_2;\ V_1^2 = \frac{w_2^2}{w_2^1}(-\rho x_2 + (1-\rho)V_1^2)$$

and plugging for these values, we can solve for the ratio $\frac{w_2^2}{w_1^2} = \frac{1}{2-\rho}$, implying that the atom is of size $\frac{1-\rho}{2-\rho}$. Thus, $V_1^2 = -\rho x_2$, $V_2^2 = -x_2$, and $V_3^2 = \frac{-3|x_3|-x_2\rho+|x_3|\rho}{3-\rho}$.

Now consider the equilibrium in which players 2 and 3 only are active at $J=3$. The willingness to win of each player is:

$$\begin{aligned} w_2^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_2^2 - V_2^3); \\ w_3^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_3^3 - V_3^2); \end{aligned}$$

Conjecture that the atom on zero is on player 3 (the opposite cannot arise). Let the size of the atom be δ. Then:

$$\begin{aligned} V_2^3 &= \delta(1-\rho)(-x_2) + (1-\delta)(-\rho(|x_3| + x_2) + (1-\rho)V_2^3 \\ V_2^3 &= \frac{\delta(1-\rho)(-x_2) - (1-\delta)\rho(|x_3| + x_2)}{1-(1-\delta)(1-\rho)} \\ V_2^2 - V_2^3 &= \frac{\rho(|x_3|(1-\delta) - \delta x_2)}{\delta(1-\rho) + \rho} \\ V_3^3 - V_3^2 &= -\rho(|x_3| + x_2 + V_3^2) \end{aligned}$$

We can solve for $\delta = 1 - \frac{w_3^3}{w_2^3}$ to find:

$$\delta(\rho) = \frac{3|x_3| + 3x_2\rho - 4|x_3|\rho - 5x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}{6|x_3| + 7x_2\rho - 5|x_3|\rho - 7x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2} \underset{\rho\to 0}{\longrightarrow} \frac{1}{2}$$

Note that $\delta(\rho) \geq 0$ for all ρ. This allows to compute

$$\begin{aligned} w_1^3 &= \rho(\frac{1+\delta(\rho)}{2}x_2 + (\frac{1-\delta(\rho)}{2})|x_3|) + \\ &\quad (1-\rho)(-\frac{1+\delta(\rho)}{2}V_1^2 - (\frac{1-\delta(\rho)}{2})V_1^3) \end{aligned}$$

[26] For brevity, we have dropped the index ρ and the index of the distribution functions.

where

$$V_1^3 = -\frac{\frac{1+\delta}{2}(\rho x_2 + (1-\rho)\rho x_2) + \frac{1-\delta}{2}\rho|x_3|}{1-(1-\rho)\frac{1-\delta}{2}}.$$

To check that $w_1^3 - w_3^3 < 0$ we note that the *lhs* is maximal for $\rho \to 0$. We therefore compute $\lim_{\rho\to 0}[w_1^3(\rho) - w_3^3(\rho)] = 3x_2 - |x_3|$ to get the required condition on the relative size of x_3 and x_2.

Simple contest functions: We first compute the equilibrium when $J = 2$ and players 1 and 2 compete. The expected utility of player 1 from bid b_1 is

$$\frac{b_1}{b_1+b_2}u_{11} + \frac{b_2}{b_1+b_2}u_{12} - b_1$$

where the first order condition is

$$\frac{b_2}{(b_1+b_2)^2}u_{11} - \frac{b_2}{(b_1+b_2)^2}u_{12} - 1 = 0 \tag{1}$$

which together with the *foc* for player 2 implies that

$$\frac{w_{12}^2}{w_{21}^2} = \frac{b_1}{b_2} \tag{2}$$

where,

$$\begin{aligned}
w_{12}^2 &= \rho x_2 + (1-\rho)(-V_1^2) \\
w_{21}^2 &= \rho x_2 + (1-\rho)(V_2^2 + x_2) \\
V_1^2 &= \frac{\frac{b_2}{b_1+b_2}(-\rho x_2) - b_1}{1 - \frac{b_2}{b_1+b_2}(1-\rho)} \\
V_2^2 + x_2 &= \frac{-b_2 + x_2\frac{b_2}{b_1+b_2}\rho}{1 - \frac{b_2}{b_1+b_2}(1-\rho)}.
\end{aligned}$$

