Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion*

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Abstract

Models of ambiguity aversion have recently found many applications in dynamic settings. This paper shows that the modeling choices that are being made in the domain of ambiguity aversion influence the set of modeling choices available in the domain of timing attitudes, in particular the preferences for the timing of the resolution of uncertainty, as defined by the classic work of Kreps and Porteus (1978). The main result of the paper is that the only model of ambiguity aversion that exhibits indifference to timing is the maxmin expected utility of Gilboa and Schmeidler (1989). This paper also examines the structure of the timing nonindifference implied by the other commonly used models of ambiguity aversion. The interdependence of ambiguity and timing that this paper identifies is of interest both conceptually and practically—especially for economists using these models in applications.

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1 Introduction

The concept of uncertainty, or ambiguity, has been studied by economists since the work of Keynes (1921) and Knight (1921). As opposed to risk, where the probability is well-specified, ambiguity is characterized by the inability of the decision maker to formulate a single probability or by his lack of trust in any unique probability estimate. As demonstrated by Ellsberg (1961), people often make choices that cannot be justified by a unique probability, exhibiting a preference for risky choices over those involving ambiguity. Ambiguity aversion has been a central topic in decision theory in the past years, which has resulted in many elegant formal models.[1]

Models of ambiguity aversion have recently found many applications in dynamic contexts to questions in finance and macroeconomics.[2] Their use typically involves a recursive formulation where uncertainty is resolving over time and a particular model of ambiguity aversion is used in each period to assess the uncertain continuation values.

In general, in situations where uncertainty does not resolve in one shot the decision maker may have an intrinsic preference for the timing of resolution of uncertainty and distinguish between prospects based on the times at which their uncertainty resolves. The standard model of discounted expected utility satisfies the reduction of compound lotteries and therefore exhibits no such intrinsic preference for timing. Choice models in which timing is important were first formally studied in the context of risk by Kreps and Porteus (1978) and subsequently extended and successfully applied to asset pricing.[3] In these models the nonindifference to timing is obtained by assuming a nonlinear aggregation of the utility of the present consumption and of the continuation value.

[1] The seminal theoretical contribution is Schmeidler (1989), followed by Gilboa and Schmeidler (1989), Epstein (1999), Epstein and Zhang (2001), Ghirardato and Marinacci (2002), Klibanoff, Marinacci, and Mukerji (2005), Maccheroni, Marinacci, and Rustichini (2006a), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008), and others.

[2] See, e.g., Epstein and Wang (1994), Maenhout (2004), Chen and Epstein (2002), Karantounias, Hansen, and Sargent (2007), Kleshchelski and Vincent (2007), Ju and Miao (2007), Collard, Mukerji, Sheppard, and Tallon (2008), Barillas, Hansen, and Sargent (2009), Chen, Ju, and Miao (2009), Benigno and Nisticò (2009), Ilut (2009), and Drechsler (2009).

[3] See, e.g., Epstein and Zin (1989, 1991), Weil (1989, 1990), and Tallarini (2000), among others.

This paper shows that even without a nonlinear aggregator, that is, with standard discounting, most models of ambiguity aversion result in timing nonindifferences, i.e., agents with such preferences would be willing to pay a premium for earlier resolution of uncertainty. The main result shows that the only ambiguity averse model which induces indifference to timing is the maxmin expected utility of Gilboa and Schmeidler (1989). This result means that assuming any other model of ambiguity aversion will result in dynamic preferences that exhibit nonindifference to timing. This paper examines the structure of this nonindifference for the most popular classes of preferences used in applications.

The interdependence of ambiguity and timing that this paper identifies is of interest both conceptually and practically, especially for economists using these models in applications because it means that the modeling choices that are being made in the domain of ambiguity attitudes influence the set of modeling choices available in the domain of timing attitudes. In applied work, holding a particular model of ambiguity constant, the degree of ambiguity aversion needed to fit the model to the data will imply a certain premium that the agent is willing to pay for earlier resolution of uncertainty. The magnitude of this premium will depend on the model in question and may be helpful in guiding modelling choices, directing attention toward models implying reasonable values of the premium.

The paper procedes as follows: Section 2 defines static ambiguity averse preferences; Section 3 defines discounted ambiguity preferences and defines the notion of preference for earlier resolution of uncertainty; Section 4 presents the main results of the paper, which show that the choice of the ambiguity model has strong consequences for the resulting preferences for timing; Section 5 compares these results to the known results for choice over lotteries; finally, Section 6 studies a more general model with a nonlinear aggregator and shows that the timing effects identified in Section 4 can be replicated by a nonlinear aggregator, but only to a limited extent.

2 Models of Ambiguity and Ambiguity Attitudes

Let S be the set of *states of nature*, Σ be an algebra of *events*, and X be a set of consequences, assumed to be a convex subset of a real vector space. An *act* is a Σ-measurable simple function $f : S \to X$; the set of such acts is denoted $\mathcal{F}$. Let $B_0(\Sigma)$ denote the set of all real-valued Σ-measurable simple functions and $B_0(\Sigma; K)$ be the set of all such functions that take values in some set $K \subseteq \mathbb{R}$. Let $\Delta(\Sigma)$ be the set of all finitely additive probability measures on $(S; \Sigma)$.

Preferences studied in this paper are represented by

$$V(f) = I(u(f)) \tag{1}$$

where $u : X \to \mathbb{R}$ is an affine utility function, and $I : B_0\big(\Sigma; u(X)\big) \to \mathbb{R}$ is a functional that represents the decision maker's "beliefs" by aggregating the utility values over states. It will be maintained throughout that u is unbounded, more specifically, that $u(X) = \mathbb{R}$ or $u(X) = \mathbb{R}_+$.

The most basic example of such a functional is coming from the familiar *subjective expected utility* preferences, where for each $\quad \in B_0\big(\Sigma; u(X)\big)$ the functional is of the form $I(\quad) = \int \quad \mathrm{d}p$ for some probability measure $p \in \Delta(\Sigma)$. Another well-known example is the functional associated with the Gilboa and Schmeidler's (1989) *maxmin expected utility* (MEU) preferences, where $I(\quad) = \min_{p \in C} \int \quad \mathrm{d}p$ for some convex and weak*-closed set of measures $C \subseteq \Delta(\Sigma)$. Other important models include:

1. *Choquet expected utility preferences* (Schmeidler, 1989), where $I(\quad) = \int \quad \mathrm{d}\quad$ for some convex capacity $\quad : \Sigma \to [0; 1]$.

2. *Second order expected utility preferences* (Neilson, 1993; Nau, 2006; Ergin and Gul, 2009), where $I(\quad) = \quad^{-1}\big(\int \quad (\quad) \, \mathrm{d}p\big)$ for some strictly increasing and concave function $\quad : u(X) \to \mathbb{R}$.

3. *Smooth ambiguity preferences* (Klibanoff et al., 2005; Seo, 2008; see also Segal, 1987), where $I(\quad) = \quad^{-1}\big(\int_{\Delta(\Sigma)} \quad (\int \quad \mathrm{d}p) \, \mathrm{d}\quad(p)\big)$ for some strictly increasing and concave function $\quad : u(X) \to \mathbb{R}$ and a Borel probability measure $\quad$ on $\Delta(\Sigma)$.

