Preference Symmetries and Utility Representations

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Abstract

A symmetry of a preference relation is a mapping from the domain of choice to itself that preserves preference comparisons; a continuous symmetry is a one-parameter family of such transformations that includes the identity; and a symmetry field is a vector field whose flow curves trace out a continuous symmetry. Any continuous symmetry or symmetry field of a preference relation imposes structure on its utility representations in the form of a system of PDEs. The solution of this system is the functional form for utility equivalent to the symmetry. This framework is seen to encompass standard results relating to quasilinear and homothetic preferences, and is used to obtain general characterizations of univariate, multivariate, and joint separability, including the special cases of Cobb-Douglas and CES utility.

1 Introduction

Every economist learns, and many will have attempted to prove (see [10, p. 96]), that a continuous preference relation is quasilinear with respect to a particular commodity if and only if it admits a utility representation that is additive in the same good. Similarly, a continuous preference relation is homothetic if and only if it admits a representation that is homogenous of degree one. These results link structural features of utility functions to symmetry properties of the underlying preferences: Specifically, the additive functional form is seen to embody *translational* symmetry of the preference relation in the relevant variable, while the homogeneous form amounts to an assumption of *dilational* symmetry centered at the origin.

The aim of this paper is to investigate the connection between preference symmetries and utility representations in an abstract and systematic manner. To facilitate this we shall strengthen the background assumption of continuity to that of "smoothness" defined by Debreu [2], where smooth preferences are shown to admit a representation of class C^2. We proceed then to formalize three general notions of preference symmetry: A "discrete symmetry" is a mapping from the domain of choice (e.g., commodity space) to itself that preserves preference comparisons, a "continuous symmetry" is a one-parameter family of

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such transformations that includes the identity, and a "symmetry field" is a vector field over the domain whose flow curves trace out a continuous symmetry.[1] For example, any preference relation over $\Re^2_{++}$ that is quasilinear with respect to x_1 possesses the discrete symmetry $\tau(x) = \langle x_1 + 1/2, x_2\rangle$, the continuous symmetry $\sigma(x, \alpha) = \langle x_1 + \alpha, x_2\rangle$, and the symmetry field $S(x) = \langle 1, 0\rangle$. (Indeed, here $\sigma(\cdot, 0)$ is the identity mapping and $\sigma(x, \alpha)$ is the flow curve of S starting from x, with α functioning as a duration parameter.)

Section 2 contains our main theoretical results. The first, Theorem 2.7, shows that any nontrivial continuous symmetry of a smooth preference relation imposes structure on its C^2 representations in the form of a system of partial differential equations.[2] Writing u for a representation of this sort, the family of translations σ above will lead to the PDE

$$\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)} = 0, \tag{1}$$

which states simply that the marginal rate of substitution between the two commodities is independent of x_1.[3] The general solution of Equation 1 is $u(x) = f(x_1 + h(x_2))$, where f and h are arbitrary functions. And the particular representation $v(x) = x_1 + h(x_2)$ then possesses the desired additive structure.

For the case of homothetic preferences over $\Re^2_{++}$, consider the continuous symmetry $\bar{\sigma}(x, \alpha) = e^\alpha x$ and associated symmetry field $T(x) = x$. This family of dilations will lead to the PDE

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right] x_1 + \left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right] x_2 = 0, \tag{2}$$

with general solution $u(x) = f(x_1 h(x_2/x_1))$ for arbitrary f and h. And the particular representation $\bar{v}(x) = x_1 h(x_2/x_1)$ is then homogenous of degree one, as desired.

Our second main result, Theorem 2.10, strengthens the first by adding a converse implication. It states that a given vector field is a symmetry field of a smooth preference relation *if and only if* its C^2 representations satisfy a system of PDEs that replicates the earlier system for the continuous symmetry tracing out the flow curves of the field. For example, Equations 1–2 are equivalent, respectively, to $S(x) = \langle 1, 0\rangle$ and $T(x) = x$ constituting symmetry fields of the preferences represented by u. And the flow curves of the latter fields are of course the original continuous symmetries $\sigma(x, \alpha) = \langle x_1 + \alpha, x_2\rangle$ and $\bar{\sigma}(x, \alpha) = e^\alpha x$.[4]

The structure of our theory is shown schematically in Figure 1. Representation results link functional forms for utility to preference axioms, which here will be expressible as

[1] Referring to the second of these notions as a "differentiable symmetry" would be more precise, since differentiability of the family of mappings with respect to its parameter will be central to our theory. But the term "continuous symmetry" is well established in, e.g., modern expositions of Noether's [11] famous results connecting the concept to physical conservation laws, and so we conform to this usage.

