Costly Self Control and Random Self Indulgence[1]

Eddie Dekel[2] Barton L. Lipman[3]

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[2]Economics Dept., Northwestern University, and School of Economics, Tel Aviv University E{mail: dekel@nwu.edu.
[3]Boston University. E{mail: blipman@bu.edu.

Abstract

We show an unexpected connection between the Gul–Pesendorfer (2001) (GP) model of temptation and a version of the Strotz (1955) model with uncertainty about the nature of the temptation that will strike. In particular, any preference over menus with a GP representation also has a representation via a Strotz model with appropriately chosen uncertainty. As a partial converse, we show that, restricting to Lipschitz continuous representations, a preference has a GP representation with uncertainty (***random GP***) if and only if it has a Strotz representation with uncertainty (***random Strotz***). Similarly, we show that choices from menus alone cannot distinguish the random GP and random Strotz models in general, though preferences over menus and choices from menus can. We give an axiomatic characterization of the Lipschitz continuous random Strotz representation, extending a result of Stovall (forthcoming), show that the representation is unique, and characterize a comparative notion of "more temptation averse."

1 Introduction

On the surface, the Strotz (1955) and Gul and Pesendorfer (2001) (henceforth GP) models of temptation, while clearly related, seem quite different. Both models consider an agent who is currently not influenced by temptation but who chooses commitments today in anticipation of how his future, tempted self will choose tomorrow. In Strotz, the untempted self anticipates that his future self will behave in a completely self–indulgent fashion, maximizing his temptation–influenced utility (paying attention to the untempted self's preferences only when indifferent). By contrast, in GP's model, the current self anticipates that his future self will, in general, exert costly self control. The current and future selves have the same objective function: the utility of the current self minus the self–control costs. Thus the future self takes into account both the current self's utility and the cost of exerting self control, so the choice made is a compromise between the factors. Because of these costs, the current self values commitment even when his future self will not succumb to temptation. GP demonstrate one relationship between these models by showing that the Strotz model is the limit of their model as the future self's preferences become more dominant.

In this paper, we show a different and unexpected connection between the GP model and a version of the Strotz model with uncertainty about the nature of the temptation that will strike. Even though the models appear very different and even though the GP model has no uncertainty, we show that the commitment choices of the agent in GP are identical to the commitment choices in a Strotz model with appropriately chosen uncertainty. Interestingly, the proof of equivalence uses standard results on incentive compatibility, a fact which highlights the potential application of such techniques to the study of random Strotz models.

In light of this result, we are led to a more detailed exploration of what we call the *random Strotz* model. We relate it to a version of the GP model which adds similar uncertainty about temptations, which we call *random GP*. Restricting attention to Lipschitz continuous versions of the two models, we find that the commitment choices they predict are identical. In this sense, there is no observable difference in commitment choices in these models of (random) costly self control and random self indulgence.[1]

We prove this equivalence in two steps. First, we give a simple way to rewrite any random GP model as a random Strotz model, establishing that random Strotz is a more general model. Second, we establish the converse via an axiomatic characterization of Lipschitz continuous versions of the two models. More specifically, we extend a result of

[1] There are, of course, other examples of preferences which have multiple interpretations, such as the overlap between multiple priors preferences and Choquet expected utility preferences | see, for example, Gilboa and Schmeidler (1994).

Stovall (2009) to give axioms on preferences over commitments — that is, over menus — which are necessary and sufficient for the existence of a Lipschitz continuous version of either model, showing that the commitment behavior of the two models is equivalent.

We also show that the random Strotz model is uniquely identified by the preference over menus (whether Lipschitz continuity is assumed or not) and use this identification to give an interesting comparative notion.

Finally, we compare the two models in terms of the choices by the tempted self. We show that observations of choice from menus ***alone*** cannot, in general, distinguish the two models. In particular, any choice function which can be generated by the random Strotz model can also be generated by the random GP model. If we focus on random choice correspondences instead, the models have a large overlap, but are not identical. We show that any choice correspondence generated by a continuous random Strotz model can also be generated by a random GP model, but the converse is not true. Hence there is a large class of choices by the tempted self that can be explained either by costly self control or by random self indulgence.

On the other hand, except in the trivial case of no temptation, the two models can ***always*** be distinguished by observing ***both*** preference over menus and choice from menus. In particular, the random Strotz model, all else equal, shows more temptation in the choices from menus than does the random GP model in a sense we make precise below.

One reason we find the random GP and random Strotz models of interest is that they are able to rationalize some very natural preferences which cannot be rationalized within GP's model. For example, consider a dieting agent who ranks menus which may contain broccoli (b), chocolate (c), and/or potato chips (p). As Dekel, Lipman, and Rustichini (2009) note, GP's representation rules out the seemingly natural ranking that has $\{b\}$ as best, then $\{b, c\}$ and $\{b, p\}$ in some order, followed by $\{b, c, p\}$. GP's model treats temptation as one–dimensional so only the most tempting item from $\{b, c, p\}$ can matter, implying that this menu must be indifferent to either $\{b, c\}$ or $\{b, p\}$. It is not hard to see that the random Strotz representation is consistent with this ranking. If there is some probability of a temptation toward chocolate and some probability of a temptation toward potato chips, then the agent breaks her diet more often with $\{b, c, p\}$ than with $\{b, c\}$ or $\{b, p\}$ and thus this menu can be strictly worse than the other two.

The basic point that the GP representation can be rewritten in terms of a random determination of which self has control has been made before, though in very different ways. In particular, Benabou and Pycia (2002) note that the GP representation can be written as the equilibrium payoff of a game between the current and future self engaging in a costly battle for control. Also, Chatterjee and Krishna (2007) show that a preference with a GP representation also has a representation where there is a probability which

depends on the menu that the choice is made by the tempted self, with the choice made by the untempted self otherwise. Unfortunately, the properties of the function relating menus to probabilities over control make it difficult to interpret in general.[2] One appealing aspect of our result is that the random Strotz representation is a natural alternative formulation.

Also, Fudenberg and Levine (2006) have proposed a dual–selves model which has some similarities in spirit to the Strotz and GP models but is different from both.

The next section defines the model and the representations considered. In Section 3, we relate random Strotz representations to GP and random GP representations. Section 4 turns to a characterization of random Strotz representations, showing the uniqueness and comparative results described above. In Section 5, we discuss choice from menus and the extent to which this information, with or without preferences over menus, enables us to distinguish the random GP and random Strotz models. Section 6 concludes. Proofs not contained in the text are in the Appendix.

2 Definitions

Fix a finite set Z of "prizes" or outcomes, let $\Delta(Z)$ denote the set of lotteries over Z, and let X denote the set of ***menus***, the set of compact, nonempty subsets of $\Delta(Z)$. The current self is modeled as having a preference over X, denoted $\succeq$, where this is interpreted as a preference regarding how much commitment to impose on subsequent choices. Later, we discuss how we represent choices from menus.

Throughout, we assume that $\succeq$ is ***nontrivial*** in the sense that there exist $x, y \in X$ such that $x \succ y$.

A function $w : \Delta(Z) \to \mathbf{R}$ is ***linear*** if $w(\lambda\alpha + (1-\lambda)\beta) = \lambda w(\alpha) + (1-\lambda)w(\beta)$ for all $\lambda \in [0,1]$ and $\alpha, \beta \in \Delta(Z)$. We say that $w : \Delta(Z) \to \mathbf{R}$ is an ***expected utility function*** if it is linear.

Both the Strotz and GP representations use two expected utility functions, $u, v : \Delta(Z) \to \mathbf{R}$. The Strotz representation uses the pair (u, v) to evaluate a menu x by

$$V_S(x) = \max_{\beta \in B_v(x)} u(\beta)$$

[2] The published version of Chatterjee and Krishna's paper, Chatterjee and Krishna (2009), considers only the case where this probability is independent of the menu. While this provides more structure, the constant probability model no longer nests GP.

where $B_v(x)$ is the set of best elements of x according to v. That is,

$$B_v(x) = \{\beta \in x \mid v(\beta) \geq v(\alpha), \quad \forall \alpha \in x\}.$$

Intuitively, v represents the preference of the future self who will be completely self indulgent, choosing from the menu as he wishes, breaking ties in favor of the current self who has utility function u.

One unfortunate feature of the Strotz model is that the agent's utility depends discontinuously on the commitments he makes. This occurs because when the choosing self is almost indifferent, the current self may still have strong preferences regarding the choices. A small change in commitments can then create indifference for the chooser. Hence we can find such small changes in commitments that have big effects on the current self's payoff.[3] This discontinuity is both intuitively implausible and analytically inconvenient. For example, because of the discontinuity, optimal policies for the current self may not exist.

The representation introduced by GP is continuous and hence avoids this problem. In their representation, the way that u and v are used to evaluate a menu x is by the function

$$V_{GP}(x) = \max_{\beta \in x}[u(\beta) + v(\beta)] - \max_{\beta \in x} v(\beta).$$

GP emphasize the idea that in their representation, the agent chooses from the menu the item which maximizes $u + v$, not v. In this sense, he shows partial self control by compromising between u and v instead of simply maximizing v. One intriguing interpretation offered by GP which highlights this idea can be seen by writing the representation as

$$V_{GP}(x) = \max_{\beta \in x}[u(\beta) - c(\beta, x)]$$

where $c(\beta, x) = [\max_{\alpha \in x} v(\alpha)] - v(\beta)$. This representation is written as if the agent chooses the β which maximizes $u(\beta) - c(\beta, x)$ which is the β which maximizes $u + v$. Under this interpretation, $c(\beta, x)$ is the cost of resisting temptation by choosing β instead of maximizing v.

As noted, we consider random versions of the GP and Strotz models. Hence we require a field for the set of EU functions. Letting K denote the number of elements of Z, we identify the set of such functions with $\mathbf{R}^K$ since for any EU function, we only need to specify the payoffs to the pure outcomes. We use the Borel field over $\mathbf{R}^K$.

Definition 1. *A* random Strotz representation *of $\succeq$ is a pair (u, μ) such that u is an expected utility function and μ is a measure over expected utility functions such that the*

[3]Note that this di culty is not eliminated by changing the tie{breaking rule.

function

$$V_{RS}(x) = \int_{\mathbf{R}^K} \max_{\beta \in B_w(x)} u(\beta)\mu(dw)$$

represents the preference.

This is the Strotz representation but where the agent is not sure what his future self's preference will be. It seems quite natural to suppose that an agent may not know exactly what temptations will strike him in the future or exactly how strong they will be. Adding uncertainty to the Strotz model also has the potential to resolve the continuity problems noted above. Intuitively, if the distribution over the chooser's utility function is suitably atomless, then the probability the chooser is indifferent will be zero. Since it is this indifference which creates the discontinuities, making such events irrelevant to the current self resolves the discontinuity problem. As Caplin and Leahy (2006) show, such atomlessness can ensure existence of an optimal policy in Strotz's sense.

A random GP representation generalizes the notion of a GP representation in a fashion exactly parallel to the way that random Strotz generalizes Strotz: specifically, the u is fixed but there is a probability measure over the "temptations."[4]

Definition 2. *A* random GP representation *is a pair* (u, ν) *such that* u *is an expected utility function and* ν *is a measure over expected utility functions such that the function*

$$V_{RGP}(x) = \int_{\mathbf{R}^K} \left\{ \max_{\alpha \in x} u(\alpha) + v(\alpha)] - \max_{\alpha \in x} v(\alpha) \right\} \nu(dv)$$

represents the preference.

3 Costly Self Control "=" Random Self Indulgence: Menu Choice

3.1 Costly Self Control $\subseteq$ Random Self Indulgence

We begin by relating the GP and random Strotz representations.

[4] There is one di erence between the way randomization enters these two representations which will become important later. Speci cally, in the random Strotz model, we could (and later will) normalize the space of EU functions, while in the random GP model, we cannot. For random GP, the scale of each v relative to u matters as it measures the \strength" of the temptation v, while for random Strotz, the choice made under a temptation is all that matters, not the scale of the temptation.

Theorem 1. *Fix any GP representation (u, v) and the corresponding V_{GP}. Then there exists a measure μ over expected utility functions such that the function V_{RS} corresponding to the random Strotz representation (u, μ) satisfies $V_{GP}(x) = V_{RS}(x)$ for every menu x.*

Proof. Let W denote the set of expected utility preferences such that $w \in W$ iff there exists $A \in [0, 1]$ with $w = v + Au$. Define a measure μ over W by taking the uniform distribution over A. That is, for a set $E \subseteq W$, we have

$$\mu(E) = \Pr[\{A \in [0, 1] \mid v + Au \in E\}],$$

where $\Pr(\cdot)$ is the uniform distribution. Finally, let V_{RS} denote the random Strotz representation generated by this measure.

Fix any menu x. Let $\beta^*(A)$ denote any element of x which maximizes u over the set $B_{v+Au}(x)$. Let $\hat{u}(A) = u(\beta^*(A))$ and let $\hat{v}(A) = v(\beta^*(A))$. Note that if multiple elements of x maximize u over $B_{v+Au}(x)$, the values of $\hat{u}(A)$ and $\hat{v}(A)$ do not depend on the particular choice of $\beta^*(A)$. Also, it is easy to show that $\hat{u}$ is nondecreasing in A and hence measurable. Since $\hat{u}$ is also bounded, it is integrable. We have

$$V_{RS}(x) = \int_0^1 u(\beta^*(A))\, dA = \int_0^1 \hat{u}(A)\, dA.$$

Define

$$\mathcal{U}(A) = \hat{v}(A) + A\hat{u}(A) = \max_{\bar{A} \in [0,1]} \hat{v}(\bar{A}) + A\hat{u}(\bar{A}).$$

From the usual argument characterizing incentive compatibility with transferrable utility (see, *e.g.*, Mas–Colell, Whinston, and Green (1995), Proposition 23.D.2, page 888, or Milgrom and Segal (2002), Theorem 2),[5] we have

$$\mathcal{U}(s) = \mathcal{U}(0) + \int_0^s \mathcal{U}'(A)\, dA = \mathcal{U}(0) + \int_0^s \hat{u}(A)\, dA.$$

Hence

$$\mathcal{U}(1) - \mathcal{U}(0) = \int_0^1 \hat{u}(A)\, dA = V_{RS}(x).$$

But $\mathcal{U}(1) = \max_{\beta \in x}[v(\beta) + u(\beta)]$, while $\mathcal{U}(0) = \max_{\beta \in x} v(\beta)$. Hence the left–hand side is $V_{GP}(x)$. ▌

Corollary 1. *Every preference with a random GP representation also has a random Strotz representation.*

[5] For intuition, consider a standard auction problem or other characterization of incentive compatibility with quasi{linear utility. View A as the type of the agent where this is his valuation for some good. Then A plays the role of the agent's report of his type, $\hat{u}(A)$ is the probability the agent obtains the good if his report is A, and $\hat{v}(A)$ is the transfer to him when his report is A.

To see explicitly why the corollary follows, note that

$$V_{RGP}(x) = \int \left\{ \max_{\alpha \in x} u(\alpha) + v(\alpha)] - \max_{\alpha \in x} v(\alpha) \right\} \nu(dv)$$

so

$$V_{RGP}(x) = \int \left\{ \int_0^1 \max_{\beta \in B_{v+Au}(x)} u(\beta)\, dA \right\} \nu(dv)$$

which is a random Strotz representation. Thus the random Strotz model is more general than the random GP model.[6]

3.2 Random Self Indulgence $\subseteq$ Costly Self Control?

This observation naturally leads one to ask what behavior random Strotz can accommodate which random GP precludes. A partial answer comes from the fact that random Strotz includes nonrandom Strotz as a special case and hence allows the possibility of discontinuous preferences. More formally, as mentioned in Section 2, a (nonrandom) Strotz representation need not be continuous. Since such representations are a special case of random Strotz, the same holds true for random Strotz representations. On the other hand, a random GP representation must be continuous. (Proofs of these claims are in Section A of the Appendix.)

Is discontinuity the only property which random Strotz allows but random GP does not? We conjecture that the answer is yes — more specifically, that the set of preferences with a *continuous* random Strotz representation equals the set with a random GP representation. Our result, however, is more limited. Instead, we show that the set of preferences with a Lipschitz continuous random Strotz representation equals the set with a Lipschitz continuous random GP representation. We show by an example in the appendix that not every random GP representation is Lipschitz continuous (even though every (nonrandom) GP representation *is* Lipschitz continuous[7]).

We prove this result via an axiomatic characterization of the class of preferences with a Lipschitz continuous random GP representation and show that these axioms also characterize the set of preferences with a Lipschitz continuous random Strotz representation.

[6] It is not hard to see that the results of this subsection do not require Z to be finite. To be specific, suppose we let Z be compact, fix a topology on Z, and let $\Delta(Z)$ be the set of distributions on the Borel field of Z. Then define an expected utility function to be a linear and continuous function from $\Delta(Z)$ to $\mathbf{R}$. With this modification of our definitions, it is easy to see that Theorem 1 and Corollary 1 carry over since the proof makes no use of the finiteness of Z. Even compactness of Z plays no role other than ensuring the representations are well–defined.

[7] Dekel, Lipman, Rustichini, and Sarver (2007a) show that every additive EU representation is Lipschitz continuous. Since, as Dekel, Lipman, and Rustichini (2009) explain, every GP representation is an additive EU representation, the claim follows.

A function $V : X \to \mathbf{R}$ is *Lipschitz continuous* if there is a $\bar{N}$ such that

$$V(y) - V(x) \leq \bar{N} d_h(x, y), \quad \forall x, y$$

where d_h denotes Hausdorff distance.

Our axiomatic characterization begins with the additive EU representation of Dekel, Lipman, and Rustichini (2001) (henceforth DLR). As shown in Dekel, Lipman, Rustichini, and Sarver (2007a) (henceforth DLRS), this representation is Lipschitz continuous. DLRS show that $\succeq$ has an additive EU representation iff it satisfies the following four axioms.

Axiom 1 (Weak Order). *$\succeq$ is complete and transitive.*

Axiom 2 (Continuity). *For every $x \in X$, the sets $\{y \in X \mid x \succ y\}$ and $\{y \in X \mid y \succ x\}$ are open in the Hausdorff topology.*

Let $\lambda x + (1 - \lambda)y = \{\gamma \in \Delta(Z) \mid \gamma = \lambda\alpha + (1 - \lambda)\beta$ for some $\alpha \in x, \beta \in y\}$.

Axiom 3 (Independence). *$x \succeq y$ implies $\lambda x + (1-\lambda)\bar{x} \succeq \lambda y + (1-\lambda)\bar{x}$ for every $\lambda \in [0, 1]$ and $\bar{x} \in X$.*

See GP or DLR for discussion of this axiom.