4. *Variational preferences* (Maccheroni et al., 2006a), where $I(\ ) = \min_{p\in\Delta(\Sigma)} \int \ \mathrm{d}p + c(p)$ for some convex and weak*-lower semicontinuous function $c : \Delta(\Sigma) \to [0, \infty]$.

5. *Multiplier preferences* (Hansen and Sargent, 2001; Strzalecki, 2008) with $I(\ ) = \min_{p\in\Delta(\Sigma)} \int \ \mathrm{d}p + \ R(p \,\|\, q)$, where $R(p \,\|\, q)$ is the relative entropy of p with respect to some fixed countably additive and nonatomic measure $q \in \Delta(\Sigma)$ and a parameter $\ \in (0, \infty]$.

6. *Confidence preferences* (Chateauneuf and Faro, 2008), defined for $u(X) = \mathbb{R}_+$, where for some quasiconcave and weak*-upper semicontinuous function $' : \Delta(\Sigma) \to [0, 1]$ and a parameter $\ \in (0, 1)$, $I(\ ) = \min_{\{p\in\Delta(\Sigma)|\varphi(p)\geq\alpha\}} \frac{1}{\varphi(p)} \int \ \mathrm{d}p$.[4]

All of these examples feature a functional I that is continuous (in the supnorm topology), monotonic (i.e., $I(\ ) \geq I(\ )$ whenever $(s) \geq \ (s)$ for all $s \in S$), normalized ($I(k) = k$ for all $k \in u(X)$, interpreted as constant functions), and quasiconcave ($I(\ \ + (1 - \ )\ ) \geq \min\{I(\ ), I(\ )\}$). The last property corresponds to the famous uncertainty (or ambiguity) aversion axiom of Schmeidler (1989), which postulates that the decision maker does not like variability of payoff across states.

Preferences that can be represented by a belief functional I with such properties are called *uncertainty averse preferences*.[5] This representation of preferences makes it convenient to study attitudes toward ambiguity. A decision maker has *constant absolute ambiguity aversion*[6] if $I(\ + k) = I(\ ) + k$ for all $k \in u(X)$ and $\ , \ + k \in B_0(\Sigma, u(X))$. The subclass of uncertainty averse preferences with this property is precisely the class of variational preferences. Similarly, a decision maker has *constant relative ambiguity aversion* if $I(b\ ) = bI(\ )$ for all $b > 0$ and $\ \in B_0(\Sigma, u(X))$. When $u(X) = \mathbb{R}_+$ the subclass of uncertainty averse preferences with this property is the

[4] An extension of these preferences to the case of $u(X) = \mathbb{R}$ was studied by Cerreia-Vioglio et al. (2008), see their Theorem 21.

[5] In a recent paper Cerreia-Vioglio et al. (2008) show that I can be written as $I(\xi) = \min_{p\in\Delta(\Sigma)} G(\int \xi \,\mathrm{d}p, p)$ for some quasiconvex function $G : \mathbb{R} \times \Delta(\Sigma) \to \mathbb{R}$ that is increasing in its first argument. The results in this paper do not depend on this (very interesting) representation.

[6] See Proposition 3 of Grant and Polak (2008) and also Definition 6 of Klibanoff et al. (2005).

class of confidence preferences.[7] Although all of the above classes of preferences have been behaviorally characterized by axioms imposed on the preference relation, the results of this paper are stated directly in the language of representations. All the results obtained in this paper can nonetheless be expressed in the language of preferences, a task which, for the sake of brevity, will not undertaken here.

3 Temporal Resolution of Uncertainty

The purpose of this section is to define formally what it means for the decision maker to care about the timing of uncertainty. In order to do so, a model will be studied where uncertainty is dated by the time of its resolution: in each period there is a state space S and the payoff at time t may depend on the realization of the period t uncertainty and/or uncertainty that has already resolved in previous periods. This model mirrors Kreps and Porteus's (1978) framework with the difference that here uncertainty is subjective and preferences may not be expected utility.[8] This recursive framework is also used in finance and macroeconomics, where in each period S is the set of possible "shocks".

Formally, time is discrete and varies over $\mathcal{T} = \{0, \ldots, T\}$. The set of states of the world is $\Omega = S^T$. Information arrival is modeled as the naturally defined filtration $\{G_t\}_{t \in T}$ where $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and for $t = 1, \ldots, T$ $\mathcal{G}_t = \Sigma^t \otimes \{\emptyset, S\}^{T-t}$ is the product sigma algebra of t copies of Σ and $T - t$ copies of the trivial sigma algebra. Thus, at time t the decision maker knows the realizations of uncertainty up to time t, but is ignorant about the future. For any $\omega = (s_1, \ldots, s_T)$ let $\omega^t = (s_1, \ldots, s_t)$ be the history of observations after time t. The consumption plans are modeled as finite-ranged X-valued adapted processes $h = (h_0, h_1, \ldots, h_T)$, where $h_t : \Omega \to X$ is $\mathcal{G}_t$-measurable for each $t \in \mathcal{T}$. Let $\mathcal{H}$ denote the set of all consumption plans. The family of relations $\{\succsim_{t,\omega}\}_{(t,\omega) \in \mathcal{T} \times \Omega}$ on $\mathcal{H}$ describes agent's conditional preferences.

[7] As the intersection of both classes, maxmin expected utility preferences are characterized by both of those properties.

[8] Epstein and Zin (1989), Chew and Epstein (1989), Segal (1990), Grant, Kajii, and Polak (1998), and Grant, Kajii, and Polak (2000) study nonexpected utility preferences in the objective risk framework of Kreps–Porteus. Section 5 compares those findings to the results obtained here.

3.1 Discounted uncertainty averse preferences

Definition 1 (Discounted uncertainty averse preferences)**.** A family $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ has a *discounted uncertainty averse representation with* (β, I, u) if it is represented by a family of functionals $V_t : \Omega \times \mathcal{H} \to \mathbb{R}$ defined recursively by $V_T(\omega, h) = u(h_T(\omega))$, and for $t < T$

$$V_t(\omega, h) = u\big(h_t(\omega)\big) + \beta I\Big(V_{t+1}\big(\cdot, h\big)\Big), \tag{2}$$

where $u : X \to \mathbb{R}$ is affine, $\beta \in (0, 1)$, and $I : B_0\big(\Sigma, u(X)\big) \to \mathbb{R}$ is normalized, monotone, continuous, and quasiconcave.[9]

Note that $V_{t+1}(\cdot, h)$ is $\mathcal{G}_{t+1}$-measurable for each $h \in \mathcal{H}$; for this reason, in period t $V_{t+1}\big(\cdot, h\big)$ defines an element of $B_0\big(\Sigma, u(X)\big)$, which represets the uncertainty about the period t+1 continuation value that the decision maker faces at period t, knowing the history of realizations ω^t.

Discounted uncertainty averse preferences include as special cases most of the models used in applications[10] but in general they allow for more flexible models of ambiguity aversion, as described in Section 2.[11]

3.2 Attitudes toward timing of resolution of uncertainty

For any $f : S \to X$ define a $\mathcal{G}_t$-measurable act $\check{f}_t : \Omega \to X$ by $\check{f}_t(s_1, \ldots, s_T) = f(s_t)$; that is, the act $\check{f}_t$ is a copy of the act f that resolves at time t, i.e., that depends only on the t-th component of the state space. Intuitively, given any $f \in \mathcal{F}$ the act $\check{f}_t$ is equally uncertain, but resolves earlier than $\check{f}_{t+1}$. This notion is now used to rank consumption plans.