[2] In K-dimensional space this system of PDEs will contain $K - 1$ independent equations; one for each marginal rate of substitution. Hence when $K = 2$ the "system" will consist of a single equation.

[3] Notation: We write $u^k(x)$ for $\partial u(x)/\partial x_k$.

[4] In light of these two examples it may not be clear why the notion of a symmetry field is needed, and why the two-way result cannot be phrased more simply in terms of continuous symmetries. The reason is that not every such symmetry is associated with a field, and two different continuous symmetries can yield the same system of PDEs (see Example 2.11). Hence it is the correspondence between symmetry fields and PDEs that is exact, with continuous symmetries comprising a larger class of properties.

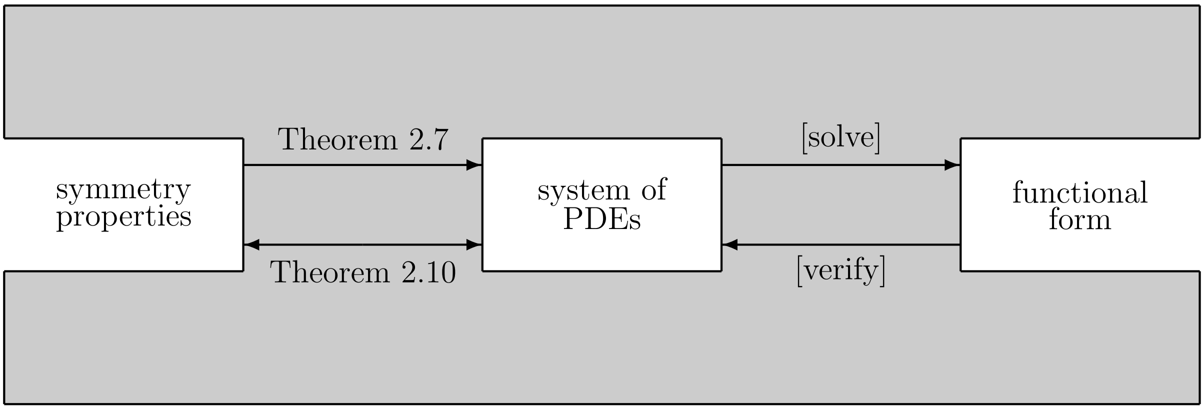

Figure 1: A schematic picture of our theory. Our main results derive a system of partial differential equations that helps to link symmetry properties of preferences to functional forms for utility. Theorem 2.7 exhibits PDEs that are necessary for a given "continuous symmetry," while Theorem 2.10 provides equations that are both necessary and sufficient for a given "symmetry field." After applying these results it remains to solve the system as well as to verify that all representations with the desired form are solutions.

symmetry properties. The PDEs we shall derive (in Equations 5 and 13) can be thought of as an island between the two sides of such a characterization, with our main results functioning as bridges that connect this island to the symmetry properties of interest. In particular, Theorem 2.7 exhibits PDEs that are necessary for a given continuous symmetry, while Theorem 2.10 provides equations that are both necessary and sufficient for a given symmetry field.

The figure also illustrates what our theory does *not* do. To complete the link between a set of symmetry properties and a particular functional form we must still show that the representations with this form are precisely those that solve the relevant system of PDEs. Verifying that representations with the desired structure are solutions is usually straightforward, but solving the equations to identify the functional form (when it is not known a priori) and to ensure that no other solutions exist need not be so simple. These tasks must be carried out case by case for each representation result we wish to construct.