Axiom 4 (L–Continuity). *There exist nonempty sets $x^*, x_* \subseteq \Delta(Z)$ and $N > 0$ such that for every $\varepsilon \in (0, 1/N)$, for every x and y with $d_h(x, y) \leq \varepsilon$,*

$$(1 - N\varepsilon)x + N\varepsilon x^* \succeq (1 - N\varepsilon)y + N\varepsilon x_*.$$

See DLRS for a discussion of this axiom.

Definition 3. *An* additive EU representation *of $\succeq$ is a countably additive, signed measure η over expected utility functions such that the function*

$$V_{AEU}(x) = \int_{\mathbf{R}^K} \max_{\beta \in x} w(\beta)\, \eta(dw)$$

represents $\succeq$.

The additive EU representation is rather general, saying nothing about temptation or other motivations of the agent. We now specialize by adding a "temptation" axiom. This axiom was first proposed by Dekel, Lipman, and Rustichini (2009). Stovall (2009) gives a different version which is equivalent given the other axioms.

Axiom 5 (Weak Set Betweenness). *If* $\{\alpha\} \succeq \{\beta\}$ *for all* $\alpha \in x$ *and* $\beta \in y$*, then* $x \succeq x \cup y \succeq y$*.*

To see the idea, fix menus x and y satisfying the hypothesis of the axiom which says that the agent would rather commit himself to *any* option in x than to *any* option in y. Intuitively, then, *everything* in x is better for the agent than

4 Properties of Random Strotz Representations

4.1 Uniqueness

In this subsection, we discuss the uniqueness properties of the random Strotz representation, while the next subsection uses this uniqueness to characterize a natural comparison notion. While our axiomatic characterization only covers Lipschitz continuous random Strotz representations, all results in Section 4 apply to ***any*** random Strotz representation, regardless of its continuity properties.

So suppose we have a preference $\succeq$ with random Strotz representations (u, μ) and $(\bar{u}, \bar{\mu})$. What is the relationship between the representations? First, it is easy to see that u and $\bar{u}$ must be the same up to a positive affine transformation. This follows from the fact that both u and $\bar{u}$ must represent the preference over singleton menus. That is, for any α, the random Strotz representation (u, μ) evaluates the menu $\{\alpha\}$ by $u(\alpha)$. Hence $u(\alpha) \geq u(\beta)$ iff $\{\alpha\} \succeq \{\beta\}$, so $u(\alpha) \geq u(\beta)$ if and only if $\bar{u}(\alpha) \geq \bar{u}(\beta)$. Thus the usual uniqueness properties for expected utility representations imply that $\bar{u}$ is a positive affine transformation of u.

To identify the measure, we must first normalize the space of expected utility functions. Since only the choices by each given w matter for the representation, we obviously cannot distinguish a representation that puts probability p on w from a representation that puts probability p on $2w$. Recall that we identified the space of EU functions with $\mathbf{R}^K$ where K is the number of pure outcomes. We take the normalized space to be

$$\mathcal{W} = \{w \in \mathbf{R}^K \mid w \cdot \mathbf{1} = 0 \text{ and } w \cdot w = 1\}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^K$. For the σ–algebra on $\mathcal{W}$, we use the Borel σ–algebra using as our topology on $\mathcal{W}$ the (relativized) usual Euclidean topology on $\mathbf{R}^K$.[10]

We refer to a random Strotz representation (u, μ) with $u \in \mathcal{W}$ and $\text{supp}(\mu) \subseteq \mathcal{W}$ as a ***canonical random Strotz representation***. It is easy to show that our assumption that $\succeq$ is nontrivial implies that if it has a random Strotz representation, then it has a canonical random Strotz representation.

It is also easy to show that if $u, \bar{u} \in \mathcal{W}$ are the same up to a positive affine transformation, then $u = \bar{u}$. Thus once we restrict attention to canonical random Strotz representations, the u is unique. As the following theorem shows, the measure is uniquely

[10]Note that $\mathcal{W}$ excludes all vectors of the form $t\mathbf{1}$ where $t \in \mathbf{R}$ and $\mathbf{1}$ is a K dimensional vector of 1's. That is, we are ruling out w's which are indi erent over every lottery. Since we have assumed $\succeq$ is nontrivial, such EU preferences are redundant.

identified given our normalization of the space of expected utility functions.[11]

Theorem 3. *If (u, μ) and $(\bar{u}, \bar{\mu})$ are canonical random Strotz representations of $\succeq$, then $(u, \mu) = (\bar{u}, \bar{\mu})$.*

4.2 A Comparative

Given that the measure is uniquely identified, we turn to the behavioral implications of properties of the measure. The following definition gives a natural version of a comparative notion of temptation[12] which relates in an interesting way to the measure. We say that $\succeq_2$ is *more temptation averse* than $\succeq_1$ if whenever $\{\alpha\} \succeq_1 x$, we have $\{\alpha\} \succeq_2 x$. In other words, whenever $\succeq_1$ prefers a commitment to leaving open the choice from x, $\succeq_2$ does as well.[13]

It is easy to see that since $\succeq_1$ and $\succeq_2$ are nontrivial, this notion requires $\succeq_1$ and $\succeq_2$ to rank singletons identically.[14] Therefore, if both have canonical random Strotz representations, the u's must be the same. Hence, the two preferences have canonical representations which differ only in terms of the measure μ. Let the measure for $\succeq_i$ be denoted μ_i. We give two statements of the relationship between μ_1 and μ_2 each of which is equivalent to $\succeq_2$ being more temptation averse than $\succeq_1$. Both involve certain first–order stochastic dominance (FOSD) comparisons.

[11]Since the measure is unique, obviously, the support is unique as well. We show in supplementary material available at http://people.bu.edu/blipman that the support consists of those $w \in \mathcal{W}$ which are *relevant* in the sense that for every neighborhood N of w, there exist menus x and x' with $x \not\sim x'$ and

$$\max_{\beta \in B_w(x)} u(\beta) = \max_{\beta \in B_w(x')} u(\beta), \quad \forall w \in \mathcal{W} \setminus N.$$

That is, w is relevant if we need it (or small neighborhoods of it) to \see" why the agent is not indi erent between x and x' since all the other w's behave the same way on x and x' as far as u is concerned. This is analogous to the characterization of the support of the measure in DLR. As observed by Chatterjee and Krishna (2009), while the characterization is analogous to that of DLR, it is not the same.

[12]Gul and Pesendorfer (2001) also give two comparative notions related to temptation for their model. Their comparatives are very di erent from ours both in spirit and formally.

[13]This de nition is also used by Ahn (2007) to compare ambiguity aversion, Sarver (2008) to compare regret attitudes, and Higashi, Hyogo, and Takeoka (2009) to compare aversion to commitment. It is also similar in spirit to the way Epstein (1999) and Ghiradato and Marinacci (2002) de ne comparisons of ambiguity aversion. Since the random Strotz representation is very di erent from the representations considered in these papers, their characterization results are quite di erent as well.

[14]To see this, note that $\{\alpha\} \sim_1 \{\beta\}$ implies $\{\alpha\} \sim_2 \{\beta\}$. Hence any $\succeq_1$ indi erence curve over singletons must be contained in a $\succeq_2$ indi erence curve over singletons. Since the $\succeq_1$ indi erence curve is a hyperplane and the $\succeq_2$ indi erence curve cannot be larger than this, the indi erence curves must coincide. Also, the direction of increase for $\succeq_1$ must be the same as that of $\succeq_2$ since $\{\alpha\} \succ_1 \{\beta\}$ implies $\{\alpha\} \succeq_2 \{\beta\}$, a preference which must be strict since $\succeq_2$ is nontrivial. Hence they have the same preference over singletons.

The first statement is based on generalizing the usual notion of FOSD. To generalize, we define an order over $\mathcal{W}$ which will replace the usual $\geq$ in our notion of FOSD. Define an order over $\mathcal{W}$ by $w \; C_u \; \hat{w}$ (read "w is closer to u than $\hat{w}$") if

$$u(\alpha) > u(\beta), \;\; \hat{w}(\alpha) \geq \hat{w}(\beta) \;\; \text{implies} \;\; w(\alpha) \geq w(\beta).$$

In other words, w is willing to "go along with" u at least as often as $\hat{w}$.

It will prove useful to give a geometric restatement of this notion. Note that we defined $\mathcal{W}$ to be the surface of a sphere. For intuition, think of $\mathcal{W}$ as the points on a globe where u is the North Pole and $-u$ the South Pole. Now let

$$\mathcal{V} = \{v \in \mathcal{W} \mid v \cdot u = 0\}.$$

Think of $\mathcal{V}$ as the equator. The following lemma shows how any given point in $\mathcal{W}$ can be rewritten in terms of a choice of an equator point and a movement along the longitude line through that point. The subsequent lemma shows that $w \; C_u \; w'$ if and only if we can move north from w' to w along a longitude line on this globe.

Lemma 1. *For every $w \in \mathcal{W}$, there exists $v \in \mathcal{V}$ and $A \in [-1, 1]$ such that $w = v\sqrt{1 - A^2} + Au$. If $w = u$, then this holds for every $(A, v) \in \{1\} \times \mathcal{V}$, while if $w = -u$, it holds for every $(A, v) \in \{-1\} \times \mathcal{V}$. For every other $w \in \mathcal{W}$, the (A, v) is unique.*

For each $v \in \mathcal{V}$, let $L(v)$ denote the "longitude" generated by v. That is, let $L(v)$ denote the set of $w \in \mathcal{W}$ such that $w = v\sqrt{1 - A^2} + Au$ for some $A \in [-1, 1]$. Intuitively, $A > 0$ corresponds to a movement north along the longitude $L(v)$, while $A < 0$ corresponds to a movement to the south. Clearly, the sets $\{L(v) \mid v \in \mathcal{V}\}$ partition $\mathcal{W} \setminus \{u, -u\}$. Note that every $L(v)$ set includes both u and $-u$, just as every longitude line on a globe runs from the North Pole to the South Pole.

To complete the argument that C_u is related to movements along these longitude lines, we have the following lemma.

Lemma 2. *$w_1 \; C_u \; w_2$ if and only if there exists $v \in \mathcal{V}$ such that $w_i = v\sqrt{1 - A_i^2} + A_i u$, $i = 1, 2$, with $A_1 \geq A_2$.*

In words, w_1 and w_2 can be compared under C_u iff they are on the same longitude. In this case, the point further north is the one closer to u (the "North Pole") in the sense of C_u.

Having provided this geometric intuition for C_u, we return to defining our first FOSD notion. Define a set $W \subseteq \mathcal{W}$ to be *closed under* C_u if $w' \in W$ and $w \; C_u \; w'$ implies $w \in W$. Geometrically, a set is closed under C_u if it equals every point north of some

curve which circumnavigates the globe (and may or may not include parts of the curve itself). Say that μ_1 C_u–FOSD μ_2 if for every closed and measurable W which is closed under C_u, we have $\mu_1(W) \geq \mu_2(W)$. This is a natural analog of the statement that a measure F on $\mathbf{R}$ FOSD G if for every s, $F(\{r \in \mathbf{R} \mid r \geq s\}) \geq G(\{r \in \mathbf{R} \mid r \geq s\})$.

We will show that $\succeq_2$ is more temptation averse than $\succeq_1$ if and only if μ_1 C_u–FOSD μ_2. Before presenting the theorem, though, we present the second version of the comparison.

In light of Lemma 1, we can obviously take a change of variables and write the distribution over $\mathcal{W}$ as a distribution over (A, v) pairs in $[-1, 1] \times \mathcal{V}$. This is more complex when $\mu(\{u, -u\}) > 0$ since, as noted, each $L(v)$ set contains both of these points. Intuitively, we can divide these mass points and allocate part of each mass point to each v in order to eliminate this ambiguity. Thus there will be a family of measures over $[-1, 1] \times \mathcal{V}$ which correspond to μ.

Given any of these distributions, we can construct a version of the conditional probabilities to write it as a marginal over $\mathcal{V}$ times a conditional on $L(v)$ given each $v \in \mathcal{V}$. Intuitively, this is just rewriting our initial measure on points on the globe as follows. First, we draw a point from the equator at random. Then, conditional on the point selected, we choose a distance to move up or down the longitude through this point.

Given Lemma 2, a natural version of a comparison of measures on the globe which seems to match our C_u–FOSD notion is that for each v, μ_1's conditional on $L(v)$ dominates μ_2's conditional in the usual first–order stochastic dominance sense.

To state this more precisely, we say that a pair $(\mu_{\mathcal{V}}, \mu_L(\cdot \mid v))$ is a ***version*** of μ if $\mu_{\mathcal{V}}$ is a probability measure over $\mathcal{V}$ and, for each v, $\mu_L(\cdot \mid v)$ is a probability measure over $[-1, 1]$ such that for every measurable $E \subseteq \mathcal{W}$,[15]

$$\mu(E) = \int_{v \in \mathcal{V}} \mu_L \left(\left\{ A \in [-1, 1] \mid Au + v\sqrt{1 - A^2} \in E \right\} \mid v \right) \mu_{\mathcal{V}}(dv). \tag{1}$$

It is easy to see that any μ has at least one such version and has essentially[16] one if $\mu(\{u, -u\}) = 0$. If $\mu(\{u, -u\}) > 0$, then μ has infinitely many versions since equation (1) only states that

$$\mu(\{u\}) = \int_{v \in \mathcal{V}} \mu_L(\{1\} \mid v) \mu_{\mathcal{V}}(dv)$$

and analogously for $\mu(\{-u\})$ without pinning down the versions any further on either subspace.

[15] The measures $\mu_{\mathcal{V}}$ and each $\mu_L(\cdot \mid v)$ are de ned on the Borel σ{algebras.

[16] In the usual sense | that is, up to sets of measure zero.

In light of Lemma 2, the versions enable us to translate the somewhat abstract notion of C_u–FOSD into the usual first–order stochastic dominance notion on A's. More specifically, we have the following result.

Theorem 4. *Fix* $\succeq_i$ *with canonical random Strotz representation* (u_i, μ_i)*,* $i = 1, 2$*. Then the following statements are equivalent:*

1. $\succeq_2$ *is more temptation averse than* $\succeq_1$*.*
2. $u_1 = u_2$ *and* μ_1 C_u{*FOSD* μ_2*.*
3. $u_1 = u_2$ *and there exists versions of* μ_i*,* $(\mu^i_{\mathcal{V}}, \mu^i_L)$*,* $i = 1, 2$*, such that* $\mu^1_{\mathcal{V}} = \mu^2_{\mathcal{V}}$ *and for almost every* $v \in \mathcal{V}$*, the conditional distribution* $\mu^1_L(\cdot \mid v)$ *rst order stochastically dominates* $\mu^2_L(\cdot \mid v)$*.*

5 Costly Self Control and Random Self Indulgence: Choice from Menus

To this point, we have focused on the random Strotz and random GP models as representations of preferences over menus. In this sense, we have treated them as models of choice *of* a menu. As we have seen, at least if we restrict attention to Lipschitz continuous models, we cannot use choice of menus to distinguish the random GP and random Strotz models.

On the other hand, each model also makes predictions about choice *from* menus. In the case of random Strotz, it is natural to interpret the representation (u, μ) as saying that with probability $\mu(w)$, the choice is the one made by w with ties broken in favor of u (where this is stated for measures with finite support for simplicity). In the case of a GP representation (u, v), Gul and Pesendorfer argue that the natural interpretation of the choice from a menu x is that it is some maximizer of $u+v$ from that menu. It is natural to interpret a random GP representation (u, ν) analogously as saying that with probability $\nu(v)$, the choice is that which maximizes $u + v$. If we adopt these interpretations as parts of the models and observe choices from menus, can we distinguish random GP and random Strotz? If we observe *both* choices of menus and choices from menus, can we distinguish the two models?

One difficulty in answering these questions is that the randomness of the choices from menus makes it difficult to say what we should assume is observed. To explain the issue in the simplest possible setting, consider a random Strotz representation where the

measure μ puts positive probability on two w's, w_1 and w_2. Clearly, if we only observe one instance of the agent choosing from a particular menu, we will see only the choice made by one of w_1 and w_2 and hence have very little information. In particular, we will not be able to infer anything about the other w or the relative probability of the two. Thus it would scarcely be surprising if this were not enough information to be able to distinguish this model from a random GP model. Hence we will assume that there are "many" observations of choices from each menu.

Even when we assume this, though, there is another problem. In general, for some menus, one or both w's will have multiple best choices, even after taking account of the breaking of ties in favor of u. That is, we could have $\alpha, \beta \in B_u(B_{w_1}(x))$, in which case the random Strotz model says that when w_1 is the chooser's utility function, either α or β could be chosen from x. If we have repeated observations of w_1 choosing from x, what determines the probabilities that w_1 chooses α rather than β?[17]

There are at least three approaches we could take to this issue. First, we could restrict attention to random Strotz and random GP models with the property that such indeterminacies have probability zero. For such models, which we call ***determinate***, there is no ambiguity about the predicted probability distribution over choices from a given menu.

Second, we could hypothesize that the chooser always breaks ties from any given menu in the same fashion. In terms of the example above, we could assume that when w_1 is the utility function of the chooser and x is the menu, w_1 always makes the same choice, either always α or always β. Under this hypothesis, the behavior of the agent is again given by a unique probability distribution over choices from each menu, so we can assume that this is what we observe. On the other hand, a single model can rationalize many such probability distributions.

Finally, we could assume that we have the maximum conceivable amount of data, namely, that we observe ***all*** possible probability distributions over choices for each menu. There are at least two ways to interpret this. First, in terms of the example above, we could hypothesize that we observe the distribution of choices from x when w_1 always chooses α ***and*** the distribution when w_1 always chooses β. Alternatively, we could hypothesize that we somehow directly observe the fact that the chooser is willing to pick either α or β. Neither interpretation is obviously compelling, making it hard to interpret how we could obtain such data. On the other hand, this case obviously identifies the upper bound on what we could identify from observing choices from menus.

[17] Of course, the multiplicity of optimal choices is a standard complication in revealed preference theory. Our point is that this multiplicity is more of a complication here. In the usual theory, different choices from the same set by the same decision maker are interpreted as indicating indifference, while here different choices could result from indifference or from the decision maker having a different w draw.

A rough summary of our results is that it is difficult to use choice from menus to distinguish the random Strotz and random GP models, but the extent of the difficulty diminishes as we make stronger assumptions about how much is observed regarding such choices. More specifically, if we restrict attention to models where indeterminacies have zero probability, then choices from menus can ***never*** distinguish random Strotz and random GP models. To be more precise, if the true model is a random GP with no indeterminacies, then we can never reject random Strotz. Similarly, if the true model is random Strotz with no indeterminacies, then we can never reject random GP.[18]

If we allow all possible models and assume that indeterminate choices from a given menu by a given w are always resolved the same way, as in the second option described above, then any random Strotz choice behavior is also possible under random GP, but the converse is not true. Thus if the true model is random Strotz, we can never reject random GP, while if random GP is true, we may or may not be able to reject random Strotz.