[9] The results of this paper hold also under the often used alternate specification of the recursion $V_t(\omega, h) = u\big(h_t(\omega)\big) + I\Big(\beta V_{t+1}\big(\cdot, h\big)\Big)$.

[10] For example the time consistent dynamic variational preferences of Maccheroni, Marinacci, and Rustichini (2006b) and recursive maxmin expected utility preferences of Epstein and Schneider (2003b), as well as the recursive smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2009)

[11] Alternative, nonrecursive, methods of extending models of ambiguity aversion to dynamic settings include the dynamically consistent updating rules investigated by Hanany and Klibanoff (2007, 2008) and the model of sophisticated dynamic choice studied by Siniscalchi (2006).

Fix a node (t, ω) and suppose that the only uncertainty that the decision maker faces is about the period $t+2$ payoff, i.e., only h_{t+2} is a non-degenerate act. Consider two scenarios. In the first one, the uncertainty resolves early, that is the decision maker learns the realizations of h_{t+2} already in period $t+1$. Formally, let $h_{t+2} = \check{f}_{t+1}$ for some $f \in \mathcal{F}$.

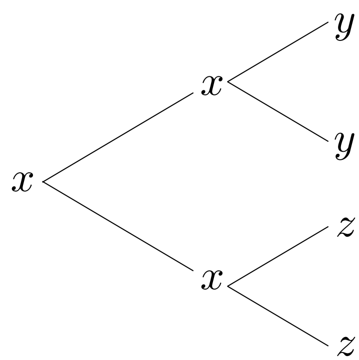

Figure 1: Uncertainty resolves today

In the second scenario, the uncertainty resolves late, that is the decision maker learns the realizations of h_{t+2} only in period $t+2$. Formally, let $h_{t+2} = \check{f}_{t+2}$ for the same $f \in \mathcal{F}$ as above.

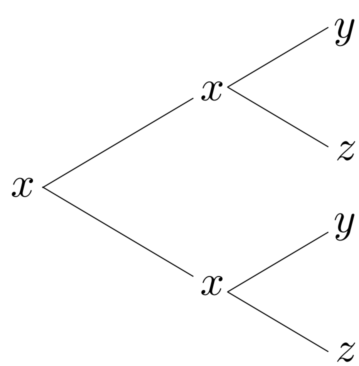

Figure 2: Uncertainty resolves tomorrow

Definition 2. A consumption plan h resolves earlier than h', denoted $h \geqq h'$, if and only if there exists $f \in \mathcal{F}$, $t \in \{0, 1, \ldots, T-2\}$, and $x_0, \ldots, x_{t+1}, x_{t+3}, \ldots, x_T \in X$ such that $h_j = h'_j = x_j$ for all $j \neq t+2$, $h_{t+2} = \check{f}_{t+1}$, and $h'_{t+2} = \check{f}_{t+2}$.

A decision maker who always respects this order is said to display a *preference for earlier resolution of uncertainty.*

Definition 3. The family of relations $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ exhibits a *preference for earlier resolution of uncertainty* if and only if for all $h, h' \in \mathcal{H}$ such that $h \geqq h'$ the preference $h \succsim_{t,\omega} h'$ holds for all $t \in \mathcal{T}$ and $\omega \in \Omega$. The notions of *indifference to timing of resolution of uncertainty* and *preference for later resolution of uncertainty* are defined analogously.

3.3 IID Ambiguity

The effects identified in this paper are also present in a formulation more general than (2), which allows for different beliefs in each period.

$$V_t(\omega, h) = u\big(h_t(\omega)\big) + \beta I_t\big(\omega, V_{t+1}(\cdot, h)\big), \tag{3}$$

However, in this more general model the attitudes toward timing of resolution of uncertainty are confounded with changing beliefs. To see that, observe that given any $f : S \to X$ the difference between the acts $\check{f}_t$ and $\check{f}_{t+1}$ is twofold. First, these two acts differ in the timing of their resolution. Second, they differ to the extent to which the beliefs about the t-th copy of S differ from the beliefs about the t+1-th copy of S. In formulation (3) a preference for $\check{f}_t$ over $\check{f}_{t+1}$ is a result of the intrinsic preference for earlier resolution of uncertainty "plus" the effect of changing beliefs.[12] By imposing a "constant beliefs" assumption, known as IID ambiguity,[13] formulation (2) eliminates this latter effect and isolates the pure attitudes toward timing.

[12]This issue does not arise in the model of Kreps and Porteus (1978) because of the objective nature of the probabilities in their formulation. The only difference between the analogues of $\check{f}_t$ and $\check{f}_{t+1}$ is the timing of their resolution because their probabilities are objectively the same.

[13]The notion of IID ambiguity was introduced by Chen and Epstein (2002) and Epstein and Schneider (2003a) in the context of the maxmin expected utility model; it means that the uncertainty that the decision maker faces in period t is identical to the uncertainty in period t+1, the only distinguishing property being the timing of their resolution. Intuitively, a decision maker has IID ambiguity if in each period he faces a new Ellsberg urn; his ex ante beliefs about each urn are identical, but because he observes only one draw from each urn, he cannot make inferences across urns and will not learn his way out of ambiguity, as opposed to a situation where he observes repeated sampling (with replacement) from the same urn. (The failure of inference in such settings

4 Discounted preferences and timing attitudes

This section takes as given a family of discounted uncertainty averse preferences $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$, defined by expression (2), and examines the relationship between the attitudes toward ambiguity, as described in Section 2, and the attitudes toward timing of resolution of uncertainty, as described in Section 3. The main message is that the modeling choices in the domain of ambiguity have strong consequences for the resulting attitudes toward timing. The starkest manifestation of this interdependence is Theorem 1, which says that the only way to ensure indifference to timing is by using the maxmin expected utility model. This means that assuming any other model of ambiguity aversion will result in a family of preferences that exhibits nonindifference to timing. The subsequent theorems examine the structure of this nonindifference implied by models of ambiguity other than MEU.

Theorem 1. *A family of dynamic uncertainty averse preferences $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ satisfies indifference toward timing of resolution of uncertainty if and only if I is a MEU functional.*

The next theorem asserts that variational preferences exhibit uniform attitudes toward timing. They all display a preference for *earlier* resolution of uncertainty, the only "knife-edge" case of indifference being the class of maxmin expected utility preferences.

Theorem 2. *If $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ is a family of dynamic variational preferences, then $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ satisfies preference for earlier resolution of uncertainty.*

The multiplicative analog of variational preferences—confidence preferences—exhibits the same behavior, provided that $\mathsf{u}(\mathsf{X}) = \mathbb{R}_+$. In contrast, when $\mathsf{u}(\mathsf{X}) = \mathbb{R}$, relative ambiguity aversion typically results in nonuniform attitudes toward timing. In this case, the only subclass with uniform attitudes are maxmin expected utility preferences.