Nevertheless, while we offer no systematic method of solving the differential equations at the heart of our theory, doing so is often not as difficult as it might first appear. Our results will yield a system of second-order PDEs that must be integrated twice in order to obtain solutions. The first integration will be made easier by the fact that the equations will involve derivatives of marginal rates of substitution; so that, for example, Equation 1 can be integrated (by inspection) to obtain

$$\frac{u^1(x)}{u^2(x)} = \eta(x_2) \tag{3}$$

for an arbitrary function η. Moreover, to accomplish the second integration we need only recall that one utility function is a monotone transformation of (and hence represents the same preferences as) another if and only if the marginal rates of substitution that they

generate coincide. Defining $h(x_2) = \int_1^{x_2} \eta(t)^{-1}\mathrm{d}t$ and $v(x) = x_1 + h(x_2)$, we then have

$$\frac{v^1(x)}{v^2(x)} = \frac{1}{h'(x_2)} = \frac{1}{\eta(x_2)^{-1}} = \eta(x_2) = \frac{u^1(x)}{u^2(x)} \tag{4}$$

in the quasilinear case. And thus $u(x) = f \circ v(x) = f(x_1 + h(x_2))$ for some monotone f, as desired. We shall employ this strategy repeatedly to construct concrete representation results from the systems of PDEs associated with various preference symmetries.

Several applications of our theory are provided in Section 3. We study first univariate separable representations of the form $v(x) = g(x_1) + h(x_2, \ldots, x_K)$, where the function g is pre-specified and h is unknown.[5] We then consider the multivariate separable form $v(x) = \sum_k \lambda_k g_k(x_k)$, where each g_k function is pre-specified, including the special cases of Cobb-Douglas and constant-elasticity-of-substitution (CES) utility.[6] Thirdly, we show how to treat variables that are jointly separable in the manner of x_1 and x_2 under the representation $v(x) = g(x_3) + h(x_1, x_2)$; an example that is readily generalized to more complicated structures. And finally we examine homogeneity and related functional forms (in a two-dimensional setting), demonstrating in particular that preferences admit both homogeneous and additively separable representations if and only if they admit a CES representation. In each application we supply an exact characterization of the designated utility functions in terms of the relevant symmetry fields. And it is then a simple matter to identify the corresponding continuous symmetries and formulate the characterization in a more traditional way.

The results stated in Section 2 are proved in situ, while those in Section 3 are proved in the Appendix.

2 Theory

2.1 Smooth preferences and representations

Fix $K \geq 2$ and let $X \subset \Re^K$ be open and path-connected. Let $\succsim$ be a preference relation on X represented by a utility function $u : X \to \Re$ (in the sense that $\forall x, y \in X$ we have $u(x) \geq u(y)$ if and only if $x \succsim y$). We partition $\succsim$ as usual into its asymmetric part $\succ$ indicating strict preference and its symmetric part $\sim$ indicating indifference.

We shall assume that u is of class C^2 and moreover that $\mathrm{G}u \gg 0$.[7] These assumptions can be transferred to $\succsim$ using the work of Debreu [2], who has shown that a preference relation admits a utility function with the desired properties if and only if it is strictly monotone and "smooth." Of course the content of this result resides largely in Debreu's definition of preferential smoothness, but we need not concern ourselves here with this

[5]This form yields additive utility when $g(x_1) = x_1$, but the variable x_1 can also enter in other ways such as when $g(x_1) = x_1^p$ or $g(x_1) = e^{px_1}$.

[6]In these cases $g_k(x_k) = \log x_k$ and $g_k(x_k) = x_k^p$, respectively. Maccheroni et al. [8, p. 1472] aver that they are "not aware of any behaviorally significant axiom that characterizes Cobb-Douglas preferences," which are merely "an analytically convenient specification of homothetic preferences." Our Corollary 3.4 will offer just such a characterization.

[7]Notation: We write G for the gradient operator; e.g., $\mathrm{G}u(x) = \langle u^k(x)\rangle_{k=1}^K$.

aspect of his contribution.[8] For our purposes the result is important because it obtains the desired features of u independently of any structural properties, and also because the implication is two-way. We can therefore consider the issue of differentiability to have been conclusively settled by Debreu, and can focus our attention on the incremental assumptions on $\succsim$ associated with particular functional forms for utility.

In addition to the function u taken as given throughout our analysis, many other C^2 functions v will represent the preference relation $\succsim$. These alternate representations can be described in various ways, four of which are recorded in the following familiar result.