Finally, if we consider the maximum conceivable amount of data on choices from menus, where we see all possible ways of resolving the indeterminacies, as in the third option described earlier, we cannot, in general, distinguish the two models but may be able to do so. That is, there will be a wide class of behavior which is possible under both models as well as behavior which is possible under one but not the other.

Formally, fix a nontrivial[19] random Strotz representation (u, μ). We define a ***selection function for*** (u, μ) to be a measurable function $\beta^* : X \times \text{supp}(\mu) \to \Delta(Z)$ such that $\beta^*(x, w) \in B_u(B_w(x))$ for all $(x, w) \in X \times \text{supp}(\mu)$. That is, the selection function $\beta^*(x, w)$ gives one way that choices could be made from menu x in the random Strotz representation as a function of w. Then we can define a ***random choice function*** $\rho : X \to \Delta(\Delta(Z))$ by

$$\rho_x(E) = \mu\left(\{w \in \mathcal{W} \mid \beta^*(x, w) \in E\}\right) \tag{2}$$

for every measurable $E \subseteq x$. We say that such a random choice function is ***RS rationalized*** by (u, μ) or is ***RS rationalizable***. Let C denote the set of random choice functions rationalized by (u, μ). We call C a ***random choice correspondence*** and say that it is RS rationalized by (u, μ) or is RS rationalizable.

We now define analogous notions for a random GP representation (u, ν). We define a selection function to be a measurable $\hat{\beta}^* : X \times \text{supp}(\nu) \to \Delta(Z)$ such that $\hat{\beta}^*(x, v) \in B_{u+v}(x)$ for all $(x, v) \in X \times \text{supp}(\nu)$. We then define a random choice function ρ generated

[18]These two situations differ slightly though. If the true model is random Strotz with no indeterminacies, we cannot reject random GP but *might* be able to reject random GP with no indeterminacies. On the other hand, if the true model is random GP with no indeterminacies, we will not be able to reject random Strotz with no indeterminacies.

[19]That is, the utility function u has the property that $u(\alpha) \neq u(\beta)$ for some $\alpha, \beta \in \Delta(Z)$.

by this selection function by

$$\rho_x(E) = \nu\left(\left\{v \in \mathbf{R}^K \mid \hat{\beta}^*(x,v) \in E\right\}\right)$$

every measurable $E \subseteq x$. We say such a ρ is RGP rationalizable or is RGP rationalized by (u, ν). Similarly, letting C denote the set of all such ρ's, we say that the random choice correspondence C is RGP rationalizable or is RGP rationalized by (u, ν).

In either case, for any given menu x, we let $C(x) = \{p \in \Delta(x) \mid p = \rho_x, \text{ for some } \rho \in C\}$. When $C(x)$ is a singleton for all x, we say that C is *determinate*.

These definitions are similar to those used in Gul and Pesendorfer (2006a, 2006b) except that they focus on the case where C is determinate so that only random choice *functions* matter.

In Section 5.1, we consider whether the models can be distinguished when our observations consist of a single random choice function. We discuss both the case where we restrict attention to models which are determinate (have a single ρ) and the case where we have repeated observations of choices from each menu where the selection function is constant over time (*i.e.*, we observe one ρ). In Section 5.2, we analyze whether the models can be distinguished when we observe the entire random choice correspondence C. Finally, in Section 5.3, we turn to the case of the maximum conceivable amount of data, where we can observe the entire random choice correspondence as well as the preference over menus. As we will see, we cannot generally separate the two models with data only on choices from menus but can with both choices from menus and preferences over menus.

5.1 Choice from Menus: Selections

In this subsection, we assume that the only data available to potentially distinguish the random GP and random Strotz models is a random choice function. We use the following theorem on random choice *correspondences* to show that if we observe only a single random choice *function*, then, whether the model is determinate or not, this data alone is not, in general, sufficient to distinguish random GP and random Strotz.

Theorem 5. *If C is a random choice correspondence which is RS rationalizable, then there exists an RGP rationalizable $\hat{C}$ such that $C(x) \subseteq \hat{C}(x)$ for all menus x. Similarly, if $\hat{C}$ is an RGP rationalizable random choice correspondence, then there exists an RS rationalizable C with $C(x) \subseteq \hat{C}(x)$ for all menus x.*

Proof. Suppose (u, μ) is a random Strotz representation that rationalizes C. Use a change of variables to construct a random GP representation where for each $w \in \text{supp}(\mu)$, we

have $v = w - u \in \text{supp}(\nu)$ with $\nu(w-u) = \mu(w)$.[20] It is then easy to see that for every $\rho \in C(x)$, ρ must be rationalized by this random GP representation since it simply embodies a particular form of tie–breaking for the $u+v$'s. That is, letting $\hat{C}$ denote the random choice correspondence generated by the random GP representation, we have $C(x) \subseteq \hat{C}(x)$ for all x.

For the other direction, suppose (u, ν) is a random GP representation that rationalizes $\hat{C}$. Define a distribution μ over w's by the same change of variables $w = u+v$. Choose any nontrivial $\hat{u}$ and consider the random Strotz representation $(\hat{u}, \mu)$. It is easy to see that this simply adds a tie–breaking rule to the random GP so that the choice correspondence C generated by $(\hat{u}, \mu)$ must satisfy $C(x) \subseteq \hat{C}(x)$ for all x. ▌

To see the implications of this result, suppose the true model describing the agent is random Strotz and we observe a single random choice function. Whether the model is determinate or not, Theorem 5 tells us that this random choice function is also rationalizable by the random GP model. The random GP model which rationalizes it may or may not be determinate, however.

Similarly, suppose that the true model is random GP and we observe a random choice function from this model. If the model is determinate, this is equivalent to observing the entire random choice correspondence. In this case, Theorem 5 tells us that this random choice function is also rationalizable by a determinate random Strotz model. To see this, simply note that if the RGP rationalizable $\hat{C}$ is determinate, then it is a singleton. Hence the RS rationalizable C with $C(x) \subseteq \hat{C}(x)$ for all x must also be a singleton, so it is determinate as well.

Theorem 5 does not tell us what happens if we observe a single random choice function from an indeterminate random GP model. That is, it leaves open the question whether there could be such a choice function which is not RS rationalizable. In Section H of the Appendix, we give an example of a random choice function which is rationalizable by an indeterminate random GP model but is not RS rationalizable.

5.2 Choice from Menus: Correspondences

We now turn to the case where the entire choice correspondence is observed. Our main result is that we achieve equality in Theorem 5 if and only if the random Strotz model generating the correspondence is continuous. To be more precise, we show that while there are random choice correspondences with an RGP rationalization and no RS ratio-

[20] For expositional simplicity, this is stated as if the supports are nite. The more general case is straightforward.

nalization and conversely, there is a very large set of random choice correspondences with both kinds of rationalization. In particular, the set of choice correspondences with both kinds of rationalization is precisely the set with an RS rationalization such that (u, μ) represents a continuous preference over menus. Put differently, if we restrict attention to continuous models, then strictly more choices can be interpreted using the random GP model than using the random Strotz model.

The key to our characterization of the overlap is the following result.

Lemma 3. *A random Strotz representation (u, μ) is continuous if and only if for every menu x, u is indi erent over all of $B_w(x)$ with probability 1. That is,*

$$\mu\left(\{w \in \mathcal{W} \mid u(\alpha) = u(\beta),\ \forall \alpha, \beta \in B_w(x)\}\right) = 1.$$

In other words, a random Strotz is continuous if and only if the tie–breaking assumption that w breaks ties in favor of u is irrelevant. If C has an RS rationalization where (u, μ) satisfies the property identified in Lemma 3, we say it has a continuous RS rationalization.

As an aside, we note that this lemma may suggest a relationship between continuity and determinacy of the choice correspondence, but such a relationship does not exist. Continuity of a random Strotz corresponds to the statement that $B_u(B_w(x)) = B_w(x)$ with probability 1 for every x. Determinacy of a random Strotz corresponds to the statement that $B_u(B_w(x))$ is a singleton with probability 1 for every x.

The following lemma shows that the change of variables construction we used in the proof of Theorem 5 is precisely how models with the same choice correspondence are related.

Lemma 4. *Suppose random choice correspondence C has an RS rationalization (u, μ) and an RGP rationalization $(\hat{u}, \nu)$. Then the measures μ and ν are related by the change of variables $w = \hat{u} + v$.*

Proof. Let x be a sphere in the interior of $\Delta(Z)$. Then every point on the sphere is the unique optimum for exactly one $w \in \mathcal{W}$. Hence the unique ρ_x in $C(x)$ identifies μ exactly up to sets of measure zero. Hence μ is identified through the change of variables defined by $\rho_x(\beta) = \mu(w)$ for the unique w which is maximized over x at β.

Similarly, ν is uniquely identified from ρ_x as well. Again, $\nu(v)$ is identified through the change of variables which replaces $\rho_x(\beta)$ with $\nu(v)$ for the unique v such that $\hat{u} + v$ is maximized over x at β. Hence ν and μ are related as described. ▌

These lemmas make it easy to show the following.

Theorem 6. *A random choice correspondence C has both an RS and a RGP rationalization if and only if it has a continuous RS rationalization.*

Proof. Suppose C has a continuous RS rationalization. By definition, the selection function β^* must satisfy $\beta^*(x, w) \in B_u(B_w(x))$. But if the RS rationalization is continuous, then $B_u(B_w(x)) = B_w(x)$ with probability 1. Hence the choice correspondence also has an RGP rationalization where we construct ν from μ by taking the change of variables $v = w - u$. Obviously, for $v = w - u$, we have $B_{u+v}(x) = B_w(x)$. Hence C has both an RS and an RGP rationalization.

We complete the proof by showing that if C does not have a continuous RS rationalization, then it cannot have both an RS and an RGP rationalization. Suppose, to the contrary, that C has an RS rationalization (u, μ) and an RGP rationalization $(\hat{u}, \nu)$. Since C has no continuous RS rationalization, there is some menu, denoted by y, such that for a positive measure set of w's, we have $u(\alpha) \neq u(\beta)$ for some $\alpha, \beta \in B_w(y)$.

We now derive a contradiction by constructing a particular selection rule for the RGP rationalization and the random choice function it generates. Since the random choice correspondences for the two models are the same, this random choice function must be generated by some selection rule for RS rationalization. We show no such selection rule can exist, generating the contradiction.

Turning to the details, fix any selection rule $\hat{\beta}^*$ for the RGP rationalization such that $\hat{\beta}^*(y, v) \in B_{-u}(B_{\hat{u}+v}(y))$. That is, $\hat{\beta}^*$ breaks $\hat{u} + v$ ties ***against*** u. Let ρ be the random choice function generated by this selection. That is, for any measurable $E \subseteq y$, $\rho_y(E) = \nu(\{v \in \mathbf{R}^K \mid \hat{\beta}^*(y, v) \in E\})$. Obviously, this ρ_y is contained in $C(y)$. Hence this ρ_y must be generated by some selection function in the RS rationalization. That is, there must exist $\beta^* : X \times \mathcal{W} \to y$ such that $\beta^*(y, w) \in B_u(B_w(y))$ and for any measurable $E \subseteq y$, $\rho_y(E) = \mu(\{w \in \mathcal{W} \mid \beta^*(w) \in E\})$.

But this is impossible. Simply note that the fact that both selection functions generate the same distribution over choices implies

$$\int_{v \in \mathbf{R}^K} u(\hat{\beta}^*(y, v))\, \nu(dv) = \int_{w \in \mathcal{W}} u(\beta^*(y, w))\, \mu(dw),$$

which cannot hold since $u(\hat{\beta}^*(y, w - u))) \leq u(\beta^*(y, w))$ for all w, strictly so on a positive measure set. ▌

It is worth noting that any choice that can be rationalized by the (nonrandom) GP model can be rationalized by a (nonrandom) Strotz model. To be specific, take the (nonrandom) choice correspondence generated by the GP (u, v), namely, $C(x) = B_{u+v}(x)$.

This is obviously rationalized by the "Strotz" model $(\hat{u}, \mu)$ where $\hat{u} = u + v$ and μ puts probability 1 on $u + v$. This is simply a reflection of the fact, noted by GP, that in their model, it is choice *of* a menu which reflects temptation, not choices from menus.

Theorem 6 obviously implies that there are random choice correspondences with an RS rationalization but no RGP rationalization since this is true of any correspondence rationalized by a discontinuous RS model. It is easy to see that there are also random choice correspondences with an RGP rationalization but no RS rationalization. To see this, recall from Section 5.1 that there are random choice functions with RGP rationalizations but no RS rationalization. Obviously, any random choice correspondence containing such a function has an RGP rationalization but no RS rationalization.

5.3 Choice of Menus Plus Choice from Menus

We have seen that as a general statement, neither preferences over menus alone nor choices from menus alone can distinguish these models. However, the two objects together *always* can, except in the degenerate case of no temptation. There are two (essentially equivalent) ways to demonstrate this. First, we could consider a random Strotz and a random GP representation of the same preference over menus and compare the choices they make from menus. Second, we could consider a random Strotz representation and a random GP representation that produce indistinguishable choices from menus and compare them in terms of preferences over menus. In doing so, we must rule out the case of no temptation which is a special case of both models and obviously one where the two representations are equivalent. We say that a preference $\succeq$ over menus *exhibits temptation* if there exist $\alpha, \beta \in \Delta(Z)$ with $\{\alpha\} \succ \{\beta\}$ and $\{\alpha\} \succ \{\alpha, \beta\}$. It's not hard to show that if (u, μ) is a random Strotz representation of $\succeq$, then $\succeq$ exhibits temptation if and only if there is a $w \in \text{supp}(\mu)$ such that w does not represent the same preference over lotteries as u. Similarly, if (u, ν) is a random GP representation of $\succeq$, then $\succeq$ exhibits temptation iff there exists $v \in \text{supp}(\nu)$ such that v does not represent the same preference over lotteries as u.

First, we show that if a random Strotz representation and a random GP representation generate the same preference over menus, then the random Strotz choices from menus exhibit strictly more temptation in a certain sense. Fix any preference over menus $\succeq$ which has both a random Strotz representation, (u, μ), and a random GP representation (u, ν).[21] Let C_{RS} be the random choice correspondence generated by the random Strotz representation and C_{RGP} the one generated by the random GP. Fix any menu x, any

[21] Note that since the two representations have the same preferences over menus, they must have the same preferences over singleton menus in particular. Hence, up to normalization, they must have the same u.

$\rho_x^{RS} \in C_{RS}(x)$, and $\rho_x^{RGP} \in C_{RGP}(x)$. Then the random Strotz choices from menus show more temptation in the sense that the agent prefers the expected behavior under the random GP. That is,

$$\left\{ \int \beta \, \rho_x^{RGP}(d\beta) \right\} \succeq \left\{ \int \beta \, \rho_x^{RS}(d\beta) \right\}.$$

Furthermore, this inequality must be strict for some x, ρ^{RGP}, and ρ^{RS} if $\succeq$ exhibits temptation.

To prove this, let $\hat{\beta}^*$ denote the selection function for (u, ν) that generates ρ^{RGP}. By definition, $\hat{\beta}^*(x, v) \in B_{u+v}(x)$. Then for any x,

$$\begin{aligned}
V_{RGP}(x) &= \int \left\{ \max_x [u(\beta) + v(\beta)] - \max_x v(\beta) \right\} \nu(dv) \\
&= \int \left\{ u(\hat{\beta}^*(x, v)) + v(\hat{\beta}^*(x, v)) - \max_x v(\beta) \right\} \nu(dv) \\
&\leq \int \left\{ u(\hat{\beta}^*(x, v)) + v(\hat{\beta}^*(x, v)) - v(\hat{\beta}^*(x, v)) \right\} \nu(dv) \\
&= \int u(\hat{\beta}^*(x, v)) \, \nu(dv) = u\left(\int \beta \, \rho_x^{RGP}(d\beta) \right) \\
&= V_{RGP}\left(\left\{ \int \beta \, \rho_x^{RGP}(d\beta) \right\} \right)
\end{aligned}$$

Hence $\left\{ \int \beta \, \rho_x^{RGP}(d\beta) \right\} \succeq x$. But $\rho_x^{RS} \in C_{RS}(x)$ implies that $x \sim \left\{ \int \beta \, \rho_x^{RS}(d\beta) \right\}$. (To see this, recall that in the random Strotz model, the agent evaluates the menu x as if he receives β with probability $\rho_x^{RS}(\beta)$.) Therefore, $\left\{ \int \beta \, \rho_x^{RGP}(d\beta) \right\} \succeq \left\{ \int \beta \, \rho_x^{RS}(d\beta) \right\}$.

To see that there must be some menu where the comparison is strict if $\succeq$ exhibits temptation, consider any α and β that satisfy $\{\alpha\} \succ \{\beta\}$ and $\{\alpha\} \succ \{\alpha, \beta\}$. It is not hard to show that this implies that there is $v \in \text{supp}(\nu)$ with $u(\alpha) > u(\beta)$ and $v(\alpha) < v(\beta)$ and to use this to show that for the menu $x = \{\alpha, \beta\}$, we must have $V_{RGP}(x) < V_{RGP}(\{\int \beta \, \rho_x^{RGP}(d\beta)\})$. Hence the inequality above is strict for such a menu.

Similarly, if we equate the choices from menus of the random Strotz and random GP agents, the random GP agent's preference over menus will show more concern about temptation. To be precise, fix random choice correspondences C_{RS} and C_{RGP} where C_{RS} has an RS rationalization (u, μ) and C_{RGP} has an RGP rationalization $(u, \hat{\nu})$. Note that we take the same u function for the two models. It is not hard to show that this hypothesis is unrestrictive in the sense that if a choice correspondence has an RGP rationalization, we can find such a rationalization with any u function we choose.[22] Suppose

[22] To see this, consider for simplicity an RGP rationalization (u, ν) where the support of ν is

these choice correspondences are (essentially) the same in the sense that for every x, $C_{RS}(x) \subseteq C_{RGP}(x)$. (Obviously, the argument to follow also works if we require these correspondences to be the same.) Let $\succeq_{RS}$ and $\succeq_{RGP}$ be the preferences over menus represented by the random Strotz and random GP rationalizations respectively. Then we claim that $\succeq_{RGP}$ is more temptation averse than $\succeq_{RS}$, strictly so if $\succeq_{RGP}$ exhibits temptation. Note that to show this result requires the hypothesis that both rationalizations use the same u since only then can preferences be compared under our notion of "more temptation averse."