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is known in econometrics as the problem of *incidental parameters*, see, e.g., Neyman and Scott, 1948).

Theorem 3. *Suppose that* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *is a family of dynamic ambiguity averse preferences that satisfy constant relative ambiguity aversion.*

(i) If $u(X) = \mathbb{R}_+$, *i.e.,* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *are dynamic confidence preferences, then* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *satisfies preference for earlier resolution of uncertainty.*

(ii) If $u(X) = \mathbb{R}$ *and if* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *displays a preference for earlier resolution of uncertainty, then* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *is a recursive multiple priors preference.*

The results so far show that the variational and confidence preferences always display a preference for earlier resolution of uncertainty, while the constant relative ambiguity aversion preferences always display a nonuniform attitude toward timing of uncertainty (if $u(X) = \mathbb{R}$). The next two important classes of preferences behave differently: for some values of parameters they display a preference for earlier resolution of uncertainty, while for others they display nonuniform attitudes. The next two theorems explore these conditions. First, Theorem 4 characterizes a subclass of second order expected utility preferences that display a preference for earlier resolution of uncertainty.

Theorem 4. *Suppose that* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *is a family of dynamic second order expected utility preferences with a twice differentiable function* .

(i) If $u(X) = \mathbb{R}$, *it displays a preference for earlier resolution of uncertainty if and only if there exists a real number* $A \geq 0$ *such that* $-^{\phi''(a)}$a a 0927 175 237(7955Ω0 36 Ω 86 8 0 98 (Ω45

condition from part (ii) permits constant relative ambiguity aversion, i.e., $\ (a) = a^{\gamma}$ for some $\ \in (0; 1)$.[15]

Finally, Theorem 5 shows that the same conditions as in Theorem 4 are sufficient for preference for earlier resolution of uncertainty in the class of smooth ambiguity preferences. Under an additional assumption it is possible to show that these conditions are also necessary.[16]

Assumption 1. S is finite with cardinality n and that the support of the measure $\ $ is finite with cardinality m. For $j = 1; : : : ; m$ let each measure $p_j \in \operatorname{supp}\ $ be represented as a row vector in $\mathbb{R}^n$ and let M be an $m \times n$ matrix of those vectors stacked on top of each other. The matrix M has rank m.

Theorem 5. *Suppose that $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ is a family of dynamic smooth ambiguity preferences with a twice differentiable function $\ $.*

(i) If $u(X) = \mathbb{R}$, it displays a preference for earlier resolution of uncertainty if there exists a real number $A \geq 0$ such that $-\frac{\phi''(x)}{\phi(x)} \in [\ A; A]$ for all $x \in \mathbb{R}$.

(ii) If $u(X) = \mathbb{R}$ and if it satisfies Assumption 1 and displays a preference for earlier resolution of uncertainty then there exists a real number $A \geq 0$ such that $-\frac{\phi''(x)}{\phi(x)} \in [\ A; A]$ for all $x \in \mathbb{R}$.

(iii) If $u(X) = \mathbb{R}_+$, it displays a preference for earlier resolution of uncertainty if $\left[-\frac{\phi''(\beta a+k)}{\phi'(\beta a+k)}\right] \leq \left[-\frac{\phi''(a)}{\phi'(a)}\right]$ for all $a; k \in \mathbb{R}_+$.

(iv) If $u(X) = \mathbb{R}_+$ and if it satisfies Assumption 1 and displays a preference for earlier resolution of uncertainty then $\ \left[-\frac{\phi''(\beta a+k)}{\phi'(\beta a+k)}\right] \leq \left[-\frac{\phi''(a)}{\phi'(a)}\right]$ for all $a; k \in \mathbb{R}_+$.

[15]Constant relative ambiguity aversion corresponds to the intersection of this class of preferences with the class of confidence preferences. The fact that those preferences satisfy a preference for earlier resolution of uncertainty follows already from Theorem 3 (i).

[16]It is not known whether these conditions are necessary in general. Likewise, it is not known whether uncertainty aversion (quasiconcavity of I) implies the concavity of ϕ (although the converse is always true).

5 Comparison to Choice over Lotteries

5.1 Intertemporal Elasticity of Substitution

The model of Kreps and Porteus (1978) allows for a separation between the elasticity of substitution between states and between time periods. However, in that model the difference betwen these two elasticities is directly related to the strength of the preference for timing of resolution of uncertainty. In other words, the three features: intertemporal elasticity of substitution, elasticity of substitution between states, and preference for timing of resolution of uncertainty are interdependent; roughly speaking, knowing two of them is sufficient to determine the third. For this reason, the Kreps–Porteus model may be seen as too restrictive because it does not allow enough freedom to specify the three parameters independently.

To see that, suppose that $u(x) = x^{\alpha}$ for some $\ \in (0; 1)$. Consider first a discounted second order expected utility model, with $\ (x) = x^{\rho}$ for some $\ \in (0; 1]$. This constitutes a (subjective) analog of the Kreps–Porteus model. In this model, the intertemporal elasticity of substitution is equal to $(1 - \ )^{-1}$, whereas the elasticity of substitution between states is equal to $(1 - \ )^{-1}$. As long as these two are different, i.e., as long as $\ \neq 1$, the decision maker will not be indifferent toward timing of resolution of uncertainty.[17]

From this point of view, dynamic ambiguity models allow more flexibility. In particular, a discounted MEU model allows for a separation of the three features. Indifference to timing is guaranteed by Theorem 1, while the intertemporal elasticity of substitution is $(1 - \ )^{-1}$, which is not identically equal to the elasticity of substitution between states. The latter varies between zero and $(1 - \ )^{-1}$ depending on the act at which it is computed. This shows that it is possible to drive a wedge between the two elasticities without forcing the timing nonindifference.

[17]According to Theorem 1, the decision maker displays an indifference to the timing of resolution of uncertainty if and only if I is MEU, The only intersection of MEU and second order expected utility preferences are expected utility preferences, i.e., $\rho = 1$.

5.2 Timing Attitudes

Dynamic models of ambiguity may be seen as more flexible than those of risk for yet another reason. As Theorem 1 of Chew and Epstein (1989) shows, when preferences are defined over lotteries rather than acts, indifference to timing implies that the certainty equivalent (the counterpart of the functional I in their model) has to be expected utility. In contrast, Theorem 1 of this paper shows that in the domain of acts the class of preferences indifferent to timing is larger—it is precisely the MEU class.

Even more restrictive is the fact that most of the known departures from expected utility in the risk domain induce a nonuniform attitude toward timing (much like the constant relative ambiguity aversion preferences of Theorem 3 (ii)). Proposition 1 of Grant et al. (2000) shows that (if preferences are rank-dependent or satisfy betweenness) expected utility is a necessary consequence of preference for earlier resolution of uncertainty.[18] In contrast, Theorems 2–5 of this paper show that in the domain of acts the class of such preferences is larger—it includes all variational and confidence preferences, as well as certain second order expected utility and smooth ambiguity preferences.

An illustrative case in point is rank-dependent expected utility (RDEU) of Quiggin (1982) and Yaari (1987) in which probability distributions are distorted by a transformation function. When the preferences are defined on acts, assuming that the probability transformation function is concave, this model reduces to Choquet expected utility with a convex capacity—a special case of MEU—and thus satisfies timing indifference. On the other hand, when the preferences are defined on lotteries the aforementioned results imply nonindifference (and nonuniform attitude) to timing.