Solving the system of the first order equations (1) and (2), we find that:

$$\begin{aligned}
b_1 &= \rho b_2 \frac{x_2}{-2b_2 + 2\rho b_2 + \rho x_2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{2} \\
b_2 &= x_2 \frac{\sqrt{-\rho + \rho^2 + 1} - 1}{2\rho - 2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{4}
\end{aligned}$$

In the limit, as $\rho \to 0$:

$$\begin{aligned}
V_1^2 &\approx -\frac{5\rho x_2}{4}, \\
V_2^2 + x_2 &\approx -\frac{3\rho x_2}{8} \\
V_3^2 &\approx (-x_3) + \frac{1}{2}\rho(-x_3 - x_2)
\end{aligned}$$

Now consider the game between players 2 and 3 at $J = 3$:

Let d_i be the equilibrium bid of player i. First order conditions for 2 and 3 are:

$$\begin{aligned}\frac{w_{23}^3}{w_{32}^3} &= \frac{d_2}{d_3}\\ \frac{d_3}{(d_2+d_3)^2}w_{23}^3 &= 1\end{aligned}$$

where,

$$\begin{aligned}w_{23}^3 &= \rho(x_2+x_3)+(1-\rho)(V_2^2-V_2^3)\\ w_{32}^3 &= \rho(x_2+x_3)+(1-\rho)(V_3^3-V_3^2)\\ V_2^2-V_2^3 &= \frac{\frac{-b_2-\frac{b_1}{b_1+b_2}x_2}{1-\frac{b_2}{b_1+b_2}(1-\rho)}\rho+d_2+\frac{d_3}{d_2+d_3}\rho(x_2+x_3)}{1-\frac{d_3}{d_2+d_3}(1-\rho)}\\ V_3^3-V_3^2 &= \frac{-d_3-\frac{d_2}{d_2+d_3}\rho(x_2+x_3)-\rho\frac{\frac{b_1}{b_1+b_2}(-x_3)+\frac{b_2}{b_1+b_2}\rho(-x_3-x_2)}{1-(1-\rho)\frac{b_2}{b_1+b_2}}}{1-\frac{d_3}{d_2+d_3}(1-\rho)}\end{aligned}$$

With these expressions we solve the set of first order conditions above in the limit as $\rho \to 0$ we find that

$$\begin{aligned}d_2 &\approx \frac{1}{2}x_3\rho\\ d_3 &\approx \frac{1}{4}x_3\rho\end{aligned}$$

Now consider a deviation from player 1.

The expected utility of player 1 from a bid b given the equilibrium is:

$$u_1(b) = \frac{d_2}{b+d_2+d_3}(\rho(-x_2)+(1-\rho)V_1^2)+\frac{d_3}{b+d_2+d_3}(\rho(-x_3)+(1-\rho)V_1^3)-b$$

where:

$$\begin{aligned}V_1^3 &= \frac{\frac{d_2}{d_2+d_3}(\rho(-x_2)+(1-\rho)V_1^2)+\frac{d_3}{d_2+d_3}\rho(-x_3)}{1-(1-\rho)\frac{d_3}{d_2+d_3}}\\ &\underset{\rho\to 0}{\approx} \frac{\frac{2}{3}(\rho(-x_2)-(1-\rho)\frac{5\rho x_2}{4})+\frac{1}{3}\rho(-x_3)}{1-(1-\rho)\frac{1}{3}}\end{aligned}$$