To show that the comparison holds, we show that $\{\alpha\} \succeq_{RS} x$ implies $\{\alpha\} \succeq_{RGP} x$. To see this, note that for any $\rho_x \in C_{RS}(x)$, $x \sim_{RS} \left\{\int \beta\, \rho_x(d\beta)\right\}$. Hence $\alpha \succeq_{RS} x$ implies $\alpha \succeq_{RS} \left\{\int \beta\, \rho_x(d\beta)\right\}$. Since both rationalizations use the same u, they rank singletons the same way, so $\{\alpha\} \succeq_{RGP} \left\{\int \beta\, \rho_x(d\beta)\right\}$. But exactly the same reasoning as above then shows that $\left\{\int \beta\, \rho_x(d\beta)\right\} \succeq_{RGP} x$. Hence $\{\alpha\} \succeq_{RGP} \left\{\int \beta\, \rho_x(d\beta)\right\} \succeq_{RGP} x$, establishing the desired conclusion. The strictness of the comparison when $\succeq_{RGP}$ exhibits temptation follows from a similar argument to that used above.

Both these results are versions of the statement that random Strotz choices from menus are "more self indulgent" from the point of view of commitment utility than random GP choices. The first comparison says this directly since it states that if preferences over menus are the same, then commitment to the expected choice under random Strotz is worse than commitment to the expected choice under random GP. The second comparison says the same thing indirectly by saying that if we change the choices to make the choice behavior (essentially) the same, we must have "improved" the behavior in the random Strotz model relative to the behavior in the random GP model in the sense that the former corresponds to evaluating menus with less concern about temptation than the latter.

To see the intuition behind this result most simply, suppose $\succeq$ has a GP representation and hence also a random Strotz representation. Suppose this preference has $\{\alpha\} \succ \{\alpha, \beta\} \succ \{\beta\}$. In the GP case, this is rationalized by having $u(\alpha) > u(\beta)$, $v(\beta) > v(\alpha)$, and $u(\alpha) + v(\alpha) > u(\beta) + v(\beta)$. These rankings imply that

$$\begin{aligned} V_{GP}(\{\alpha, \beta\}) &= \max\{u(\alpha) + v(\alpha), u(\beta) + v(\beta)\} - \max\{v(\alpha), v(\beta)\} \\ &= u(\alpha) - [v(\beta) - v(\alpha)]. \end{aligned}$$

Thus the predicted choice is α, the same as the "choice" from the menu $\{\alpha\}$, but the menu is ranked lower than $\{\alpha\}$ because of the self–control cost of $v(\beta) - v(\alpha)$. By contrast, the random Strotz representation would have $V_{RS}(\alpha, \beta) = pu(\alpha) + (1 - p)u(\beta)$ for some

$\{v_1, \ldots, v_J\}$. Fix *any* $\hat{u}$. For $j = 1, \ldots, J$, let $\hat{v}_j = u - \hat{u} + v_j$, so that $\hat{u} + \hat{v}_j = u + v_j$. Define $\hat{\nu}$ by $\hat{\nu}(\hat{v}_j) = \nu(v_j)$. The RGP rationalization (u, ν) says that given menu x, the choice is the one which maximizes $u + v_j$ with probability $\nu(v_j)$. Clearly, this is the same thing as saying it is the choice which maximizes $\hat{u} + \hat{v}_j$ with probability $\hat{\nu}(\hat{v}_j)$. Hence $(\hat{u}, \hat{\nu})$ is also an RGP rationalization.

$p \in (0, 1)$. Thus the random Strotz representation "explains" the fact that $\{\alpha\} \succ \{\alpha, \beta\}$ not by self–control costs but by a nonzero probability of "self–indulgent" behavior under the latter menu.

In other words, the random Strotz model explains the desire for commitment entirely in terms of a fear of succumbing to temptation, while the random GP model explains it in part by this but in part by a desire to avoid self–control costs. Hence the choice from menus in the random Strotz model must involve succumbing to temptation more frequently.

6 Conclusion

In summary, we have shown that the random Strotz and random GP models are, in general, indistinguishable in terms of commitment behavior or in terms of subsequent choices but that with both kinds of data do they have different predictions. In addition, we have shown that the random Strotz representation is unique up to an appropriate normalization, provided a characterization of a notion of "more temptation averse than," and axiomatically characterized the Lipschitz continuous version of the representation.

There are many interesting directions for further research. First, it would be natural to consider dynamic versions of the random Strotz model analogous to the way Gul and Pesendorfer (2004) extend the static GP model to dynamic environments. It is easy to see that the dynamic version of GP can also be rewritten as a dynamic random Strotz model. This fact shows that there are at least some interesting recursive random Strotz models, suggesting that a broader exploration of such models may be fruitful.

Second, the results here may have other interpretations of interest. For example, Olszewski (2007) and Ahn (2007) suggest models of ambiguity where an act is viewed not as a function from states to consequences but as a set of lotteries, where this is interpreted as a set of consequences. (See also related work by Gajdos, Hayashi, Tallon, and Vergnaud (2008).[23] In other words, we interpret a menu not as a set of options that the agent will choose from later but as a set of possible outcomes that "Nature" will choose from later. Under this interpretation, the random Strotz model represents the agent as forming various theories about what guides Nature's choices. The weak set betweenness axiom which characterizes random Strotz is arguably even more plausible

[23] The Steiner point, which plays a signi cant role in the analysis of Gajdos, Hayashi, Tallon, and Vergnaud, provides an interesting connection between their work and random Strotz. One de nition of the Steiner point of a set of lotteries is that it is the expected value of the lottery chosen by an expected utility preference which is drawn at random from a uniform distribution. Thus it is the expected choice by a particular random Strotz agent.

in this context than in the context of temptation.

Finally, while numerous versions of the Strotz model have been used in applications, particularly the special case of quasi–hyperbolic discounting (e.g., Laibson (1997), O'Donoghue and Rabin (1999), or Benabou and Tirole (2002)), we think exploration of applications of the random Strotz model may be of great interest. As noted above and in Caplin and Leahy (2006), the commitment behavior of random Strotz is, in general, better behaved than that of the nonrandom version of the model as one can avoid problems with discontinuities.

A Proof of Continuity Claims from Section 3.2

Fix a random GP representation and the associated V_{RGP}. Fix a sequence of menus x_n converging to x. Then

$$|V_{RGP}(x) - V_{RGP}(x_n)| \leq \int \left| \max_x (u+v) - \max_x v - \max_{x_n}(u+v) + \max_{x_n} v \right| \nu(dv).$$

Let V denote the support of ν. For any $\bar{V} \subseteq V$,

$$\begin{aligned} |V_{RGP}(x) - V_{RGP}(x_n)| \leq & \int_{\bar{V}} \left| \max_x (u+v) - \max_x v - \max_{x_n}(u+v) + \max_{x_n} v \right| \nu(dv) \\ & + \int_{V \setminus \bar{V}} [\max_{\Delta(Z)} u - \min_{\Delta(Z)} u] \, \nu(dv) \end{aligned}$$

where the second integral uses the fact that

$$\max_{\Delta(Z)} u \geq \max_x u \geq \max_x (u+v) - \max_x v \geq \min_x u \geq \min_{\Delta(Z)} u.$$

By Theorem 1.8.11 of Schneider (1993),

$$\left| \max_x w - \max_{x_n} w \right| \leq \quad \| w \| \, d_h(x, x_n)$$

for any expected utility function w where $\| \cdot \|$ denotes sup norm. Hence

$$\begin{aligned} |V_{RGP}(x) - V_{RGP}(x_n)| \leq & \int_{\bar{V}} [\| u+v \| + \| v \|] \, d_h(x, x_n) \nu(dv) \\ & + \int_{V \setminus \bar{V}} [\max_{\Delta(Z)} u - \min_{\Delta(Z)} u] \, \nu(dv) \end{aligned}$$

The measure ν is on Borel sets and so is regular. Hence for every $\varepsilon > 0$, there exists a compact subset of v's, say $\bar{V}_\varepsilon$, such that $\nu(\bar{V}_\varepsilon) \geq 1-\varepsilon$. Fix any $\delta > 0$. Take ε sufficiently small that

$$\varepsilon[\max_{\Delta(Z)} u - \min_{\Delta(Z)} u] < \frac{\delta}{3}.$$

Note that this is independent of x. Given this ε and the implied $\bar{V}_\varepsilon$, take $\bar{n}$ sufficiently large that for all $n \geq \bar{n}$,

$$d_h(x, x_n)(1-\varepsilon) \max_{v \in \bar{V}_\varepsilon} [\| u + v \| + \| v \|] < \frac{\delta}{3}.$$

(Note that the compactness of $\bar{V}_\varepsilon$ implies that the maximum on the left–hand side is finite, ensuring that this is possible.) Hence for all $n \geq \bar{n}$,

$$\begin{aligned} |V_{RGP}(x) - V_{RGP}(x_n)| &\leq (1-\varepsilon) \max_{v \in \bar{V}_\varepsilon} [\| u + v \| + \| v \|]\, d_h(x, x_n) + \varepsilon[\max_{\Delta(Z)} u - \min_{\Delta(Z)} u] \\ &< \frac{\delta}{3} + \frac{\delta}{3} < \delta. \end{aligned}$$

Hence $V_{RGP}(x)$ is continuous in x.

To show that a random Strotz representation can be discontinuous, consider the following example. Assume there are three outcomes in Z, let $u = (2, 0, 0)$, and let $w = (0, 1, 0)$. Consider the (non)random Strotz with $\mu(\{w\}) = 1$. Let $\alpha = (1/2, 1/2, 0)$, $\beta = (0, 1/2, 1/2)$, and $\beta_n = (0, (1/2) + (1/n), (1/2) - (1/n))$. Finally, let $x = \{\alpha, \beta\}$ and $x_n = \{\alpha, \beta_n\}$. It is easy to see that $x_n \to x$ as $n \to \infty$. Also, $V_{RS}(x) = 1$, while $V_{RS}(x_n) = 0$ for all n. Obviously, the preference represented by V_{RS} is discontinuous since this function is discontinuous.

B Example of Random GP which is not Lipschitz Continuous

In a moment, we give specific numbers but for clarity, we begin by specifying the key properties of the construction. We will give a u and v such that $u \cdot \mathbf{1} = v \cdot \mathbf{1} = u \cdot v = 0$ and $u \cdot u = v \cdot v = 1$. We construct lotteries α, β, and for each n, β_n with the following properties. First, $u(\alpha) = c$ for some $c > 0$ and $v(\alpha) = 0$. Second, $u(\beta) = v(\beta) = 0$. Finally, $v(\beta_n) = 1/n$, $u(\beta_n) = 0$, and $d_h(x, x_n) = 1/n$ where $x = \{\alpha, \beta\}$ and $x_n = \{\alpha, \beta_n\}$.

A specific example which generates these numbers is the following. Suppose there are three pure outcomes, $v = (q_1, -q_1/2, -q_1/2)$ and $u = (0, q_2, -q_2)$ where $q_1 = \sqrt{6}/3$

and $q_2 = \sqrt{2}/2$. For the lotteries, let $\alpha = (1/2, 1/2, 0)$ (which gives $c = \sqrt{6}/6$), $\beta = (1/3, 1/3, 1/3)$, and

$$\beta_n = \left(\frac{1}{3} + \frac{2}{3nq_1}, \frac{1}{3} - \frac{1}{3nq_1}, \frac{1}{3} - \frac{1}{3nq_1} \right)$$

To define the representation, for any menu y, let

$$V(y) = \frac{\lambda - 1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda} \right)^k \left[\max_y (u + \lambda^k v) - \max_y \lambda^k v \right],$$

where $\lambda > 1$. It is easy to see that $V(x) = c$ since $v(\alpha) = v(\beta) = 0$, while $u(\alpha) = c > 0 = u(\beta)$.

To compute $V(x_n)$, note that

$$u(\alpha) + \lambda^k v(\alpha) \geq u(\beta_n) + \lambda^k v(\beta_n)$$

if and only if $c \geq \lambda^k (1/n)$ or $k \leq \log(nc)/\log(\lambda)$. Let k_n denote the largest integer less than or equal to $\log(nc)/\log(\lambda)$. Then

$$V(x_n) = \frac{\lambda - 1}{\lambda} \left[\sum_{k=0}^{k_n} \left(\frac{1}{\lambda} \right)^k \left[c - \lambda^k \frac{1}{n} \right] + \sum_{k=k_n+1}^{\infty} \left(\frac{1}{\lambda} \right)^k (0) \right].$$

Simplifying,

$$V(x_n) = \frac{\lambda - 1}{\lambda} \left[\left(\sum_{k=0}^{k_n} \left(\frac{1}{\lambda} \right)^k c \right) - \frac{k_n + 1}{n} \right]$$

or

$$V(x_n) = \left[1 - \left(\frac{1}{\lambda} \right)^{k_n+1} \right] c - \left(\frac{\lambda - 1}{\lambda} \right) \left(\frac{k_n + 1}{n} \right).$$

So

$$V(x) - V(x_n) = \left(\frac{1}{\lambda} \right)^{k_n+1} c + \left(\frac{\lambda - 1}{\lambda} \right) \left(\frac{k_n + 1}{n} \right).$$

Hence

$$\frac{V(x) - V(x_n)}{d_h(x, x_n)} = n[V(x) - V(x_n)] = nc \left(\frac{1}{\lambda} \right)^{k_n+1} + (k_n + 1) \left(\frac{\lambda - 1}{\lambda} \right).$$

The first term on the far right is always positive and the second goes to infinity as $n \to \infty$. Hence this representation is not Lipschitz continuous.

C Proof of Theorem 2

To see that (2) implies (3), recall that we showed in the text that any random GP representation can be rewritten as a random Strotz representation. Obviously, if the random GP representation is Lipschitz continuous, the random Strotz is as well since it is simply a different way of writing the same function. Hence (2) implies (3).

It is easy to show that any preference with a random Strotz representation must satisfy weak order, independence, and mixture continuity (i.e., $x \succ x' \succ x''$ implies that there exists $\lambda, \bar{\lambda} \in (0,1)$ with $\lambda x + (1-\lambda)x'' \succ x' \succ \bar{\lambda}x + (1-\bar{\lambda})x''$). Lemma 1 of DLRS implies that any preference with a Lipschitz continuous random Strotz representation must satisfy L–continuity. Theorem 1 of Dekel, Lipman, Rustichini, and Sarver (2007b) then implies that such a preference must satisfy continuity. Hence (3) implies (1).

We now show that (1) implies (2) — that is, that any preference which satisfies weak order, continuity, independence, L–continuity, and weak set betweenness has a random GP representation. So fix such a preference $\succeq$. By DLRS, we have a representation of the form

$$V(x) = \int \max_{\beta \in x} w(\beta)\, \mu(dw),$$

where μ is a countably additive signed measure on the Borel σ–algebra over $\mathcal{W}$. The rest of the proof rewrites this representation in a series of steps, turning it into a random GP representation.

The following simple lemma will aid in this rewriting.

Lemma 5. *Fix* $w, \bar{w} \in \mathcal{W}$. *Then* $w \cdot \bar{w} \in [-1, 1]$. *Furthermore,* $w \cdot \bar{w} = 1$ *i* $w = \bar{w}$ *and* $w \cdot \bar{w} = -1$ *i* $w = -\bar{w}$.

Proof. Obviously, $\sum_k (w_k - \bar{w}_k)^2 \geq 0$ with equality iff $w = \bar{w}$. Hence $\sum_k w_k^2 - 2\sum_k w_k \bar{w}_k + \sum_k \bar{w}_k^2 \geq 0$ or

$$w \cdot w - 2w \cdot \bar{w} + \bar{w} \cdot \bar{w} \geq 0.$$

But $w \cdot w = \bar{w} \cdot \bar{w} = 1$, so $w \cdot \bar{w} \leq 1$ with equality if and only if $w = \bar{w}$. A similar argument covers the comparison to -1. ▌

Let $u = \int w\mu(dw)$. Clearly, $w \cdot \mathbf{1} = 0$ for all $w \in \mathcal{W}$ implies $u \cdot \mathbf{1} = 0$.

Suppose $u \cdot u = 0$, so u is the zero vector. Then $V(\{\alpha\}) = V(\{\beta\})$ for all α and β. By weak set betweenness, then, $x \sim y$ for all menus x and y. (To see this, note that x must be indifferent to any singleton in x by weak set betweenness and the fact that all these singletons are indifferent. Hence x has to be indifferent to every singleton and likewise for

y.) But we have assumed that $\succeq$ is nontrivial, a contradiction.[24] So henceforth, assume $u \cdot u > 0$.

Without loss of generality, we can assume $u \cdot u = 1$. If not, we can divide V by the constant $\sqrt{u \cdot u}$, giving a representation where the measure is $\hat{\mu} = \mu/\sqrt{u \cdot u}$. It is easy to see that this representation will have $\int w \, \hat{\mu}(dw) = u/\sqrt{u \cdot u} = \hat{u}$ where $\hat{u} \cdot \hat{u} = 1$. Hence we simply take $u \cdot u = 1$. Thus $u \in \mathcal{W}$.

Lemma 6. *Weak set betweenness implies* $\mu(\{u\}) \geq 0$ *and* $\mu(\{-u\}) \leq 0$.

Proof of Lemma. For each $w \in \mathcal{W}$, let

$$\alpha_w = \frac{1}{K}\mathbf{1} + \varphi w$$

where $\varphi \in (0, 1/K)$. Recall that K is the number of pure outcomes. Clearly, $\alpha_w \cdot \mathbf{1} = 1$ since $w \cdot \mathbf{1} = 0$. Hence this is a well defined lottery as long as each component is nonnegative. Let $w(k)$ denote the kth component of the vector w. So we require

$$\frac{1}{K} + \varphi \min_k w(k) \geq 0.$$

But $\sum_k [w(k)]^2 = 1$, so $\min_k w(k) \geq -1$. Hence

$$\frac{1}{K} + \varphi \min_k w(k) \geq \frac{1}{K} - \varphi.$$

Hence the assumption that $\varphi < 1/K$ implies that this is positive. Let $x = \{\alpha_w \mid w \in \mathcal{W}\}$.

Clearly, $w \cdot \alpha_w = \varphi$. For any $w \neq w'$, $w \cdot \alpha_{w'} = \varphi w \cdot w' < \varphi$ by Lemma 5. So $w \cdot \alpha$ is uniquely maximized over $\alpha \in x$ at $\alpha = \alpha_w$.