It should be stressed that these differences are not a consequence of the conceptual distinction between risk and ambiguity, but rather they are caused by the

[18] More precisely, Proposition 1 of Grant et al. (2000), as well as Theorem 1 of Chew and Epstein (1989) allow for the certainty equivalent in each time period to differ and show that all but the first or last certainty equivalents have to be EU. These results imply the above statements in the context of a model with a constant certainty equivalent, like the one here.

dissimilarity of the two choice domains. In particular, the definition of early resolution (relation $\geqq$) in the subjective domain is less restrictive than in the model with objective probabilities. This is caused by the fact that in the objective setting earlier resolution is defined through probability mixtures of lotteries, while in the subjective setting the—less flexible—eventwise mixtures are used. For this reason, although RDEU does not preserve the indifferences to timing for all comparable pairs of temporal lotteries in the objective domain, it does so in the subjective domain because there are fewer such $\geqq$-comparable pairs.

6 Recursive preferences and timing attitudes

A more general class of preferences can be defined by relaxing the standard discounting assumption implicit in expression (2).

Definition 4 (Recursive uncertainty averse preferences)**.** A family $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ has a *recursive uncertainty averse representation with* $(W; I; v)$ if it is represented by a family of functionals $V_t : \Omega \times \mathcal{H} \to \mathbb{R}$ defined recursively by $V_T(\omega; h) = v(h_T(\omega))$, and for $t < T$

$$V_t(\omega; h) = W\Big(h_t(\omega); I\Big(V_{t+1}(\cdot; h)\Big)\Big); \tag{4}$$

where $v : X \to \mathbb{R}$, the *aggregator* $W : X \times D_0 \to \mathbb{R}$ is continuous, strictly increasing and unbounded in the second argument, and $I : B_0(\Sigma; D_0) \to \mathbb{R}$ is normalized, monotone, continuous, and quasiconcave.[19]

These preferences are a natural generalization of Koopmans (1960), Kreps and Porteus (1978), and Epstein and Zin (1989) to subjective uncertainty.[20] The discounted preferences defined by (2) correspond to the special case of recursive preferences defined by (4) with $W^{disc}(x; \ ) = u(x) + \quad$ and an affine function $u : X \to \mathbb{R}$.

[19]The set D_0 is defined recursively by $D_T := v(X)$ and $D_t := D_{t+1} \cup \bigcup_{x\in X} W(x, D_{t+1})$.

[20]Other subjective extensions have been studied and axiomatized by Hayashi (2005), Klibanoff and Ozdenoren (2007), and Skiadas (1998). Skiadas (1998) also studies attitudes toward timing by assuming that preferences are defined over pairs consisting of a consumption plan and exogenously given information in form of a filtration that the consumption plan is adapted to.

In the model of Kreps and Porteus (1978), where $I(\ ) = \int \ \mathrm{d}p$, the standard discounting aggregator W^{disc} implicit in expression (2) characterizes indifference to timing of resolution of uncertainty, while more general aggregators lead to nonindifference (in particular, convexity of W in the second argument corresponds to the case of preference for earlier resolution). The main result of this paper is that even with standard discounting most models of ambiguity lead to timing nonindifference, maxmin expected utility preferences being the only case of indifference to timing. Thus, attitudes toward timing arise in a certain sense "endogenously", without explicitly imposing them through a nonstandard aggregator W.

From this point of view, the aggregator, $W;$ and the "belief functional", $I;$ are responsible for the same phenomenon of timing nonindifference. This section studies the extent to which W and I are substitutes. The main result is a characterization of the class of preferences where the timing nonindifference resulting from a non-MEU belief functional I is exactly the same as the one resulting from the nonstandard aggregator W.

Consider a MEU preference with $I^{MEU}(\ ) = \min_{p \in C} \int \ \mathrm{d}p$ for some set of measures C. As established in Theorem 1, discounted MEU preferences are the only ones that satisfy indifference to timing. In principle, there are two ways of obtaining timing nonindifference: first, by changing the functional I^{MEU} to some other uncertainty averse functional I; second, by changing the standard discounting aggregator W^{disc}, implicit in expression (2), to a nonstandard aggregator W. The following theorem characterizes the class of preferences where the first method is equivalent to the second.

Theorem 6. *Suppose that* $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ *is a family of recursive uncertainty averse preferences with* $(W^{disc}; I; u)$, $u(X) = \mathbb{R}$, *and there exists an essential event* $E \in \Sigma$.[21] *It has a recursive uncertainty averse representation with* $(W; I^{MEU}; v)$ *if and only if* $I(\ ) = \min_{p\in C} \ ^{-1}\Big(\int \ (\ )\,\mathrm{d}p\Big)$ *for some strictly increasing and concave function* $: \mathbb{R} \to \mathbb{R}$ *and some convex and weak*-closed set* $C \subseteq \Delta(\Sigma)$. *In this case* $I^{MEU}(\ ) = \min_{p\in C} \int \ \mathrm{d}p$ *and* $W(x; d) = \ \big(u(x) + \ ^{-1}(d)\big)$ *for all* $x \in X$ *and all* $d \in \mathrm{Range}(\ )$.

[21] An event E is essential iff there exists $\overline{k}, \underline{k} \in \mathbb{R}$ such that $\overline{k} > I\big(\overline{k}E\underline{k}\big) > \underline{k}$.

The class of preferences characterized by Theorem 6 generalizes both the MEU preferences (with being an affine function) and second-order expected utility preferences (with C being a singleton). This is precisely the subclass of uncertainty averse preferences for which W and I are perfect substitutes. The timing effects induced by the non-MEU nature of the belief functional are exactly like the effects induced by an appropriate choice of the aggregator. For any other uncertainty averse preference this equivalence breaks down and the timing effects of I cannot be mimicked by W.

A Appendix: Proofs

Lemma 1. *The family $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ displays a preference toward earlier resolution of uncertainty if and only if $I(\ + k) \geq \ I(\ ) + k$ for all $\ \in B_0\big(\Sigma, u(X)\big)$ and all $k \in u(X)$. The family $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ displays indifference toward timing of resolution of uncertainty if and only if $I(\ + k) = \ I(\ ) + k$ for all $\ \in B_0\big(\Sigma, u(X)\big)$ and all $k \in u(X)$.*

Proof. Fix $x_0, \dots, x_T \in X$ and $f \in \mathcal{F}$. Let $l := {}^{0}u(x_{t+3}) + \dots + {}^{T-t-3}u(x_T)$. Observe that

$$V_t(\omega, (x_0, \dots, x_{t+1}, f_{t+1}, x_{t+3}, \dots, x_T)) = u(x_t) + \ I\left(u(x_{t+1}) + \ I\big(u(f_{t+1}(\omega^{t+1}, \cdot)) + \ l\big)\right).$$

Because f_{t+1} is $\mathcal{G}_{t+1}$-measurable $I\big(u(f_{t+1}(\omega^{t+1}, \cdot))\big) = u(f_{t+1}(\omega)) = u(f(s_{t+1}))$ for all $\omega \in \Omega$, thus by denoting $\ := u(f)$,

$$V_t(\omega, (x_0, \dots, x_{t+1}, f_{t+1}, x_{t+3}, \dots, x_T)) = u(x_t) + \ I\left(u(x_{t+1}) + \ (\ + \ l)\right). \tag{5}$$