Plugging this and V_1^2 in his expected utility, assuming that $b = \gamma\rho x_3$, dividing utility by ρ, and taking the limit, we get:

$$\lim_{\rho\to 0}\frac{U_1(b)}{\rho} = -\gamma x_3 - 0.25\frac{x_3}{0.75x_3+\gamma x_3}(2.25x_2+1.5x_3) - 1.125x_2\frac{x_3}{0.75x_3+\gamma x_3}$$

The derivative of this expression evaluated at $\gamma = 0$ is,

$$\frac{\partial(\lim_{\rho\to 0}\frac{U_1(\gamma)}{\rho})|_{\gamma=0}}{\partial\gamma} < 0 \Leftrightarrow x_3 > 9x_2$$

Note that if $\frac{\partial(\lim_{\rho\to 0}\frac{U_1(\gamma)}{\rho})}{\partial\gamma}$ is positive, it is positive for $\gamma = 0$. Therefore we conclude that for small ρ an equilibrium will exist only for $x_3 > 9x_2$.■

Proof of Proposition 5:

Claim 2: Suppose $x_3 > 0$ and that at $J = 2$, players 1 and 2 compete and that at $J = 3$, players 2 and 3 compete. In both the all pay auction and the simple contest player 3 wins the stage game at $J = 3$ with a higher probability than he does when $x_3 < 0$.

Proof of Claim 2: Let x denote the distance between players 2 and 3. We want to show that the probability that 3 wins the game with 2 is decreasing with x when $|x_3| > x_2$ and x_2 is fixed. Consider the equilibrium for some x and now decrease x. If the players use the same strategies, for both the instantaneous gain increases by $\rho\Delta x$. Let z be the probability that 2 wins. The future payoffs:

$$V_2^2 - V_2^3 = V_2^2 - \frac{z(1-\rho)V_2^2 - (1-z)\rho x - \tilde{b}_2}{1-(1-z)(1-\rho)} = \frac{\rho V_2^2 + (1-z)\rho x + \tilde{b}_2}{1-(1-z)(1-\rho)}$$

as V_2^2 remains the same, this decreases with x when strategies remain the same. Thus it decreases by $\frac{(1-z)\rho}{1-(1-z)(1-\rho)}\Delta x$. For player 3:

$$\begin{aligned} V_3^3 - V_3^2 &= \frac{z(1-\rho)V_3^2 - z\rho x - \tilde{b}_3}{1-(1-z)(1-\rho)} - V_3^2 = \frac{-\rho V_3^2 - z\rho x + \tilde{b}_3}{1-(1-z)(1-\rho)} \\ &= \frac{-\rho\frac{z_1|x_3| - z_2\rho x}{1-(1-\rho)z_2} - z\rho x + \tilde{b}_3}{1-(1-z)(1-\rho)} \end{aligned}$$

which changes by $\frac{\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho}{1-(1-z)(1-\rho)}\Delta x$ when x decreases. If $\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho < 0$, then we know that $\frac{w_{32}}{w_{23}}$ increases, implying by both Lemma A2 and Lemma A3 that 3 has to bid more aggressively relatively so that 3 will win more often. Suppose that it is positive, we then want to show that $\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho < (1-z)\rho$ which holds as $z_2 < 1$ and so we have the same result.■

This implies that the willingness to win of 1 increases. The willingness to 3 converges when $\rho \to 0$ to ρx_3 disregarding its distance form x_2 and so we know that the condition for gradualism becomes harder to sustain in the all-pay-auction. Moreover, we can follow the same strategy as in the proof of Proposition 4, and find

that the fully gradual equilibrium holds for all ρ iff $4x_2 < x_3$ and that player 3 wins the stage game when $J = 3$ with a higher probability than when $x_3 < 0$.

In the simple contest function, note that as H_2 decreases and w_3 is roughly the same, again it is easier to have the condition for deviation which is $w_1 > H_2 w_3$ as w_1 is higher.■

Proof of Proposition 6:

By Lemma A2, A3 and Lemma 1 in the text, there exists a deviation in both the all-pay auction and the simple contest function whenever player 1 wins with probability converging to zero.