Let $\alpha_\varepsilon = \alpha_u + \varepsilon u$ for each $\varepsilon > 0$. Just as before, this is a lottery for all ε sufficiently small. Let $x_\varepsilon = x \cup \{\alpha_\varepsilon\}$. Note that $u \cdot (\alpha_u + \varepsilon u) = u \cdot \alpha_u + \varepsilon > u \cdot \alpha_u$. Hence $\{\alpha_\varepsilon\} \succ \{\alpha_u\} \succ \{\beta\}$ for all $\beta \in x \setminus \{\alpha_u\}$. Hence by weak set betweenness, $x_\varepsilon \succeq x$ for every $\varepsilon > 0$. Let

$$W_\varepsilon = \{w \in \mathcal{W} \mid w \cdot \alpha_\varepsilon \geq \max_{\beta \in x} w \cdot \beta\}.$$

Thus

$$W_\varepsilon = \{w \in \mathcal{W} \mid (\varphi + \varepsilon) w \cdot u \geq \varphi\}.$$

So

$$V(x_\varepsilon) - V(x) = \int_{W_\varepsilon} [(\varphi + \varepsilon) w \cdot u - \varphi] \, \mu(dw) \geq 0.$$

[24] It is easy to drop the requirement that $\succeq$ is nontrivial since for any probability measure ν on $\mathcal{W}$, $\int [\max_x [u + v] - \max_x v] \, \nu(dv) = 0$. Thus we still obtain a (trivial) random Strotz representation for $\succeq$.

Since $w \cdot u \leq 1$ for all $w \in \mathcal{W}$ by Lemma 5, we know that $(\varphi + \varepsilon)w \cdot u - \varphi \leq \varepsilon w \cdot u$. Hence for every sufficiently small $\varepsilon > 0$, we have

$$\int_{W_\varepsilon} \varepsilon w \cdot u \, \mu(dw) \geq 0.$$

so $\int_{W_\varepsilon} w \cdot u \, \mu(dw) \geq 0$ for all ε. It is easy to see, though, that $W_\varepsilon \to \{u\}$ as $\varepsilon \downarrow 0$. Hence $\mu(\{u\}) \geq 0$.

To show that $\mu(\{-u\}) \leq 0$, replace α_ε in the argument above with $\hat{\alpha}_\varepsilon = \alpha_{-u} - \varepsilon u$. ▌

In light of these results, let $c_1 = \mu(\{u\}) \geq 0$ and $c_2 = -\mu(\{-u\}) \geq 0$. Then we have

$$V(x) = \int w \, \hat{\mu}(dw) + c_1 \max u - c_2 \max(-u)$$

where $\hat{\mu}$ is the measure defined by $\hat{\mu}(E) = \mu(E \setminus \{u, -u\})$.

To rewrite further, for each $w \in \mathcal{W} \setminus \{-u, u\}$, let

$$k_w = \sqrt{1 - (w \cdot u)^2}$$

$$\theta_w = \frac{w \cdot u}{k_w}$$

$$h_w = (1/k_w)w - \theta_w u$$

Note that θ_w and h_w are well defined iff k_w is well defined and strictly positive. Since $w \notin \{-u, u\}$, we have $w \cdot u \in (-1, 1)$ by Lemma 5, so this holds. Note that $h_w \cdot \mathbf{1} = (1/k_w)w \cdot \mathbf{1} - \theta_w u \cdot \mathbf{1} = 0$. Also,

$$\begin{aligned} h_w \cdot h_w &= (1/k_w)^2 - 2(\theta_w/k_w)w \cdot u + \theta_w^2 \\ &= \frac{1}{1 - (w \cdot u)^2} - \theta_w^2 \\ &= \frac{1}{1 - (w \cdot u)^2} - \frac{(w \cdot u)^2}{1 - (w \cdot u)^2} \\ &= 1. \end{aligned}$$

Hence each $h_w \in \mathcal{W}$. Also, $h_w \cdot u = (1/k_w)w \cdot u - \theta_w = 0$. Hence for all $w \in \mathcal{W} \setminus \{-u, u\}$, $h_w \in H \equiv \{h \in \mathcal{W} \mid h \cdot u = 0\}$. Finally, note that $k_w w = \theta_w u + h_w$.

Now we can write the representation as

$$V(x) = \int k_w \max[\theta_w u + h_w] \, \hat{\mu}(dw) + c_1 \max u - c_2 \max(-u).$$

Note that since $k_w\hat{\mu}$ is a signed measure, we can write it as the difference between two positive measures, say F and G, so

$$V(x) = \int \max[\theta_w u + h_w]\,[F(dw) - G(dw)] + c_1 \max u - c_2 \max(-u).$$

For the most part, we will be interested in the first term of V, so it will prove convenient to define

$$V_1(x) = \int \max[\theta_w u + h_w]\,[F(dw) - G(dw)],$$

so that $V(x) = V_1(x) + c_1 \max u - c_2 \max(-u)$.

Since w is only relevant through the measurable functions θ_w and h_w, we can rewrite the measures as measures over (θ, h) pairs to obtain (with some abuse of notation)

$$V_1(x) = \int \max[\theta u + h]\,F(d(\theta, h)) - \int \max[\theta u + h]\,G(d(\theta, h)).$$

Let Θ denote the set of possible values of θ_w for $w \in \mathcal{W}$.

Lemma 7. *For every closed, measurable* $\hat{H} \subseteq H$*,* $F(\Theta \times \hat{H}) = G(\Theta \times \hat{H})$.

Proof of Lemma. For each $h \in H$, let

$$\alpha_h = \frac{1}{K}\mathbf{1} + \varphi h$$

where $\varphi \in (0, 1/K)$. Just as in the proof of Lemma 6, α_h is a lottery for every h. Let x denote the collection of the α_h's over $h \in H$.

As in the proof of Lemma 6, $h \cdot \alpha$ is uniquely maximized over $\alpha \in x$ at $\alpha = \alpha_h$. Since $u \cdot \alpha_h = 0$ for all h, this implies that $\theta u + h$ is uniquely maximized over $\alpha \in x$ at $\alpha = \alpha_h$. Also, u and $-u$ are indifferent over $\alpha \in x$.

Fix any closed $\hat{H} \subseteq H$ and any $\varepsilon \in (0, (1/K) - \varphi)$. For each $h \in \hat{H}$, let $\beta_h = \alpha_h + \varepsilon h$. As before, β_h is a lottery as long as $\varphi + \varepsilon < 1/K$, which holds by assumption.

Let y be x together with all the β_h's for $h \in \hat{H}$. Just as above, it is easy to see that for any $h \in \hat{H}$, $\theta u + h$ is uniquely maximized over $\beta \in y$ at $\beta = \beta_h$. Again, u and $-u$ are indifferent over all of y.

Consider any $h \in H \setminus \hat{H}$. Clearly, for $h' \in \hat{H}$, $h \cdot \beta_{h'} = (\varepsilon + \varphi) h \cdot h'$. Let

$$r_h^* = \max_{h' \in \hat{H}} h \cdot h'.$$

Clearly, the best payoff in y for $\theta u + h$ is the larger of φ and $r_h^*(\varphi + \varepsilon)$.

It is easy to see that for any $\beta \in y$, we have $u \cdot \beta = 0$. Hence weak set betweenness implies that $x \sim y$. Thus the expected gain to the various $\theta u + h$'s in moving from x to the larger set y must integrate to zero. For any $h \in \hat{H}$, the gain is ε. For any $h \in H \setminus \hat{H}$, the gain is $\max\{0, (\varphi + \varepsilon)r_h^* - \varphi\}$. Note that the latter expression also applies for $h \in \hat{H}$ since $r_h^* = 1$ for $h \in \hat{H}$. Hence

$$\int_{h \in H | r_h^*(\varphi+\varepsilon) \geq \varphi} \int_\theta [(\varphi + \varepsilon) r_h^* - \varphi][F(d(\theta, h)) - G(d(\theta, h))] = 0.$$

Divide both sides by ε to obtain

$$\int_{h \in H | r_h^*(\varphi+\varepsilon) \geq \varphi} \int_\theta \left[\frac{(\varphi + \varepsilon) r_h^* - \varphi}{\varepsilon}\right] [F(d(\theta, h)) - G(d(\theta, h))] = 0.$$

Note that $r_h^* \leq 1$ always with equality if and only if $h \in \hat{H}$. Hence

$$\begin{aligned}
\int_{h \in H \setminus \hat{H} | r_h^*(\varphi+\varepsilon) \geq \varphi} \int_\theta F(d(\theta, h)) + \int_{h \in \hat{H}} \int_\theta [F(d(\theta, h)) - G(d(\theta, h))] \\
\geq \int_{h \in H | r_h^*(\varphi+\varepsilon) \geq \varphi} \int_\theta \left[\frac{(\varphi + \varepsilon) r_h^* - \varphi}{\varepsilon}\right] [F(d(\theta, h)) - G(d(\theta, h))] = 0 \\
\geq - \int_{h \in H \setminus \hat{H} | r_h^*(\varphi+\varepsilon) \geq \varphi} \int_\theta G(d(\theta, h)) + \int_{h \in \hat{H}} \int_\theta [F(d(\theta, h)) - G(d(\theta, h))]
\end{aligned}$$

As $\varepsilon \downarrow 0$, the set $\{h \in H \setminus \hat{H} \mid r_h^*(\varphi + \varepsilon) \geq \varphi\}$ converges to the empty set. To be more precise, for every $h' \in H \setminus \hat{H}$, there exists $\bar{\varepsilon} > 0$ such that $h' \notin \{h \in H \setminus \hat{H} \mid r_h^*(\varphi+\varepsilon) \geq \varphi\}$ for all $\varepsilon \in (0, \bar{\varepsilon})$. Hence as $\varepsilon \downarrow 0$, this equation converges to

$$\int_{h \in \hat{H}} \int_\theta [F(d(\theta, h)) - G(d(\theta, h))] \geq 0 \geq \int_{h \in \hat{H}} \int_\theta [F(d(\theta, h)) - G(d(\theta, h))],$$

so for every $\hat{H}$, we have

$$\int_{h \in \hat{H}} \int_\theta [F(d(\theta, h)) - G(d(\theta, h))] = 0,$$

or $F(\Theta \times \hat{H}) = G(\Theta \times \hat{H})$. ▌

Note in particular that Lemma 7 implies that $F(\Theta \times H) = G(\Theta \times H)$. At the end of the proof, we consider the case where $F(\Theta \times H) = G(\Theta \times H) = 0$. In the meantime,

we assume $F(\Theta \times H) = D > 0$. Thus, abusing notation by redefining F and G, we can write $V_1(x)$ as

$$V_1(x) = D \int \max[\theta u + h][F(d(\theta, h)) - G(d(\theta, h))]$$

where F and G are now probability measures. Now we see that Lemma 7 says that F and G have the same marginal distribution over H.

Lemma 8. *There are versions of the conditionals such that for almost every $h \in H$, $F(\cdot \mid h)$ dominates $G(\cdot \mid h)$ in the sense of rst{order stochastic dominance.*

Proof of Lemma. Return to the α_h's constructed above. Fix any closed measurable $\hat{H} \subseteq H$. Fix any $\hat{\theta} \in \Theta$ and any $\varepsilon > 0$ such that $\varphi + \varepsilon\hat{\theta} > 0$. (Recall that $\varphi > 0$ so this holds for ε sufficiently small.) For each $h \in \hat{H}$, let

$$\beta^*_h = \alpha_h + \varepsilon \left(\hat{\theta} h - \frac{1}{u \cdot u} u \right).$$

It is easy to see that each β^*_h is a lottery if ε is sufficiently small. For $\hat{\varepsilon} > 0$, let

$$\beta^{**}_h = \beta^*_h + \hat{\varepsilon} h,$$

which again is a lottery for sufficiently small $\hat{\varepsilon}$. Let x^* denote the set consisting of all α_h's for $h \in H$ and β^*_h's for $h \in \hat{H}$. Let x^{**} denote the set consisting of x^* and all β^{**}_h's for $h \in \hat{H}$.

Obviously, u is maximized over x^* at any of the α_h's, while $-u$ is maximized at any of the β^*_h's. Neither of these functions achieves a higher value on x^{**}. Also, $\theta u + h$ is maximized over x^* either at α_h or at one of the $\beta^*_{\bar{h}}$'s and is maximized over x^{**} either at α_h, or one of the $\beta^*_{\bar{h}}$'s or $\beta^{**}_{\bar{h}}$'s. The payoff to $\theta u + h$ from $\beta^*_{\bar{h}}$ is

$$(\varphi + \varepsilon\hat{\theta}) h \cdot \bar{h} - \varepsilon\theta.$$

Since $\varphi + \varepsilon\hat{\theta} > 0$, this is strictly increasing in $h \cdot \bar{h}$. Hence the best of the $\beta^*_{\bar{h}}$'s for h must maximize $h \cdot \bar{h}$. As before, define

$$r^*_h = \max_{\bar{h} \in \hat{H}} h \cdot \bar{h}.$$

As before, for $h \in \hat{H}$, $r^*_h = 1$, while for $h \notin \hat{H}$, we have $r^*_h < 1$. Hence the best payoff to $\theta u + h$ from any $\beta^*_{\bar{h}}$ is $(\varphi + \varepsilon\hat{\theta}) r^*_h - \varepsilon\theta$. A similar argument shows that the best payoff to $\theta u + h$ from any $\beta^{**}_{\bar{h}}$ is $(\varphi + \varepsilon\hat{\theta} + \hat{\varepsilon}) r^*_h - \varepsilon\theta$.

The table below summarizes the best payoff to $\theta u + h$ from each "category" of lotteries:

Lottery	Payoff
α	φ
β^*	$(\varphi + \varepsilon\hat{\theta}) r^*_h - \varepsilon\theta$
β^{**}	same $+ \hat{\varepsilon} r^*_h$

We can compute the gains from adding the β_h^{**}'s as follows.

(θ, h)	Change	Gain
$(\varphi + \varepsilon\hat{\theta})r_h^* < \theta\varepsilon + \varphi \le (\varphi + \varepsilon\hat{\theta} + \hat{\varepsilon})r_h^*$	$\alpha_h \to \beta^{**}$	$(\varphi + \hat{\varepsilon} + \varepsilon\hat{\theta})r_h^* - \varphi - \varepsilon\theta$
$\varepsilon\theta + \varphi \le (\varepsilon\hat{\theta} + \varphi)r_h^*,\ r_h^* \ge 0$	$\beta^* \to \beta^{**}$	$\hat{\varepsilon}r_h^*$

So we can write $V(x^{**}) - V(x^*)$ as D times

$$\int_{\hat{\varepsilon}r_h^* \ge \max\{0, \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*\}} \left[\hat{\varepsilon}r_h^* - \max\{0, \varphi + \varepsilon\theta - (\varphi + \varepsilon\hat{\theta})r_h^*\}\right][F(d(\theta,h)) - G(d(\theta,h))].$$

By weak set betweenness, this must be less than or equal to zero. Dividing the implied inequality by $D\hat{\varepsilon}$ and rearranging yields

$$\int_{\hat{\varepsilon}r_h^* \ge \max\{0, \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*\}} \left[r_h^* - \frac{\max\{0, \varphi + \varepsilon\theta - (\varphi + \varepsilon\hat{\theta})r_h^*\}}{\hat{\varepsilon}}\right] F(d(\theta,h))$$
$$\le \int_{\hat{\varepsilon}r_h^* \ge \max\{0, \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*\}} \left[r_h^* - \frac{\max\{0, \varphi + \varepsilon\theta - (\varphi + \varepsilon\hat{\theta})r_h^*\}}{\hat{\varepsilon}}\right] G(d(\theta,h)).$$

Note that the function being integrated on both sides is always weakly less than r_h^*. Also, throughout the range being integrated over, the function is nonnegative. Hence the right–hand side increases if we replace the function with r_h^*, while the left–hand side decreases if we integrate over a smaller range. Therefore,

$$\int_{r_h^* \ge 0 \ge \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*} \left[r_h^* - \frac{\max\{0, \varphi + \varepsilon\theta - (\varphi + \varepsilon\hat{\theta})r_h^*\}}{\hat{\varepsilon}}\right] F(d(\theta,h))$$
$$\le \int_{\hat{\varepsilon}r_h^* \ge \max\{0, \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*\}} r_h^*\, G(d(\theta,h)).$$

Note that in the range being integrated over on the left–hand side, the function is just r_h^*. Furthermore, if $r_h^* < 0$, then we cannot have $0 \ge \varphi + \varepsilon\theta - (\varphi + \varepsilon\hat{\theta})r_h^*$. Hence this is equivalent to

$$\int_{(\varphi+\varepsilon\hat{\theta})r_h^* \ge \varphi+\varepsilon\theta} r_h^*\, F(d(\theta,h)) \le \int_{\hat{\varepsilon}r_h^* \ge \max\{0, \varphi+\varepsilon\theta-(\varphi+\varepsilon\hat{\theta})r_h^*\}} r_h^*\, G(d(\theta,h)).$$

Now the left–hand side is independent of $\hat{\varepsilon}$ and only the range of integration on the right–hand side depends on $\hat{\varepsilon}$. If we take $\hat{\varepsilon}$ to 0, the range being integrated over on the right–hand side converges to the same range as on the left–hand side. More precisely, for any h not in the range on the left–hand side, h is not contained in the range on the right–hand side for all sufficiently small $\hat{\varepsilon} > 0$. Hence we obtain

$$\int_{(\varphi+\varepsilon\hat{\theta})r_h^* \ge \varphi+\varepsilon\theta} r_h^*\, F(d(\theta,h)) \le \int_{(\varphi+\varepsilon\hat{\theta})r_h^* \ge \varphi+\varepsilon\theta} r_h^*\, G(d(\theta,h)).$$

Note that we are integrating over the set of (θ, h) such that

$$\theta - \hat{\theta} r_h^* \le \frac{\varphi(r_h^* - 1)}{\varepsilon}.$$

If $h \notin \hat{H}$, the right–hand side is strictly negative. Hence as $\varepsilon \downarrow 0$, the set of (θ, h) satisfying this inequality with $h \notin \hat{H}$ converges to the empty set. That is, for all $h \notin \hat{H}$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, (θ, h) is not in the range being integrated over for any θ. Hence this converges as $\varepsilon \downarrow 0$ to the set of (θ, h) with $h \in \hat{H}$ and $\theta \le \hat{\theta}$. Since $r_h^* = 1$ for all $h \in \hat{H}$, we see that

$$\int_{h \in \hat{H}, \theta \le \hat{\theta}} F(d(\theta, h)) \le \int_{h \in \hat{H}, \theta \le \hat{\theta}} G(d(\theta, h)).$$

If $F(\hat{H}) = 0$ (where $F(\hat{H}) \equiv F(\Theta \times \hat{H})$), then this inequality simply says $0 \le 0$. (Recall that F and G have the same marginals on H.) Otherwise, divide both sides of this inequality by $F(\hat{H})$. Then we have $F(\hat{\theta} \mid \hat{H}) \le G(\hat{\theta} \mid \hat{H})$.