On the other hand,

$$V_t(\omega, (x_0, \dots, x_{t+1}, f_{t+2}, x_{t+3}, \dots, x_T)) = u(x_t) + \ I\left(u(x_{t+1}) + \ I\big(u(f_{t+2}(\omega^{t+1}, \cdot)) + \ l\big)\right).$$

Because f_{t+2} does not depend on ω_{t+1}, $I\big(u(f_{t+2}(\omega^{t+1},\cdot))\big) = I(u(f)))$. Thus,

$$V_t(\omega,(x_0,\dots,x_{t+1},f_{t+2},x_{t+3},\dots,x_T)) = u(x_t) + \quad \Big(u(x_{t+1}) + \quad I(\quad + \quad I)\Big). \tag{6}$$

Suppose I displays a preference for earlier resolution of uncertainty. Then the expression (5) is bigger than (6) for any choice of $x_0,\dots,x_T \in X$ and $\quad \in B_0(\Sigma)$, in particular such that $u(x_{t+1}) = k$, and $\quad = \quad + \quad I$. The converse and the proof of the statement about indifference follow similarly. □

Lemma 2. *Let* $\Phi := \quad (u(X))$ *and for each* $k \in u(X)$ *define* $F_k : \Phi \to \mathbb{R}$ *by* $F_k(\quad) = \quad \big(\quad^{-1}(\quad) + k\big)$*. Suppose that* $\quad$ *is twice differentiable. Then* F_k *is convex for each* $k \in u(X)$ *if and only if* $\Big[-\frac{\phi''(\beta a+k)}{\phi'(\beta a+k)}\Big] \leq \Big[-\frac{\phi''(a)}{\phi'(a)}\Big]$ *for all* $a, k \in u(X)$*.*

Proof. Because $\quad$ is twice differentiable, $\quad^{-1}$ is twice differentiable. Convexity of F_k for each $k \in u(X)$ is equivalent to

$$F_k''(\quad) \geq 0 \text{ for all } k \in u(X) \text{ and } \quad \in \Phi. \tag{7}$$

A direct computation reveals that

$$F_k''(\quad) = \quad''\big(\quad^{-1}(\quad)+k\big)\left[\frac{}{\quad'\big(\quad^{-1}(\quad)\big)}\right]^2 - \quad'\big(\quad^{-1}(\quad)+k\big)\frac{\quad''\big(\quad^{-1}(\quad)\big)}{\big[\quad'\big(\quad^{-1}(\quad)\big)\big]^3}.$$

Thus, $F_k''(\quad) \geq 0$ iff $\left[-\frac{\phi''\left(\beta\phi^{-1}(\gamma)+k\right)}{\phi'\left(\beta\phi^{-1}(\gamma)+k\right)}\right] \leq \left[-\frac{\phi''\left(\phi^{-1}(\gamma)\right)}{\phi'\left(\phi^{-1}(\gamma)\right)}\right].$ □

Lemma 3. *Let* $u(X) = \mathbb{R}$ *and* $\Phi := \quad(\mathbb{R})$ *and for each* $k \in \mathbb{R}$ *define* $F_k : \Phi \to \mathbb{R}$ *by* $F_k(\quad) = \quad\big(\quad^{-1}(\quad) + k\big)$*. Suppose that* $\quad$ *is twice differentiable. Then* F_k *is convex for each* $k \in \mathbb{R}$ *if and only if there exists* $A \geq 0$ *such that* $-\frac{\phi''(a)}{\phi'(a)} \in [\quad A, A]$ *for all* $a \in \mathbb{R}$*.*

Proof. Let $a, b \in \mathbb{R}$. Define $k := b - \quad a$. By Lemma 2 convexity of F_k is equivalent to

$$\left[-\frac{\quad''(b)}{\quad'(b)}\right] \leq \left[-\frac{\quad''(a)}{\quad'(a)}\right] \text{ for all } a, b \in \mathbb{R}. \tag{8}$$

It is immediate that condition (8) is implied if there exists $A \geq 0$ such that $-\frac{\phi''(a)}{\phi'(a)} \in [\ \ A; A]$ for all $a \in \mathbb{R}$. Conversely, let $A := \sup_{b \in \mathbb{R}} \left[-\frac{\phi''(b)}{\phi'(b)} \right]$. The number A is finite because otherwise condition (8) is violated by fixing a and letting the left hand side diverge. Let $A' := \inf_{a \in \mathbb{R}} \Big[-\frac{\phi''(a)}{}$

$I(\ ^{n-1}\ ) = \cdots = \ ^{n} I(\ )$ for any $n \in \mathbb{N}$. Choose n such that $\ ^{n} < b$. For this n it follows that $\ ^{n} I(\ ) = I(\ ^{n}\ ) = I\left(\frac{\beta^n}{b} b\ + \frac{b-\beta^n}{b} 0\right) \geq \frac{\beta^n}{b} I(b\ ) > \ ^{n} I(\ )$. Contradiction.

As a consequence is positively homogeneous (homogeneity for $b > 1$ follows trivially). Thus it satisfies the assumptions of Lemma 3.5 of Gilboa and Schmeidler (1989); therefore, there exists a closed and convex set $C \subseteq \Delta(S)$ such that $I(\ ) = \min_{p \in C} \int\ \mathrm{d}p$ for all $\ : S \to \mathbb{R}$. □

A.2 Proof of Theorem 2

By Lemma 1 the preference for earlier resolution of uncertainty is equivalent to $I(\quad + k) \geq \ I(\ ) + k$ for all $\ \in B_0\big(\Sigma; u(X)\big)$ and $k \in u(X)$. By concavity of I, $I(\quad + k) = I(\quad + (1 - \ )\frac{k}{1-\beta}) \geq \ I(\ ) + (1 - \ ) I(\frac{k}{1-\beta}) = \ I(\ ) + k:$ □

A.3 Proof of Theorem 3

(i): The functional I is concave because $I(\quad + (1 - \ )\ ) = \min_{\{p \in \Delta(\Sigma) | \varphi(p) \geq \alpha\}} \frac{1}{\varphi(p)} \int\quad + (1 - \ )\ \mathrm{d}p = \ \frac{1}{\varphi(p^*)} \int\ \mathrm{d}p^* + (1 - \ ) \frac{1}{\varphi(p^*)} \int\ \mathrm{d}p^*$ for some $p^* \in \Delta(\Sigma)$ with $\ '(p^*) \geq\ $. This expression is then weakly bigger than $\ I(\ ) + (1 - \ ) I(\ )$, which proves concavity. From concavity, it follows that $I(\quad) = I\big(\quad + (1 - \ ) 0\big) \geq \ I(\ ) + (1 - \ ) I(0) = \ I(\ ):$

By Lemma 1 the preference for earlier resolution of uncertainty is equivalent to $I(\quad + k) \geq \ I(\ ) + k$ for all $\ \in B_0(\Sigma; \mathbb{R}_+)$ and $k \in \mathbb{R}_+$. By concavity of I, $I(\quad + k) = I(\quad + (1 - \ )\frac{k}{1-\beta}) \geq \ I(\ ) + (1 - \ ) I(\frac{k}{1-\beta}) = \ I(\ ) + k:$ □