To show existence in the all-pay-auction (which is not covered under Proposition 1), we now find the equilibria for N players. Note that players' continuation values are at least 0 at any stage game. Second, consider $J = 2$ and note that the only players that may potentially submit strictly positive bids are 1 and 2. The atom must be on 2 and the solution is the same as in the standard model, and we have that $V_1^2 = v(1-\rho)$, $w_1^2 = \rho v(2-\rho)$, $V_2^2 = 0$ and $w_2^2 = \rho v$. Suppose, by way of induction, that for every state $l < J$, the equilibrium is as in Baye et al (1996) in which player 1 has the highest willingness to pay. In particular this implies that there is an atom on bid zero for all players beside player 1 and so $v > V_1^l > 0$ and $V_i^l = 0$ for all $1 < i \leq l$, implying that $w_1^l > \rho v$ and that $w_i^l = \rho v$ for all $1 < i \leq l$.

Suppose we are now at state J. Note that players who are more extreme than J do not participate. We then have that $w_i^J = \rho v + (1-\rho)(V_i^i - \sum_{j \neq i} p_{i,j} V_i^j)$ but by the induction, $V_i^i = 0$, and $V_i^j = 0$ for all j that participate, thus $w_i^J = \rho v$ for all i that are weakly more moderate than J.

On the other hand, $w_1^J = \rho v + (1-\rho)(v - \sum_{j \neq 1} p_{1,j} V_1^j) > \rho v$ as $V_1^j < v$ by the induction hypothesis. We can therefore apply Baye et al (1996) for this stage to find that, (i) $\Pi_i F_i(0) = \frac{1-\rho}{2-\rho}$ for all $i \neq 1$ that participates, and so $F_i(0) > 0$ for any such i, (ii) player 1 must participate in every stage and he wins with a higher probability than any other. Also, there is a continuum of equilibria as in Baye et al (1996).■

Proof of "cost sharing" equilibrium in the discussion: Consider an all-pay-auction and $J = 2$. Suppose that player 1 places an atom or that there is no atom. Then: $w_1 = \rho x_2 + (1-\rho)(-V_1^2) = x_2$ as player 1 loses. For the 2-player who bids: $w_i = \rho x_2 + (1-\rho)(V_i^2 + x_2)$; $V_i^2 = \frac{z(-x_2)}{1-(1-z)(1-\rho)}$ as he does not bid later on, where z is the probability that player 1 wins any stage when $J = 2$. Thus, $w_i = \rho x_2 + (1-\rho)(\rho x_2 \frac{1-z}{z+\rho-z\rho}) = \rho(\frac{x_2}{z+\rho-z\rho})$. Obviously the willingness to win of player

1 is higher unless $z = 0$, a contradiction to 1 placing an atom or no atom.

Thus the 2-players must place an atom. We have:

$$\begin{aligned}
V_1^2 &= \frac{-(1-\delta)\rho x_2}{1-(1-\delta)(1-\rho)} \\
w_1 &= \rho x_2[1 + \frac{(1-\rho)(1-\delta)}{1-(1-\delta)(1-\rho)}] \\
V_i^2 &= \frac{z(-x_2)}{1-(1-z)(1-\rho)};\ z = \delta + \frac{1-\delta}{2} \\
w_i &= \rho x_2 + (1-\rho)(-\rho x_2 \frac{z-1}{z+\rho-z\rho}) = \rho x_2 \frac{2}{\delta+\rho-\delta\rho+1} \\
\delta &= -\frac{1}{\rho-1}\left(\sqrt{-2\rho+\rho^2+2}-1\right)
\end{aligned}$$

The probability that 1 wins is roughly between $(0.5, 0.71)$ and converges to $0.707\,11$ when $\rho \to 0$ as compared to 0.75 when only player 2 is active in any stage.■

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