Hence for almost every $\hat{h}$, the conditional distribution of $F(\theta \mid \hat{h})$ dominates $G(\theta \mid \hat{h})$ in the sense of first–order stochastic dominance. ▮

We use this lemma to rewrite the representation in the form of a random GP. Recall that we have

$$V_1(x) = D \int \max[\theta u + h] \, F(d(\theta, h)) - D \int \max[\theta u + h] \, G(d(\theta, h))$$

and $V(x) = V_1(x) + c_1 \max u - c_2 \max(-u)$ where $c_1 \ge 0$ and $c_2 \ge 0$. By the disintegration theorem (e.g., Theorem 1, Appendix F, in Pollard (2002)) and the fact that F and G have equal marginals on H, we can take a version of the conditionals for F and G and write V_1 as

$$\begin{aligned} V_1(x) &= D \int \max[\theta u + h] \, F(d\theta | h) F(dh) - D \int \max[\theta u + h] \, G(d\theta | h) G(dh) \\ &= D \int_h \int_\theta \{\max[\theta u + h] F(d\theta \mid h) - \max[\theta u + h] G(d\theta \mid h)\} \, F(dh). \end{aligned}$$

Now we do two more changes of variables. First, we change variables by replacing $D\theta$ with θ and Dh with h. We again abuse notation by continuing to use F and G to denote the distributions over the redefined variables. Note that this change of variables does not affect the conclusion of Lemma 8. Second, we write V_1 as

$$V_1(x) = \int_h \int_t \Big\{ \max[F^{-1}(t \mid h) u + h] \, dt - \max[G^{-1}(t \mid h) u + h] \, dt \Big\} \, F(dh).$$

Let $q(t \mid h) = F^{-1}(t \mid h) - G^{-1}(t \mid h)$. By Lemma 8, $F^{-1}(t \mid h) \geq G^{-1}(t \mid h)$ for all (t, h), so $q(t \mid h) \geq 0$.

Substituting:

$$V_1(x) = \int_h \int_t \left\{ \max[q(t \mid h)u + G^{-1}(t \mid h)u + h] \, dt - \max[G^{-1}(t \mid h)u + h] \, dt \right\} F(dh).$$

For (t, h) such that $q(t \mid h) \neq 0$, let $v(t, h) = [G^{-1}(t \mid h)u + h]/q(t \mid h)$. For (t, h) such that $q(t \mid h) = 0$, define $v(t, h)$ arbitrarily. Then if $q(t \mid h) \geq 0$ for all t and h, then we have

$$V_1(x) = \int_h \int_t q(t \mid h) \left\{ \max[u + v(t, h)] - \max v(t, h) \right\} \, dt \, dF(h).$$

Note that

$$\int_{h,t} q(t \mid h) \, dt \, F(dh) = \int_h \int_t F^{-1}(t \mid h) F(dh) - \int_h \int_t G^{-1}(t \mid h) G(dh)$$

which equals

$$\int_{(\theta,h)} \theta F(d(\theta, h)) - \int_{(\theta,h)} \theta G(d(\theta, h)).$$

We claim this equals $1 - c_1 - c_2$. To see this, recall that

$$u = \int [\theta u + h][dF - dG] + c_1 u + c_2 u$$

or

$$u(1 - c_1 - c_2) = \int [\theta u + h][dF - dG].$$

Hence

$$(1 - c_1 - c_2) u \cdot u = \int [\theta u \cdot u][dF - dG],$$

since $u \cdot h = 0$ for all h in the support. Hence since $u \cdot u = 1$, we have $\int \theta [dF - dG] = 1 - c_1 - c_2$.

Define v_1 to be identically zero and $v_2 = -2u$. Note that

$$\max(u + v_1) - \max v_1 = \max u$$

while

$$\begin{aligned} \max(u + v_2) - \max v_2 &= \max(u - 2u) - \max(-2u) \\ &= \max(-u) - 2\max(-u) = -\max(-u). \end{aligned}$$

Let $q_1 = c_1$ and $q_2 = c_2$. Hence

$$V(x) = \int_h \int_t q(t \mid h) \left\{ \max[u + v(t,h)] - \max v(t,h) \right\} \, dt \, dF(h)$$
$$+ q_1 \left\{ \max[u + v_1] - \max v_1 \right\} + q_2 \left\{ \max[u + v_2] - \max v_2 \right\},$$

Now $\int_h \int_t q(t \mid h) dt F(dh) + q_1 + q_2 = 1$, so this is a random GP representation.

Note that if $\int F(dw) = \int G(dw) = 0$, then we have only the last two terms. The proof that $q(t \mid h)$ integrates to $1 - c_1 - c_2$ still holds but $q(t \mid h)$ is identically zero, proving that $c_1 + c_2 = 1$. Hence $q_1 + q_2 = 1$ so we again have a random GP representation. ∎

D Proof of Lemma 1

The claims about $w = u$ and $w = -u$ are obvious.

To complete the proof, fix any w different from u and $-u$ and let $A = w \cdot u$, $a = \sqrt{1 - (w \cdot u)^2}$, and $v = (w - Au)/a$. By Lemma 5, $w \cdot u \in (-1, 1)$ so a is well–defined and strictly positive, implying that v is well–defined. Obviously $w = av + Au = v\sqrt{1 - A^2} + Au$. To see that $v \in \mathcal{V}$, note that $v \cdot \mathbf{1} = 0$ and

$$v \cdot v = \frac{w \cdot w - 2Aw \cdot u + A^2 u \cdot u}{a^2} = \frac{1 - (w \cdot u)^2}{a^2}$$

where the last equality uses the definition of A and the normalization that $w \cdot w = u \cdot u = 1$. By the definition of a, then, $v \cdot v = 1$. So $v \in \mathcal{W}$. To complete the proof that $v \in \mathcal{V}$, note that

$$v \cdot u = \frac{w \cdot u - A}{a} = 0.$$

To see that (v, A) is unique, suppose $w = av + Au = \hat{a}\hat{v} + \hat{A}u$ where $a = \sqrt{1 - A^2}$ and $\hat{a} = \sqrt{1 - \hat{A}^2}$. Hence

$$w \cdot u = Au \cdot u = \hat{A}u \cdot u,$$

so $A = \hat{A}$. Therefore, $a = \hat{a}$. Finally,

$$w \cdot \hat{v} = av \cdot \hat{v} = \hat{a}\hat{v} \cdot \hat{v},$$

so $v \cdot \hat{v} = 1$. But this implies $v = \hat{v}$. ∎

E Proof of Lemma 2

For brevity, let $f(A) = \sqrt{1 - A^2}$. The key facts to note about f are that $f(A) \geq 0$ for all A, strictly so for $A \in (-1, 1)$, and that f is strictly decreasing in A for $A > 0$ and strictly increasing for $A < 0$.

Lemma 9. *If $w_1 \; C_u \; w_2$ and $w_2 \notin \{-u, u\}$, then there exists $C, c \geq 0$, at least one strictly positive, such that $w_1 = Cu + cw_2$.*[25]

Proof. Suppose not. Let

$$W = \{w' \mid w' = Cu + cw_2 + e\mathbf{1}, \text{ for some } C, c \geq 0, \text{ some } e\}.$$

Obviously, W is closed, convex, and nonempty. Since $w_1 \notin W$ by hypothesis, there is a separating hyperplane. So there exists a vector $p \neq 0$, such that $p \cdot w_1 < p \cdot w'$ for all $w' \in W$. That is,

$$p \cdot w_1 < Cp \cdot u + cp \cdot w_2 + ep \cdot \mathbf{1}$$

for all $C, c \geq 0$ and all e.

Since the sign of e is arbitrary, this implies that $\sum_k p_k = 0$. Otherwise, we can take $e \to -\infty$ or $e \to \infty$ to make $ep \cdot \mathbf{1}$ arbitrarily negative and force a contradiction. Similarly, $p \cdot u \geq 0$ and $p \cdot w_2 \geq 0$. To see this, suppose to the contrary that $p \cdot u < 0$. Then we can take C arbitrarily large to generate a contradiction. Obviously, w_2 is analogous. Finally, we must have $p \cdot w_1 < 0$. Otherwise, take $C = c = e = 0$ for all i to get a contradiction.

Hence there exists a vector p, such that $\sum_k p_k = 0$, $p \cdot u \geq 0$, $p \cdot w_2 \geq 0$, and $p \cdot w_1 < 0$. It is not difficult to show that we can rewrite the vector p as a difference between two interior lotteries, α and β to obtain the conclusion that $u \cdot \alpha \geq u \cdot \beta$, $w_2 \cdot \alpha \geq w_2 \cdot \beta$, but $w_1 \cdot \alpha < w_1 \cdot \beta$.

Since $w_1 \; C_u \; w_2$, it must be true that $u \cdot \alpha = u \cdot \beta$. We can write $w_2 = f(A_2)v_2 + A_2 u$. Fix $\varepsilon > 0$ and let

$$\alpha^* = \alpha + \varepsilon \left[f(A_2)u - A_2 v_2 \right].$$

It is not hard to show that if ε is sufficiently small, then α^* is a lottery. Note that $u \cdot \alpha^* = u \cdot \alpha + \varepsilon f(A_2)$ as $u \cdot u = 1$ and $u \cdot v_2 = 0$. Since $w_2 \notin \{-u, u\}$, we have $A_2 \in (-1, 1)$, so $f(A_2) > 0$. Hence $u \cdot \alpha^* > u \cdot \alpha = u \cdot \beta$.

Also,

$$w_2 \cdot \alpha^* = w_2 \cdot \alpha + \varepsilon [f(A_2)A_2 - A_2 f(A_2)] = w_2 \cdot \alpha = w_2 \cdot \beta.$$

[25] This is a version of the Harsanyi Aggregation Theorem. See Weymark (1991).

For ε sufficiently small, the fact that $w_1 \cdot \alpha < w_1 \cdot \beta$ implies $w_1 \cdot \alpha^* < w_1 \cdot \beta$, contradicting $w_1 \; C_u \; w_2$. ▌

(If.) First, suppose there exists $v \in \mathcal{V}$ such that $w_i = f(A_i)v + A_i u$, $i = 1, 2$, with $A_1 \geq A_2$. If $A_2 = 1$, this requires $A_1 = 1$ also, in which case $w_1 = w_2 = u$ and $w_1 \; C_u \; w_2$. If $A_2 = -1$, then it is easy to see that every w satisfies $w \; C_u \; w_2$, so w_1 certainly does.

So suppose $A_2 \in (-1, 1)$, implying $f(A_2) > 0$. Obviously, if $A_1 = A_2$, then $w_1 = w_2$, so $w_1 \; C_u \; w_2$. So without loss of generality, assume $A_1 > A_2$. Then we have

$$\begin{aligned} w_1 = A_1 u + f(A_1)v &= \left[A_1 - A_2 \frac{f(A_1)}{f(A_2)}\right] u + A_2 \frac{f(A_1)}{f(A_2)} u + f(A_1)v \\ &= \left[A_1 - A_2 \frac{f(A_1)}{f(A_2)}\right] u + \frac{f(A_1)}{f(A_2)} \left[A_2 u + f(A_2)v\right] \\ &= \left[A_1 - A_2 \frac{f(A_1)}{f(A_2)}\right] u + \frac{f(A_1)}{f(A_2)} w_2 \end{aligned}$$

The coefficient on w_2 is nonnegative. Also, $A_1 > A_2$ implies that the coefficient on u is strictly positive. To see this, note that the conclusion is obvious if $A_1 > 0 \geq A_2$ since $f(A_1)/f(A_2) \geq 0$. If $A_1 > A_2 > 0$, the fact that f is strictly decreasing in A in this range implies

$$A_1 - A_2 \frac{f(A_1)}{f(A_2)} > A_1 - A_2 > 0.$$

If $0 \geq A_1 > A_2$, the fact that f is strictly increasing in A in this range implies exactly the same conclusion. So the coefficient on u is strictly positive. Hence if $u(\alpha) > u(\beta)$ and $w_2(\alpha) \geq w_2(\beta)$, we must have $w_1(\alpha) > w_1(\beta)$. Hence $w_1 \; C_u \; w_2$.

(Only if.) Suppose $w_1 \; C_u \; w_2$. If $w_2 = u$, then this requires $w_1 = u$ and the claim follows trivially. If $w_2 = -u$, again, the claim follows trivially since for any $v \in \mathcal{V}$, we have $w_2 = f(A_2)v + A_2 u$ with $A_2 = -1$. So suppose $w_2 \notin \{-u, u\}$. Then by Lemma 9, there exists $C, c \geq 0$, at least one strictly positive, such that $w_1 = Cu + cw_2$. Since $w_2 \notin \{-u, u\}$, there is a unique $v \in \mathcal{V}$ and $A_2 \in (-1, 1)$ such that $w_2 = f(A_2)v + A_2 u$. Hence $w_1 = cf(A_2)v + (C + cA_2)u$. If $c = 0$, then $w_1 = u$, implying that $w_1 = f(A_1)v + A_1 u$ with $A_1 = 1 \geq A_2$, so the conclusion follows. If $C = 0$, we must have $c = 1$ implying $w_1 = w_2$, so again the conclusion follows. Hence we can assume that $C > 0$ and $c > 0$. Thus we have $w_1 = f(A_1)v + A_1 u$. So we only need to show that $A_1 \geq A_2$.

So suppose $1 > A_2 > A_1$. If $w_1 = -u$, then we cannot have $w_1 \; C_u \; w_2$, so $A_1 > -1$. Hence $f(A_i) > 0$, $i = 1, 2$. Fix any interior α and $\varepsilon > 0$. Let

$$\beta = \alpha + \varepsilon \left[f(A_2)u - A_2 v\right].$$

It is easy to show that β is a lottery for all sufficiently small ε. Then $u \cdot \beta = u \cdot \alpha + \varepsilon f(A_2)$. Since $f(A_2) > 0$, then $u(\beta) > u(\alpha)$. Also, it is easy to see that $w_2 \cdot \beta = w_2 \cdot \alpha$. Finally,

$w_1 \cdot \beta = w_1 \cdot \alpha + \varepsilon[f(A_2)A_1 - A_2 f(A_1)]$. Hence $w_1 \cdot \beta < w_1 \cdot \alpha$ iff $A_1/f(A_1) < A_2/f(A_2)$ which holds as $A_1 < A_2$. Thus there is a pair of lotteries for which w_2 agrees with u and w_1 does not, so we cannot have $w_1 \; C_u \; w_2$, a contradiction. ▌

F Proof of Theorem 3

Given a function $A^* : \mathcal{V} \to [-1, 1]$, let

$$W(A^*) = \bigcup_{v \in \mathcal{V}} \{w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au, \text{ for some } A \geq A^*(v), \; A \neq -1\}.$$

Note that by excluding $A = -1$, the definition of $W(A^*)$ ensures that $-u \notin W(A^*)$ for any A^*.

Lemma 10. *A set $W \subseteq \mathcal{W}$, $W \neq \mathcal{W}$, is closed and closed under C_u if and only if there exists a lower semi{continuous function A^* such that $W = W(A^*)$ and $A^*(v) > -1$ for all $v \in \mathcal{V}$.*

Proof. Fix any lower semi–continuous function A^* such that $A^*(v) > -1$ for all $v \in \mathcal{V}$. Let $W = W(A^*)$. Since the definition of $W(A^*)$ prevents $-u \in W(A^*)$, $W \neq \mathcal{W}$. From Lemma 2, it is easy to see that W is closed under C_u if and only if $v\sqrt{1 - A^2} + Au \in W$ implies $v\sqrt{1 - \hat{A}^2} + \hat{A}u \in W$ for all $\hat{A} \in (A, 1]$. Obviously, the definition of $W(A^*)$ implies W is closed under C_u. Finally, to show that W is closed, fix any sequence w_n converging to w such that $w_n \in W$ for all n. We can write $w_n = \sqrt{1 - A_n^2}v_n + A_n u$ for each n. Since $w_n \in W$ for all n, we have $A_n \geq A^*(v_n)$ for all n. Let v denote the limit of v_n and A the limit of A_n. It is easy to see from the proof of Lemma 1 that the v and A associated with a given w depend continuously on w, so we must have $w = \sqrt{1 - A^2}v + Au$. Hence $w \in W(A^*)$ if and only if $\lim_{n\to\infty} A^*(v_n) \geq A^*(\lim_{n\to\infty} v_n)$. Since A^* is lower semi–continuous, this holds. Hence W is closed.

For the converse, suppose $W \neq \mathcal{W}$ is closed and closed under C_u. For each v, let

$$A^*(v) = \min\{A \in [-1, 1] \mid v\sqrt{1 - A^2} + Au \in W\}.$$

Since W is closed, $A^*(v)$ is well–defined. Since $w \; C_u \; -u$ for all w, the fact that $W \neq \mathcal{W}$ implies $-u \notin W$. Hence $A^*(v) > -1$ for all v. Since W is closed under C_u, for every $A \geq A^*(v)$, we have $v\sqrt{1 - A^2} + Au \in W$, implying that $v\sqrt{1 - A^2} + Au \in W$ if and only if $A \geq A^*(v)$. Hence $W = W(A^*)$.

Finally, to see that A^* is lower semi–continuous, again, consider the sequence constructed above. As noted, for any such sequence, $w \in W$ if and only if $\lim_{n\to\infty} A^*(v_n) \geq$

$A^*(\lim_{n\to\infty} v_n)$. Since W is closed, we must have $w \in W$. Hence any jumps in A^* must be downward, so A^* is lower semi–continuous. ▮

We note that if A^* is lower semi–continuous, then it is measurable.

Lemma 11. *Fix any measurable function $A^* : \mathcal{V} \to [-1, 1]$ such that $W(A^*)$ is closed. Then there exists a sequence of positive numbers $\{\varepsilon_n\}$ and a sequence of menus $\{x_n\}$ such that for every random Strotz representation,*

$$\lim_{n\to\infty} \frac{V(x_n)}{\varepsilon_n} = \mu(W(A^*)).$$

Proof. Fix such an A^* function.

Part 1. First, suppose that A^* is bounded in the sense that $A^*(v)/\sqrt{1-(A^*(v))^2}$ is bounded from above and below.

For each $v \in \mathcal{V}$, let

$$\alpha_v = \frac{1}{K}\mathbf{1} + \varphi v$$
$$\beta_v(\varepsilon) = \alpha_v + \varphi\varepsilon\left[u - \frac{A^*(v)}{a^*(v)}v\right]$$

where $a^*(v) = \sqrt{1-(A^*(v))^2}$. By the boundedness of A^*/a^*, there exists $\varphi > 0$ such that for all sufficiently small $\varepsilon > 0$, every α_v and $\beta_v(\varepsilon)$ is a lottery.