(ii): Constant relative ambiguity aversion means that $I(\quad) = \ I(\ )$ for all $\ \in B_0(\Sigma)$. By Lemma 1 $I(\quad + k) \geq \ I(\ ) + k$ for all $\ \in B_0(\Sigma)$ and $k \in \mathbb{R}$. Thus $I(\quad + k) \geq \ I(\ ) + k = I(\quad) + k$ for all $\ \in B_0(\Sigma)$ and $k \in \mathbb{R}$, which means that

$$I(\ + k) \geq I(\ ) + k \text{ for all } \ \in B_0(\Sigma) \text{ and } k \in \mathbb{R} \tag{10}$$

Suppose, anticipating a contradiction, that $I(\ + k) > I(\ ) + k$ for some $\ \in B_0(\Sigma)$ and $k \in \mathbb{R}$. Then $I(\ + k) > I(\ ) + k = I\big((\ + k) - k\big) + k \geq I(\ + k) - k + k = I(\ + k)$, where the last inequality follows from (10). Contradiction. Thus I satisfies satisfies the assumptions of Lemma 3.5 of Gilboa and Schmeidler (1989), so there exists a

closed, convex set $C \subseteq \Delta(S)$ with $I(\ ) = \min_{p \in C} \int \ \mathrm{d}p$ for all $\ : S \to \mathbb{R}$. □

A.4 Proof of Theorem 4

Let $\Phi := \ (u(X))$. Preference for earlier resolution of uncertainty is equivalent to

$$\int \ (\ +k)\,\mathrm{d}p \geq \ \Big(\ ^{-1}\Big(\int \ (\ )\,\mathrm{d}p\Big)+k\Big) \text{ for all } \ \in B_0\big(\Sigma; u(X)\big) \text{ and } k \in u(X). \tag{11}$$

For each $k \in u(X)$ define $F_k : \Phi \to \mathbb{R}$ by $F_k(\ ) = \ \big(\ ^{-1}(\ )+k\big)$. With this notation, (11) becomes $\int F_k\big(\ (\ )\big)\,\mathrm{d}p \geq F_k\big(\int \ (\ )\,\mathrm{d}p\big)$ for all $\ \in B_0\big(\Sigma; u(X)\big)$ and $k \in u(X)$. By letting $\ = \ (\ )$, this is equivalent to

$$\int F_k(\ )\,\mathrm{d}p \geq F_k\Big(\int \ \mathrm{d}p\Big) \text{ for all } \ \in B_0(\Sigma; \Phi) \text{ and } k \in u(X). \tag{12}$$

Next, condition (12) is equivalent to convexity of F_k for all $k \in u(X)$. To see that, observe that sufficiency follows from Jensen's inequality. Conversely, suppose that (12) holds and find an event $E \in \Sigma$ such that $0 < p(E) < 1$, denoting $\ := p(E)$. For any $\ ;\ ' \in \Phi$ condition (12) applied to $\ = \ _E\ '$ implies that $\ F_k(\ )+(1-\ )F_k(\ ') \geq F_k(\ \ +(1-\ )\ ')$ for each $k \in u(X)$. By Theorem 88 of Hardy, Littlewood, and Pólya (1952) for each $k \in u(X)$ function F_k is convex (for any k the function F_k is continuous). An application of Lemmas 2 and 3 leads to the desired conclusions. □

A.5 Proof of Theorem 5

Let $\Phi := \ (\mathbb{R})$. By Lemma 1 preference for earlier resolution of uncertainty is equivalent to

$$\int_{\Delta(\Sigma)} \ \Big(\ \int \ \mathrm{d}p + k\Big)\,\mathrm{d}\ (p) \geq \ \Big(\ ^{-1}\Big(\int_{\Delta(\Sigma)} \ \Big(\int \ \mathrm{d}p\Big)\,\mathrm{d}\ (p)\Big) + k\Big)$$
$$\text{for all } \ \in B_0\big(\Sigma; u(X)\big) \text{ and } k \in u(X).$$

For each $k \in u(X)$ define $F_k : \Phi \to \mathbb{R}$ by $F_k(\ ) = \ \big(\ ^{-1}(\ ) + k\big)$. With this notation this condition becomes

$$\int_{\Delta(\Sigma)} F_k\Big(\ \big(\int \ \mathrm{d}p\big)\Big)\,\mathrm{d}\ (p) \geq F_k\Big(\int_{\Delta(\Sigma)} \ \big(\int \ \mathrm{d}p\big)\,\mathrm{d}\ (p)\Big)$$

$$\text{for all } \ \in B_0\big(\Sigma; u(X)\big) \text{ and } k \in u(X).$$

By defining $\Upsilon := \{\ : \Delta(\Sigma) \to \mathbb{R} \mid \ (p) = \ (\int \ \mathrm{d}p) \text{ for some } \ \in B_0\big(\Sigma; u(X)\big)\}$ this condition which can be rewritten as

$$\int_{\Delta(\Sigma)} F_k(\ )\,\mathrm{d}\ \geq F_k\Big(\int_{\Delta(\Sigma)} \ \mathrm{d}\ \Big) \text{ for all } \ \in \Upsilon \text{ and } k \in u(X).$$

Sufficiency follows from Lemmas 2 and 3 and Jensen's inequality. For necessity, observe that under Assumption 1 the above condition is equivalent to

$$\sum_{j=1}^{m} F_k(\ _j)\ _j \geq F_k\Big(\sum_{j=1}^{m} \ _j\ _j\Big) \text{ for all } \ \in \Upsilon \text{ and } k \in u(X);$$

where $\Upsilon = \{\ \in \mathbb{R}^m \mid \ _j = \ \big((M\ )_j\big) \text{ for some } \ \in \big(u(X)\big)^n\} = \{\ \in \mathbb{R}^m \mid \ _j = \ \big(\ _j\big) \text{ for some } \ \in \big(u(X)\big)^m\}$. Taking $\ = (a; b; \dots; b)$ for all $a; b \in u(X)$ ensures that for all $c; d \in \Phi$ the set Υ includes all vectors of the form $(c; d; \dots; d)$. Hence, the preference for earlier resolution of uncertainty implies that

$$F_k(c)\ _1 + F_k(d)(1 - \ _1) \geq F_k(c\ _1 + d(1 - \ _1)) \text{ for all } c; d \in \Phi.$$

By Theorem 88 of Hardy et al. (1952) for each $k \in u(X)$ function F_k is convex (for any k the function F_k is continuous). An application of Lemmas 2 and 3 leads to the desired conclusions. □

A.6 Proof of Theorem 6

Sufficiency is trivial. For necessity, assume that $\widehat{V}_t$ is the representation of $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ in terms of $(W^{disc}; I; u)$ and let $\widetilde{V}_t$ be its representation in terms of $(W; I^{MEU}; v)$.

First, fix $x_0; x_1; : : : ; x_{T-1} \in X$ and let h range over $(x_0; x_1; : : : ; x_{T-1}; x)$ for $x \in X$. Then $\widehat{V}_T(!; h) = u(x)$, while $\widetilde{V}_T(!; h) = v(x)$. Because $\widehat{V}_T$ and $\widetilde{V}_T$ represent the same order $\succsim_{T,\omega}$ they have to be ordinally equivalent; thus, there exists a strictly increasing function $\ : \mathbb{R} \to \mathbb{R}$ such that $v(x) = \ (u(x))$ for all $x \in X$.

Second, fix $x_0; x_1; : : : ; x_{T-1} \in X$ and let h range over $(x_0; x_1; : : : ; x_{T-1}; \check{f}_T)$ for $f \in \mathcal{F}$. Observe, that $\widehat{V}_{T-1}(!; h) = u(x_{T-1}) + \ I(u(f))$, while

$$\widetilde{V}_{T-1}(!; h) = W\big(x_{T-1}; \min_{p \in C} \int v(f)\,\mathrm{d}p\big) = W\big(x_{T-1}; \min_{p \in C} \int \ (u(f))\,\mathrm{d}p\big):$$

Because $\widehat{V}_{T-1}$ and $\widetilde{V}_{T-1}$ represent the same order, $\succsim_{T-1,\omega}$, they have to be ordinally equivalent; in particular, their restrictions to $\mathcal{F}$ have to be ordinally equivalent. Of course the restriction of $\widehat{V}_{T-1}$ is ordinally equivalent to $I(u(f))$ and the restriction of $\widetilde{V}_{T-1}$ is ordinally equivalent to $\ ^{-1}\big(\min_{p \in C} \int \ (u(f))\,\mathrm{d}p\big)$. Observe that they coincide on constant acts; for this reason it is possible to define induced preferences over "utility acts", $B_0(\Sigma)$. The representation of this preference induced by $\widehat{V}_{T-1}$ is simply I, while the representation induced by $\widetilde{V}_{T-1}$ is $J(\ ) = \ ^{-1}\big(\min_{p \in C} \int \ (\ )\,\mathrm{d}p\big)$. They are both ordinally equivalent, so there exists a strictly increasing function $\ : \mathbb{R} \to \mathbb{R}$ such that $I(\ ) = \ (J(\ ))$ for all $\ \in \mathcal{B}_0(\Sigma)$. In particular, this must hold for all constant utility acts: $I(r) = \ (J(r))$ for all $r \in \mathbb{R}$. However, I is normalized (by assumption), whereas J is normalized (by direct verification). Thus, for all $r \in \mathbb{R}$ it must be that $r = I(r) = \ (J(r)) = \ (r)$; hence, $\ $ is identity and $I \equiv J$.

Now, recall that $\widehat{V}_{T-1}$ and $\widetilde{V}_{T-1}$ are ordinally equivalent on $X \times \mathcal{F}$. For this reason, there exists a strictly increasing function $\ ' : \mathbb{R} \to \mathbb{R}$ such that

$$W\big(x_{T-1}; \min_{p \in C} \int \ (u(f))\,\mathrm{d}p\big) = \ '\big(u(x_{T-1}) + \ I(u(f))\big)$$

for all $x_{T-1} \in X$ and all $f \in \mathcal{F}$. Because $I \equiv J$ it follows that $W\big(x_{T-1}; \ (I(u(f)))\big) = \ '\big(u(x_{T-1}) + \ I(u(f))\big)$ for all $x_{T-1} \in X$ and all $f \in \mathcal{F}$. This means that $W\big(x; \ (\ )\big) = \ '\big(u(x) + \ \ \big)$ for all $x \in X$ and all $\ \in \mathbb{R}$. Hence, $W\big(x; d\big) = \ '\big(u(x) + \ \ ^{-1}(d)\big)$ for all $x \in X$ and all $d \in \mathrm{Range}(\ )$.

Finally, fix $x_0; x_1; : : : ; x_{T-2} \in X$ and $x_T \in X$ such that $u(x_T) = 0$; let h range

over $(x_0; x_1; \dots; x_{T-2}; \check{f}_{T-1}; x_T)$ for $f \in \mathcal{F}$. Observe that

$$\widehat{V}_{T-2}(!;h) = u(x_{T-2}) + \quad I\left(\widehat{V}_{T-1}\big((!^{T-1};\cdot);h\big)\right) = u(x_{T-2}) + \quad I\big(u(f)\big):$$

Thus, $\widehat{V}_{T-2}$ induces a quasiconcave order on $B_0(\Sigma)$ represented by $\quad \mapsto \min_{p \in C} \int \ (\ ) \, \mathrm{d}p$. On the other hand,

$$\begin{aligned} \widetilde{V}_{T-2} &= W\left(x_{T-2}; I^{MEU}\left(\widetilde{V}_{T-1}\big((!^{T-1};\cdot);h\big)\right)\right) \\ &= {}'\left(u(x_{T-2}) + \quad {}^{-1}\left(I^{MEU}\big(\widetilde{V}_{T-1}\big((!^{T-1};\cdot);h\big)\big)\right)\right) \\ &= {}'\left(u(x_{T-2}) + \quad {}^{-1}\left(I^{MEU}\big({}'(u(f))\big)\right)\right) \end{aligned}$$

Thus, $\widetilde{V}_{T-2}$ induces an order on $B_0(\Sigma)$ represented by $\quad \mapsto \min_{p \in C} \int {}'(\ ) \, \mathrm{d}p$.

Because $\widehat{V}_{T-2}$ and $\widetilde{V}_{T-2}$ are ordinally equivalent, the mappings $\quad \mapsto \min_{p \in C} \int \ (\ ) \, \mathrm{d}p$ and $\quad \mapsto \min_{p \in C} \int {}'(\ ) \, \mathrm{d}p$ represent the same quasiconcave order on $\mathcal{F}$. Because there exists an essential event $E \in \Sigma$ by letting $\quad := \min_{p \in C} p(E)$ it follows that $0 < \quad < 1$. Both mappings induce mappings on the set $\{(\overline{k}; \underline{k}) \in \mathbb{R}^2 \mid \overline{k} \geq \underline{k}\}$ by considering utility acts of the form $\overline{k} E \underline{k}$. These induced mappings are $(\overline{k}; \underline{k}) \mapsto \quad (\overline{k}) + (1 - \quad) \quad (\underline{k})$ and $(\overline{k}; \underline{k}) \mapsto \quad {}'(\overline{k}) + (1 - \quad) \, {}'(\underline{k})$; observe that they inherit quasiconcavity. Standard arguments from expected utility theory imply that $'$ is an positive affine transformation of $\quad$. Because I^{MEU} preserves positive affine transformations, $'$ can be chosen to equal $\quad$.

To show concavity of $\quad$, for any $k \in \mathbb{R}$ consider the restriction of the mapping $(\overline{k}; \underline{k}) \mapsto \quad (\overline{k}) + (1 - \quad) \quad (\underline{k})$ to the set $(-\infty; k) \times (k; \infty)$. By Theorem 1 of Debreu and Koopmans (1982), the function $\quad$ is continuous on $(k; \infty)$; hence it is continuous on $\mathbb{R}$ in light of the arbitrary choice of k. By the Proposition of Yaari (1977), $\quad$ is a concave function on $(k; \infty)$, hence it is concave on $\mathbb{R}$ in light of the arbitrary choice of k. $\square$

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