Suppose $w = av + Au$ and consider some $\bar{v}$ which may or may not equal v. Then

$$w \cdot \alpha_{\bar{v}} = a\varphi v \cdot \bar{v}$$

while

$$w \cdot \alpha_v = a\varphi.$$

Since $v \cdot \bar{v} \leq 1$, strictly if $\bar{v} \neq v$, we see that

$$w \cdot \alpha_{\bar{v}} \leq w \cdot \alpha_v,$$

strictly so for any $\bar{v} \neq v$. Hence if w picks any α, he must pick α_v.

Also, for any v,

$$u \cdot \alpha_v = 0 < \varphi\varepsilon = u \cdot \beta_v(\varepsilon).$$

So u is indifferent among the α's, indifferent among the β's, and prefers the β's to the α's. Hence, letting x_ε denote the menu consisting of all the α's and β's, we see that

$$V(x_\varepsilon) = \varphi\varepsilon\mu(W_\varepsilon),$$

where

$$W_\varepsilon = \bigcup_{v \in \mathcal{V}} \{w \in L(v) \mid w \cdot \beta_{\bar{v}}(\varepsilon) \geq w \cdot \alpha_v, \text{ for some } \bar{v} \in \mathcal{V}\}.$$

We now show that

$$\lim_{\varepsilon \downarrow 0} \mu(W_\varepsilon) = \mu(W(A^*)).$$

Note that if $w = av + Au$, then $a = \sqrt{1 - A^2}$ and

$$w \cdot \beta_v(\varepsilon) = w \cdot \alpha_v + \varphi\varepsilon \left[A - \frac{A^*(v)}{a^*(v)} a \right],$$

so $w \cdot \beta_v(\varepsilon) \geq w \cdot \alpha_v$ iff $(A/a) \geq (A^*(v)/a^*(v))$. It is not hard to show that this holds iff $A \geq A^*(v)$. Hence for every ε, we have $W(A^*) \subseteq W_\varepsilon$.

Next, we show that if $w \notin W(A^*)$, then there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, we have $w \notin W_\varepsilon$. To show this, suppose not. Then there exists a sequence ε_n converging to zero such that $w \in W_{\varepsilon_n} \setminus W(A^*)$ for all n.

Write $w = av + Au$. Then there exists a sequence $\bar{v}_n$ such that

$$w \cdot \beta_{\bar{v}_n}(\varepsilon_n) \geq w \cdot \alpha_v$$

or

$$a\varphi v \cdot \bar{v}_n + \varphi\varepsilon_n \left[A - \frac{A_n}{a_n} av \cdot \bar{v}_n \right] \geq a\varphi$$

where $A_n = A^*(\bar{v}_n)$ and $a_n = \sqrt{1 - A_n^2}$. Rearranging yields

$$\varepsilon_n \left[\frac{A}{a} - \frac{A_n}{a_n} v \cdot \bar{v}_n \right] \geq 1 - v \cdot \bar{v}_n.$$

Since $v \cdot \bar{v}_n \leq 1$ and A_n/a_n is bounded from below, we must have $v \cdot \bar{v}_n \to 1$ as $n \to \infty$. Note for future use that this implies $\bar{v}_n \to v$. Also, the fact that the right–hand side is nonnegative for all n implies that

$$\frac{A}{a} \geq \frac{A_n}{a_n} v \cdot \bar{v}_n$$

for all n. Recall that $w \in L(v)$ and $w \notin W^*$. Hence $A < A^*(v)$. So we have

$$\frac{A^*(v)}{a^*(v)} > \frac{A^*(\bar{v}_n)}{a^*(\bar{v}_n)} v \cdot \bar{v}_n.$$

Since $v \cdot \bar{v}_n \to 1$, we have

$$\frac{A^*(v)}{a^*(v)} \geq \lim_{n \to \infty} \frac{A_n}{a_n},$$

or $A^*(v) \geq \lim_{n\to\infty} A_n$.

By Lemma 10, the fact that $W(A^*)$ is closed that A^* is lower semi–continuous. Hence we have the opposite weak inequality, so $A^*(v) = \lim_{n\to\infty} A_n$.

But recall that

$$\frac{A}{a} \geq \frac{A_n}{a_n} v \cdot \bar{v}_n$$

for all n. Hence

$$\frac{A}{a} \geq \lim_{n\to\infty} \frac{A_n}{a_n} v \cdot \bar{v}_n = \frac{A^*(v)}{a^*(v)}.$$

But this implies $A \geq A^*(v)$ or $w \in W(A^*)$, a contradiction.

Hence

$$\lim_{n\to\infty} \mu(W_\varepsilon) = \mu(W^*).$$

Therefore,

$$\lim_{n\to\infty} \frac{V(x_\varepsilon)}{\varphi\varepsilon_n} = \lim_{n\to\infty} \mu(W_\varepsilon) = \mu(W^*).$$

Taking the sequence referred to in the statement of the lemma to be $\{\varphi\varepsilon_n\}$ gives the desired conclusion.

Part 2. Now we drop the assumption that A^* is bounded. Fix a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ for all n such that $\varepsilon_n \to 0$. Define a new function

$$A_n^*(v) = \begin{cases} A^*(v), & \text{if } -1+\varepsilon_n \leq A^*(v) \leq 1-\varepsilon_n \\ -1+\varepsilon_n, & \text{if } A^*(v) < -1+\varepsilon_n \\ 1-\varepsilon_n, & \text{if } A^*(v) > 1-\varepsilon_n \end{cases}$$

Clearly, A_n^* is bounded for every n. It is tedious but not difficult to show that the fact that $W(A^*)$ is closed implies that $W(A_n^*)$ is closed for every n. Hence for each n, we can find sequences $\{\varepsilon_m^n\}$ and $\{x_m^n\}$ such that

$$\lim_{m\to\infty} \frac{V(x_m^n)}{\varepsilon_m^n} = \mu(W(A_n^*)).$$

That is, for any $\delta > 0$, there exists $M_n(\delta)$ such that

$$\left| \frac{V(x_m^n)}{\varepsilon_m^n} - \mu(W(A_n^*)) \right| < \delta, \quad \forall m \geq M_n(\delta).$$

Rewriting,

$$\left| \frac{V(x_m^n)}{\varepsilon_m^n} - \mu(W(A^*)) + \mu(W(A^*) \setminus W(A_n^*)) - \mu(W(A_n^*) \setminus W(A^*)) \right| < \delta, \quad \forall m \geq M_n(\delta).$$

Clearly, though, if $w \in W(A^*) \setminus W(A_n^*)$, then there must be some N such that $w \in W(A^*) \cap W(A_{\bar{n}}^*)$ for all $\bar{n} \geq N$. The analogous statement is true for $W(A_n^*) \setminus W(A^*)$.

So fix any sequence $\{\delta_n\}$ converging to zero. For each n, fix $m_n \geq M_n(\delta_n)$. Consider the sequence $\{\hat{\varepsilon}_n\} = \{\varepsilon_{m_n}^n\}$ and $\{\hat{x}_n\} = \{x_{m_n}^n\}$. Clearly, for every n,

$$\left| \frac{V(\hat{x}_n)}{\hat{\varepsilon}_n} - \mu(W(A^*)) + k\mu(W(A^*) \setminus W(A_n^*)) - \mu(W(A_n^*) \setminus W(A^*)) \right| < \delta_n.$$

So

$$\lim_{n \to \infty} \frac{V(\hat{x}_n)}{\hat{\varepsilon}_n} = \mu(W(A^*)).$$

∎

To complete the proof of Theorem 3, suppose (u, μ) and $(\bar{u}, \bar{\mu})$ are canonical random Strotz representations of $\succeq$. Let V and $\bar{V}$ denote the utility functions over menus generated by (u, μ) and $(\bar{u}, \bar{\mu})$ respectively. As explained in the text, we must have $u = \bar{u}$. Also, since V and $\bar{V}$ represent the same preference, there exists $a > 0$ and b such that $V(x) = a\bar{V}(x) + b$ for all menus x. For $x = \{\alpha\}$, then, $V(\{\alpha\}) = a\bar{V}(\{\alpha\}) + b$ or $u(\alpha) = a\bar{u}(\alpha) + b$. Since $u = \bar{u}$, then, $a = 1$ and $b = 0$. In other words, we must have $V = \bar{V}$.

Note that the sequence of menus constructed in the proof of Lemma 11 is independent of the representation. Hence for any set W which is closed and closed under C_u, we must have $\mu(W) = \bar{\mu}(W)$.

Fix any measurable set $E \subseteq \mathcal{W}$ which is closed under C_u. By Theorem 12.3 of Billingsley (1995, page 174),

$$\mu(E) = \sup_{W \subseteq E | W \text{ closed}} \mu(W)$$

and similarly for $\bar{\mu}$. It is easy to see that if $W \subseteq E$, then if we close W under C_u, the resulting set will be contained in E. That is,

$$W^* \equiv \{w \in \mathcal{W} \mid w \ C_u \ w', \text{ for some } w' \in W\} \subseteq E.$$

Obviously, $\mu(W^*) \geq \mu(W)$. Hence

$$\mu(E) = \sup_{W \subseteq E | W \text{ closed, closed under } C_u} \mu(W).$$

Since $\mu(W) = \bar{\mu}(W)$ for any W which is closed and closed under C_u, this implies $\mu(E) = \bar{\mu}(E)$. Thus μ and $\bar{\mu}$ coincide for any measurable set which is closed under C_u.

Let $\mathcal{P}$ denote the collection of measurable sets E which are closed under C_u. So we have established that μ and $\bar{\mu}$ coincide on $\mathcal{P}$. It is easy to show that $\mathcal{P}$ is a π–system. To see this, suppose $E_1, E_2 \in \mathcal{P}$. Then E_1 and E_2 are measurable, so $E_1 \cap E_2$ is measurable. Also, fix any $w \in E_1 \cap E_2$ and any w' such that $w' \; C_u \; w$. Since $w \in E_i$ and E_i is closed under C_u, we must have $w' \in E_i$, $i = 1, 2$. Hence $w' \in E_1 \cap E_2$. So $E_1 \cap E_2$ is closed under C_u and hence is an element of $\mathcal{P}$. Hence $\mathcal{P}$ is a π–system.

Theorem 3.3 of Billingsley (1995) then implies $\mu = \bar{\mu}$ on $\sigma(\mathcal{P})$, the σ–algebra generated by $\mathcal{P}$. We now show that $\sigma(\mathcal{P})$ is the Borel σ–algebra, completing the proof of uniqueness.

Fix any open set $W \subseteq \mathcal{W}$ and any $w \in W \setminus \{-u, u\}$. It is not hard to see that there exists a closed $\hat{\mathcal{V}} \subseteq \mathcal{V}$ and rational numbers $r_1, r_2 \in (-1, 1]$ such that $w \in W(A_1^*) \setminus W(A_2^*) \subseteq W$ where

$$A_i^*(v) = \begin{cases} r_i, & \text{for } v \in \hat{\mathcal{V}} \\ 1, & \text{otherwise} \end{cases}$$

Obviously, $W(A_1^*), W(A_2^*) \in \mathcal{P}$ implies $W(A_1^*) \setminus W(A_2^*) \in \sigma(\mathcal{P})$. Note that $\{u\} \in \mathcal{P}$ and that $\mathcal{W} \setminus \{-u\} \in \mathcal{P}$ implies $\{-u\} \in \sigma(\mathcal{P})$. Hence W is the union of a countable collection of sets in $\sigma(\mathcal{P})$ so $W \in \sigma(\mathcal{P})$. Hence $\sigma(\mathcal{P})$ contains all open sets and so contains the Borel σ–algebra. Since all sets in $\mathcal{P}$ are in the Borel σ–algebra, $\sigma(\mathcal{P})$ cannot be larger than the Borel field, so it must equal it. ▌

G Proof of Theorem 4

Lemma 12. *Suppose* $\succeq_i$ *has a random Strotz representation* (u, μ_i), $i = 1, 2$. *Then* $\succeq_2$ *is more temptation averse than* $\succeq_1$ *if and only if for every menu* x*,*

$$V_2(x) \equiv \int \max_{\beta \in B_w(x)} u(\beta) \, \mu_2(dw) \leq \int \max_{\beta \in B_w(x)} u(\beta) \, \mu_1(dw) \equiv V_1(x).$$

Proof of Lemma. Suppose $\succeq_2$ is more temptation averse than $\succeq_1$. Fix any menu x. It is easy to see that there must be a lottery α such that $\{\alpha\} \sim_1 x$, so $u(\alpha) = V_1(x)$. Since $\succeq_2$ is more temptation averse, we have $\{\alpha\} \succeq_2 x$, so $u(\alpha) \geq V_2(x)$ implying $V_1(x) \geq V_2(x)$. For the converse, suppose$V_1(x) \geq V_2(x)$ for all x. Then whenever $u(\alpha) \geq V_1(x)$, we have $u(\alpha) \geq V_2(x)$, so $\succeq_2$ is more temptation averse than $\succeq_1$. ▌

First, we show the equivalence of (1) and (2). To show (2) implies (1), suppose $u_1 = u_2 = u$ and that $\mu_1 \; C_u$–FOSD μ_2. Fix any menu x. For each number $r \in \mathbf{R}$, let W_r denote the set of $w \in \mathcal{W}$ such that

$$\max_{\beta \in B_w(x)} u(\beta) \geq r.$$

Obviously, W_r is closed and hence measurable. Note that each W_r is closed under C_u. To see this, suppose $w' \in W_r$ and $w\ C_u\ w'$. Fix any $\alpha \in B_u(B_w(x))$ and $\beta \in B_u(B_{w'}(x))$. Clearly, since $\beta \in B_{w'}(x)$, we have $w'(\beta) \geq w'(\alpha)$. Hence if $u(\beta) > u(\alpha)$, the fact that $w\ C_u\ w'$ implies $w(\beta) > w(\alpha)$, contradicting $\alpha \in B_w(x)$. Hence we must have $u(\alpha) \geq u(\beta)$. Since $w' \in W_r$, we have $u(\beta) \geq r$, so $u(\alpha) \geq r$, implying $w \in W_r$.

Letting V_i denote the utility function over menus for the random Strotz representation (u, μ_i), we have

$$V_i(x) = \int_0^\infty \mu_i(W_r)\, dr + \int_{-\infty}^0 [\mu(W_r) - 1]\ dr.$$

Since μ_1 C_u–FOSD μ_2, we see that $V_1(x) \geq V_2(x)$. Hence by Lemma 12, $\succeq_2$ is more temptation averse than $\succeq_1$.

For the converse, suppose $\succeq_2$ is more temptation averse than $\succeq_1$. Fix an arbitrary closed $W \subseteq \mathcal{W}$ which is closed under C_u. If $W = \emptyset$, obviously, $\mu_1(W) \geq \mu_2(W)$, so assume $W \neq \emptyset$. If $-u \in W$, then, since $w\ C_u\ -u$ for all w, the fact that W is closed under C_u implies that $W = \mathcal{W}$. Again, in this case, obviously $\mu_1(W) \geq \mu_2(W)$. So assume $-u \notin W$, so $W \neq \mathcal{W}$. In this case, Lemma 10 implies that there is an A^* function such that $W = W(A^*)$. Hence Lemma 11 implies that there is a sequence of menus x_n and a sequence of positive numbers ε_n such that

$$\lim_{n\to\infty} \frac{V_i(x_n)}{\varepsilon_n} = \mu_i(W).$$

(Note in particular that the lemma shows that the sequence of menus does not depend on the preference.) Since $V_1(x) \geq V_2(x)$ for all x, we see that $\mu_1(W) \geq \mu_2(W)$.

Now we show that (1) and (3) are equivalent. First, we show that (1) implies (3) by constructing the versions of μ_1 and μ_2. To begin, we fix a partition $\mathcal{V}_1, \ldots, \mathcal{V}_N$ of $\mathcal{V}$ with the property that each $\mathcal{V}_i$ is measurable. We refer to such a partition as a ***measurable partition***. For any $\mathcal{V}_n$ and any $A_n \in [-1, 1]$, let

$$\mu_i(A_n, \mathcal{V}_n) = \mu_i\left(\{w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au,\ \text{for } 1 > A \geq A_n,\ A \neq -1,\ v \in \mathcal{V}_n\}\right).$$

Note that $\mu_i(A_n, \mathcal{V}_n)$ is defined so that it does not include the measure of u or $-u$. In particular, $\mu_i(1, \mathcal{V}_n) = 0$ and

$$\mu_i(-1, \mathcal{V}_n) = \mu_i\left(L(\mathcal{V}_n))\right) - \mu_i(\{u\}) - \mu_i(\{-u\})$$

where

$$L(\mathcal{V}_n) = \left\{w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au,\ \text{for } A \in [-1, 1],\ v \in \mathcal{V}_n\right\}.$$

Fix any $A_1, \ldots, A_N$ such that the set

$$W^* = \bigcup_{n=1}^{N} \bigcup_{v \in \mathcal{V}_\mathcal{N}} \{w \in \mathcal{W} \mid w = v\sqrt{1-A^2} + Au,\ A \geq A_n,\ A \neq -1\}$$

is closed. By Lemma 11, we know that there is a sequence of menus x_m and numbers ε_m such that

$$\lim_{m\to\infty} \frac{V_i(x_m)}{\varepsilon_m} = \mu_i(W^*).$$

Since this sequence is independent of the preference and since $V_1(x_m) \geq V_2(x_m)$ for all m, we have $\mu_1(W^*) \geq \mu_2(W^*)$.

Suppose, though, that W^* is not closed. In this case, Theorem 12.3 of Billingsley (1995, page 174) implies that

$$\mu_2(W^*) = \sup_{W \subseteq E | W \text{ closed}} \mu_2(W).$$

As shown in the proof of Theorem 3, we can rewrite this as

$$\mu_2(W^*) = \sup_{W \subseteq W^* | W \text{ closed, closed under } C_u} \mu_2(W).$$

But we know that for every W which is closed and closed under C_u, we have $\mu_2(W) \leq \mu_1(W)$. Hence

$$\begin{aligned}\mu_2(W^*) &= \sup_{W \subseteq E | W \text{ closed}} \mu_2(W) \\ &\leq \sup_{W \subseteq W^* | W \text{ closed, closed under } C_u} \mu_1(W) \\ &= \mu_1(W^*).\end{aligned}$$

Summarizing, we have

$$\mu_1(u) + \sum_{n=1}^{N} \mu_1(A_n, \mathcal{V}_n) \geq \mu_2(u) + \sum_{n=1}^{N} \mu_2(A_n, \mathcal{V}_n). \tag{3}$$

for any $(A_1, \ldots, A_N) \in [-1, 1]^N$.

Note that this implies $\mu_1(u) \geq \mu_2(u)$ and $\mu_1(-u) \leq \mu_2(-u)$. The former is implied by taking $A_n = 1$ for all n and the latter by $A_n = -1$ for all n.

Let $\mu^*(u) = \mu_1(u) - \mu_2(u)$. First, assume $\mu^*(u) > 0$. For each n, let

$$\lambda_n^* = \sup_{A_n \in [-1,1]} \frac{\mu_2(A_n, \mathcal{V}_n) - \mu_1(A_n, \mathcal{V}_n)}{\mu^*(u)}.$$

By assumption, $\mu^*(u) > 0$, so this is well–defined. Also, note that for $A_n = 1$, the difference on the right–hand side is zero. Hence $\lambda_n^* \geq 0$.

Also,

$$\sum_n \lambda_n^* = \frac{1}{\mu^*(u)} \sum_n \sup_{A_n \in [-1,1]} [\mu_2(A_n, \mathcal{V}_n) - \mu_1(A_n, \mathcal{V}_n)].$$

Suppose this is strictly greater than 1. Then for each n, there is a sequence $\{A_n^m\}$ such that

$$\lim_{m \to \infty} \sum_n \mu_2(A_n^m, \mathcal{V}_n) > \mu^*(u) + \lim_{m \to \infty} \sum_n \mu_1(A_n^m, \mathcal{V}_n).$$

Substituting for $\mu^*(u)$ and rearranging,

$$\lim_{m \to \infty} \left[\mu_2(u) + \sum_n \mu_2(A_n^m, \mathcal{V}_n)\right] > \lim_{m \to \infty} \left[\mu_1(u) + \sum_n \mu_1(A_n^m, \mathcal{V}_n)\right].$$

But this contradicts equation (3). Hence $\sum_n \lambda_n^* \leq 1$.

Fix any $\lambda_1^1, \ldots, \lambda_N^1$, summing to 1, such that $\lambda_n^1 \geq \lambda_n^*$ for all n. Obviously, such a λ^1 exists. Then the definition of λ_n^* and the fact that $\lambda_n^1 \geq \lambda_n^*$ implies

$$\lambda_n^1 \mu^*(u) + \mu_1(A_n, \mathcal{V}_n) \geq \mu_2(A_n, \mathcal{V}_n), \quad \forall A_n \in [-1, 1], \ \forall n.$$

Substituting for $\mu^*(u)$, then,

$$\lambda_n^1 \mu_1(u) + \mu_1(A_n, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(A_n, \mathcal{V}_n), \quad \forall A_n \in [-1, 1], \ \forall n. \tag{4}$$

Next, suppose $\mu^*(u) = 0$. In this case, define $\lambda_n^1 = 1/N$ for $n = 1, \ldots, N$. Then equation (3) evaluated at any fixed A_n with $A_m = 1$ for all $m \neq n$ implies equation (4).

Next, define $\mu^*(-u) = \mu_2(-u) - \mu_1(-u)$. First, assume $\mu^*(-u) > 0$. Then define $\lambda_1^2, \ldots, \lambda_N^2$ by

$$\lambda_n^1 \mu_1(u) + \mu_1(-1, \mathcal{V}_n) = \lambda_n^1 \mu_2(u) + \mu_2(-1, \mathcal{V}_n) + \lambda_n^2 \mu^*(-u). \tag{5}$$

By equation (4) at $A_n = -1$, $\lambda_n^2 \geq 0$ for all n. Also, summing both sides over n and using $\sum_n \lambda_n^1 = 1$, we obtain

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) = \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n) + \mu^*(-u) \sum_n \lambda_n^2.$$

The left–hand side is $\mu_1(\mathcal{W}) - \mu_1(-u) = 1 - \mu_1(-u)$. The right–hand side is $1 - \mu_2(-u) + \mu^*(-u) \sum_n \lambda_n^2$. Hence we have

$$\mu^*(-u) = \mu^*(-u) \sum_n \lambda_n^2.$$

Since $\mu^*(-u) > 0$ by assumption, we must have $\sum_n \lambda_n^2 = 1$.

Second, suppose $\mu^*(-u) = 0$. In this case, the definition of λ^1 implies

$$\lambda_n^1 \mu_1(u) + \mu_1(-1, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(-1, \mathcal{V}_n) \tag{6}$$

for every n. Summing both sides over n and using $\sum_n \lambda_n^1 = 1$, we obtain

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) \geq \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n).$$

But since $\mu^*(-u) = 0$, we have $\mu_1(-u) = \mu_2(-u)$, so

$$\mu_1(u) + \sum_n \mu_1(-1, \mathcal{V}_n) + \mu_1(-u) \geq \mu_2(u) + \sum_n \mu_2(-1, \mathcal{V}_n) + \mu_2(-u).$$

But both each side of this inequality must equal 1. Hence equation (6) must hold with equality for all n. In light of this, we can define $\lambda_n^2 = 1/n$ for all n and equation (5) still holds.

This implies that we can rewrite μ_i as a measure $\hat{\mu}_i$ over $[-1, 1] \times \mathcal{V}$, $i = 1, 2$, as follows. For any measurable $E \subseteq (-1, 1) \times \mathcal{V}$, let

$$\hat{\mu}_i(E) = \mu_i \left(\left\{ w \in \mathcal{W} \mid w = v\sqrt{1 - A^2} + Au, \ (A, v) \in E \right\} \right).$$

For $E = \{1\} \times \mathcal{V}_n$, let

$$\hat{\mu}_i(E) = \lambda_n^1 \mu_i(u)$$

and for $E = \{-1\} \times \mathcal{V}_n$, let

$$\hat{\mu}_i(E) = \lambda_n^2 \mu_i(-u).$$

To see that such a measure exists, for each n, choose an arbitrary $v_n \in \mathcal{V}_n$ and assign probability $\lambda_n^1 \mu_i(u)$ to $\{1\} \times v_n$ and probability $\lambda_n^2 \mu_i(-u)$ to $\{-1\} \times v_n$. Extend this to the Borel field on $[-1, 1] \times \mathcal{V}$ in the obvious manner. That is, for each $E \subseteq [-1, 1] \times \mathcal{V}_n$, let

$$\begin{aligned}\hat{\mu}_i(E) = \hat{\mu}_i(E \cap [(-1, 1) \times \mathcal{V}]) + \hat{\mu}_i(E \cap (\{1\} \times \{v_i \mid i = 1, \ldots, N\})) \\ + \hat{\mu}_i(E \cap (\{-1\} \times \{v_i \mid i = 1, \ldots, N\})).\end{aligned}$$

The key point to observe about these measures is that for every n and every $A_n \in [-1, 1]$, we have

$$\hat{\mu}_1([A_n, 1] \times \mathcal{V}_n) = \lambda_n^1 \mu_1(u) + \mu_1(A_n, \mathcal{V}_n) \geq \lambda_n^1 \mu_2(u) + \mu_2(A_n, \mathcal{V}_n) = \hat{\mu}_2([A_n, 1] \times \mathcal{V}_n)$$

and

$$\begin{aligned}\hat{\mu}_1([-1,1]\times\mathcal{V}_n) =&\lambda_n^1\mu_1(u)+\mu_1(-1,\mathcal{V}_n)+\lambda_n^2\mu_1(-u)\\ =&\lambda_n^1\mu_2(u)+\mu_2(-1\mathcal{V}_n)+\lambda_n^2\mu_2(-u)\\ =&\hat{\mu}_2([-1,1]\times\mathcal{V}_n).\end{aligned}$$

Generalizing, given any finite measurable partition Π of $\mathcal{V}$, let $\mathcal{M}_\Pi$ be the set of pairs of measures $(\hat{\mu}_1,\hat{\mu}_2)$ over $[-1,1]\times\mathcal{V}$ such that

$$\begin{gathered}\hat{\mu}_i(E)=\mu_i\left(\left\{w\in\mathcal{W}\mid w=v\sqrt{1-A^2}+Au,\ (A,v)\in E\right\}\right),\\ \forall \text{ measurable } E,\ i=1,2,\end{gathered} \tag{7}$$

$$\hat{\mu}_1([A_n,1)\times\mathcal{V}_n)\geq\hat{\mu}_2([A_n,1]\times\mathcal{V}_n),\quad \forall A_n\in[-1,1],\ \forall n, \tag{8}$$

and

$$\hat{\mu}_1([-1,1]\times\mathcal{V}_n)=\hat{\mu}_2([-1,1]\times\mathcal{V}_n),\quad \forall n. \tag{9}$$

We have shown that for every finite measurable partition Π, $\mathcal{M}_\Pi$ is nonempty. It is also not hard to see that it must be closed. Clearly, if Π' is a refinement of Π, then $\mathcal{M}_{\Pi'}\subseteq\mathcal{M}_\Pi$.

Each $\mathcal{M}_\Pi$ is a closed nonempty subset of the space of pairs of measures over $\mathcal{V}$, obviously a compact set. Fix a finite collection of finite measurable partitions, say $\Pi_1,\ldots,\Pi_T$. Let Π be the coarsest common refinement of these partitions. Then $\mathcal{M}_\Pi\subseteq\mathcal{M}_{\Pi_t}$ for all t. Since $\mathcal{M}_\Pi$ must be nonempty, we see that $\cap_t\mathcal{M}_{\Pi_t}\neq\emptyset$. By Kelly (1955, Chapter 5, Theorem 1), this implies that $\cap_\Pi\mathcal{M}_\Pi$ is nonempty where the intersection is taken over the set of all finite measurable partitions. Hence there is at least one pair of measures which satisfies equations (7), (8), and (9) for every finite measurable partition.

Hence we have shown that we can rewrite μ_1 and μ_2 as distributions $\hat{\mu}_1$ and $\hat{\mu}_2$ over $(A,v)\in[-1,1]\times\mathcal{V}$ with the following properties. First, equation (7) implies that for every menu x,

$$\int_w \max_{\beta\in B_w(x)} u(\beta)\,\mu_i(dw)=\int_{(A,v)}\max_{\beta\in B_{v\sqrt{1-A^2}+Au}(x)} u(\beta)\,\hat{\mu}_i(d(A,v)),\quad i=1,2.$$

This holds since we have only specified how mass at u and $-u$ is spread across the sets $\{1\}\times\mathcal{V}$ and $\{-1\}\times\mathcal{V}$ respectively. Since $(1,v)$ and $(1,v')$ both correspond to utility function u, this has no effect on the calculation of the utility of any menu.

Second, equation (8) implies that for every measurable function $A^*:\mathcal{V}\to[-1,1]$,

$$\int_v\hat{\mu}_1([A^*(v),1]\times\{v\})\,dv\geq\int_v\hat{\mu}_2([A^*(v),1]\times\{v\})\,dv.$$

To see this, simply note since A^* is bounded and measurable, there exists an increasing sequence of simple functions A_n^* converging to A^* pointwise from below.[26] Letting $W^* = \{(A,v) \mid A \geq A^*(v)\}$ and $W_n = \{(A,v) \mid A \geq A_n^*(v)\}$, we see that $W^* = \cap_n W_n$, so $\hat{\mu}_i(W^*) = \lim_{n\to\infty} \hat{\mu}_i(W_n)$. Hence

$$\hat{\mu}_1(W^*) = \lim_{n\to\infty} \hat{\mu}_1(W_n) \geq \lim_{n\to\infty} \hat{\mu}_2(W_n) = \hat{\mu}_2(W^*),$$

where the inequality follows from equation (8).

Third, it is easy to see that equation (9) implies that the marginals of $\hat{\mu}_1$ and $\hat{\mu}_2$ over $\mathcal{V}$ are the same.

Letting $\mu_{\mathcal{V}}^i$ denote the marginal on $\hat{\mu}_i$ on $\mathcal{V}$ and $\mu_L^i(\cdot \mid v)$ a regular version of the conditional, we see that $(\mu_{\mathcal{V}}^i, \mu_L^i(\cdot \mid v))$ is a version of μ_i for $i = 1, 2$.

Now we complete the proof that (1) implies (3) by showing that $\mu_L^1(\cdot \mid v)$ FOSD $\mu_L^2(\cdot \mid v)$ for almost all v.

Let

$$\mathcal{V}(\bar{A}) = \left\{v \in \mathcal{V} \mid \hat{\mu}_1([\bar{A},1] \mid v) < \hat{\mu}_2([\bar{A},1] \mid v)\right\}$$

and

$$\mathcal{V}^* = \{v \in \mathcal{V} \mid \exists A_v \text{ with } \mu_L^1([A_v,1] \mid v) < \mu_L^2([A_v,1] \mid v)\}.$$

If there is an A_v such that

$$\mu_L^1([A_v,1] \mid v) < \mu_L^2([A_v,1] \mid v),$$

then there must be a rational A_v with this property. Obviously, if the distributions are continuous in a neighborhood of A_v, this is true. If a distribution has a mass point at A_v, then we can perturb the A_v slightly in one direction and maintain the inequality.

Hence $\mathcal{V}^* = \cup_{\bar{A}\in\mathcal{R}} \mathcal{V}(\bar{A})$ where $\mathcal{R}$ denotes the rationals. But for any $\bar{A}$, $\mathcal{V}(\bar{A})$ is measurable. Hence $\mathcal{V}^*$ is a countable union of measurable sets and so is measurable. Hence

$$\mu_{\mathcal{V}}^i(\mathcal{V}^*) \leq \sum_{\bar{A}\in\mathcal{R}} \mu_{\mathcal{V}}^i\left(\mathcal{V}(\bar{A})\right).$$

Since the marginals over $\mathcal{V}$ are the same, the right–hand side must be zero. To see this, suppose it is not zero. Then there must be some rational $\bar{A}$ such that $\hat{\mu}_i(\mathcal{V}(\bar{A})) > 0$. Since the marginals are the same, let $\mu_{\mathcal{V}}$ denote the marginal on v. Then for every $v \in \mathcal{V}(\bar{A})$, we have

$$\mu_L^1([\bar{A}_v,1] \mid v)\mu_{\mathcal{V}}(v) < \mu_L^2([\bar{A}_v,1] \mid v)\mu_{\mathcal{V}}(v).$$

[26] It is straightforward to modify the proof of Theorem 13.5 in Billingsley (1995, page 185) to show this.

Integrating over $v \in \mathcal{V}(\bar{A})$, we get

$$\hat{\mu}_1([\bar{A}, 1] \times \mathcal{V}(\bar{A})) < \hat{\mu}_2([\bar{A}, 1] \times \mathcal{V}(\bar{A})),$$

a contradiction.

This concludes the proof that if $\succeq_2$ is more temptation averse than $\succeq_1$, then there exist versions of μ_1 and μ_2 such that for almost all v, $\mu_L^1(\cdot \mid v)$ first–order stochastically dominates $\mu_L^2(\cdot \mid v)$ and $\mu_{\mathcal{V}}^1 = \mu_{\mathcal{V}}^2$.

For the converse, suppose we have the conditional FOSD property and equal marginals. Fix any menu and any $v \in \mathcal{V}$. Since the utility u gets from the choice is weakly increasing in A, we know that the expected utility of the menu conditional on v is higher under μ_L^1 than under μ_L^2. Since this is true for (almost) every v, the same is true when we take expectations over v since the marginals are the same. Hence $V_1(x) \geq V_2(x)$ for all x. Hence $\succeq_2$ is more temptation averse than $\succeq_1$. ▌

H Example for Section 5

We give an example of a random choice function with an RGP rationalization but no RS rationalization.

Fix a random GP representation with support on $\{v_1, v_2\}$ with $\nu(v_1) \neq \nu(v_2)$. Take v_1 and v_2 to have the property that $(u + v_1) \cdot (u + v_2) = 0$. Let w_i be the unique $w \in \mathcal{W}$ such that w_i represents the same preference over lotteries as $u + v_i$, $i = 1, 2$. Of course, we must have $w_1 \cdot w_2 = 0$.

Let w_3 be any $w \in \mathcal{W}$ such that $w_1 \cdot w_3 = w_2 \cdot w_3 = 0$, $w_3 \notin \{w_1, w_2, -w_1, -w_2\}$. Such a w_3 must exist if $K \geq 4$.

Construct a selection ρ from the random choice correspondence this representation induces as follows. Fix a sphere in the interior of $\Delta(Z)$. For $i = 1, 2$, let x_i and y_i be two distinct menus each of which is generated by taking the intersection of the sphere with some w_i indifference curve that intersects the interior of the sphere. It is easy to see that $u + v_2$ has a unique maximizer from x_1 and from y_1, but that $u + v_1$ is indifferent over every point in either. Similarly, $u + v_1$ has a unique maximizer from x_2 and from y_2, but $u + v_2$ is indifferent over every point in either. Construct $\rho(x_1)$ by assuming that $u + v_1$ chooses the best $\beta \in x_1$ according to w_3. Construct $\rho(y_1)$ by assuming that $u + v_1$ chooses the best $\beta \in y_1$ according to $-w_3$. Similarly, construct $\rho(x_2)$ by assuming $u + v_2$ chooses the best $\beta \in x_2$ according to w_3 and construct $\rho(y_2)$ by assuming $u + v_2$ chooses the best

$\beta \in y_2$ according to $-w_3$. Otherwise, ρ can be any selection from the random GP choice correspondence. Note that the constructed choice from x_i or y_i by $u + v_i$ differs from the choice from x_i or y_i by $u + v_j$, $i \neq j$.

Suppose there is some random Strotz representation $(\hat{u}, \mu)$ that rationalizes this ρ. The argument in Lemma 4 shows that we must have $\text{supp}(\mu) = \{w_1, w_2\}$ and $\mu(w_i) = \nu(v_i)$, $i = 1, 2$.

First, suppose that $\hat{u} \notin \{w_1, -w_1, w_2, -w_2\}$. Let

$$\mathcal{V}_i = \{v \in \mathcal{W} \mid v \cdot w_i = 0\}, \quad i = 1, 2.$$

Since $\hat{u} \notin \{w_i, -w_i\}$, just as in Lemma 1, there exists a unique $\hat{v}_i \in \mathcal{V}_i$ such that $\hat{u} = f(A)\hat{v}_i + Aw_i$, $i = 1, 2$.

Note that the choice from x_i by $u + v_i$ must be some maximizer of $\hat{u}$ from this menu and similarly for y_i. Since w_i is indifferent over all of x_i and y_i, the choice must be some maximizer of $\hat{v}_i$. But since $\hat{v}_i$ is orthogonal to w_i, the maximizer for $\hat{v}_i$ from x_i is unique and different for each distinct $\hat{v} \in \mathcal{V}_i$ and similarly for y_i. Since the choice from x_1 is the one which maximizes w_3, we must have $\hat{v}_1 = w_3$. But the symmetric argument for y_1 implies $\hat{v}_1 = -w_3$, a contradiction. Hence it must be true that $\hat{u} \in \{w_1, -w_1\}$. However, the symmetric argument for x_2 and y_2 implies that $\hat{u} \in \{w_2, -w_2\}$. Since $\{w_1, -w_1\} \cap \{w_2, -w_2\} = \emptyset$, this is impossible.

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