Proposition 2.1. *Let $v : X \to \Re$ be C^2. Then the following statements are equivalent:*

- **(i)** *The function v represents $\succsim$.*
- **(ii)** *There exists a C^1 function $\rho : X \to \Re_{++}$ such that, $\forall x \in X$, $\mathrm{G}v(x) = \rho(x)\mathrm{G}u(x)$.*
- **(iii)** *For each $1 \leq k < K$ and $\forall x \in X$, $u^k(x)/u^K(x) = v^k(x)/v^K(x)$.*
- **(iv)** *There exists a C^1 function $f : v[X] \to u[X]$ such that $f' > 0$ and $u = f \circ v$.*

Proof (Sketch). The equivalence of (i) and (ii) follows from Debreu [2, pp. 606–611], who shows that the preferences represented by a C^2 function v with no critical points are characterized by the normalized gradient map $x \mapsto \mathrm{G}v(x)/\left\|\mathrm{G}v(x)\right\|$.[9] The equivalence of (ii) and (iii) is immediate, as is the fact that (iv) implies (iii). To see that (iv) follows from the other statements, note first that since X is open and path-connected the sets $u[X], v[X] \subset \Re$ are open intervals. Selecting for each $\xi \in v[X]$ an arbitrary point $x_\xi \in X$ such that $v(x_\xi) = \xi$, we can define $f(\xi) = u(x_\xi)$ unambiguously in consequence of (i). The resulting f will be an increasing homeomorphism from $v[X]$ to $u[X]$ with $u = f \circ v$.[10] And it is then straightforward to show that the map $\xi \mapsto u^1(x_\xi)/v^1(x_\xi) > 0$ is continuous and constitutes the derivative of f. □

2.2 Discrete symmetries

A preference symmetry is a mapping from the domain of choice to itself that preserves preference comparisons. This concept is formalized in the following definition, which will serve as the starting point for our analysis.

Definition 2.2. A C^2 function $\tau : X \to X$ is a *discrete symmetry* of $\succsim$ if $\forall x, y \in X$ we have $x \succsim y \Longleftrightarrow \tau(x) \succsim \tau(y)$.

Example 2.3. Let $X = \Re^2$ and $u(x) = 2[x_1 + x_2] + \sin[x_1 - x_2]$. Then the transformations $\tau(x) = \langle x_1 + \pi, x_2 + \pi \rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi \rangle$ are both discrete symmetries of $\succsim$. Note that in the first case we have $u \circ \tau(x) = u(x) + 4\pi$ and hence $\tau(x) \succ x$, while in the second case we have $u \circ \bar{\tau}(x) = u(x)$ and hence $\bar{\tau}(x) \sim x$.

[8] Roughly speaking, preferences over X are smooth in the sense of Debreu if they are continuous and the indifference relation is a differentiable manifold when viewed as a subset of $\Re^{2K}$. (For further details, see [2, p. 610].)

[9] See also Debreu [3] and Mas-Colell [9, p. 1389].

[10] The function f is increasing and one-to-one by (i), is continuous since its codomain $u[X]$ is an interval, and (as an injective, continuous mapping from an open interval to $\Re$) is therefore a homeomorphism.

While Definition 2.2 expresses the idea of a discrete symmetry most directly, a different characterization of these transformations will at times prove more convenient.

Proposition 2.4. *A C^2 function $\tau : X \to X$ is a discrete symmetry of $\succsim$ if and only if there exists a $\rho : X \to \Re_{++}$ such that $\forall x \in X$ we have $\mathrm{G}[u \circ \tau](x) = \rho(x)\mathrm{G}u(x)$.*

Proof. By Proposition 2.1 a suitable ρ exists if and only if $u \circ \tau$ represents $\succsim$, which is to say that $\forall x, y \in X$ we have $u \circ \tau(x) \geq u \circ \tau(y) \iff x \succsim y$. But since u also represents $\succsim$ this is equivalent to $\tau(x) \succsim \tau(y) \iff x \succsim y$, the condition for τ to be a discrete symmetry of $\succsim$. □

In other words, the discrete symmetries of $\succsim$ are those and only those transformations τ for which the normalized gradients of u and $u \circ \tau$ are identical; for the reason that either of these functions represents the preference relation equally well.

Example 2.5. Let $X = \Re^2$ and $u(x) = \|x\|$. Then the transformations $\tau(x) = \langle 2x_2, 2x_1 \rangle$ and $\bar{\tau}(x) = \left\langle [x_1^2 + 1]^{1/2}, x_2 \right\rangle$ are both discrete symmetries of $\succsim$. Indeed, here $\mathrm{G}[u \circ \tau](x) = [2]\mathrm{G}u(x)$ and $\mathrm{G}[u \circ \bar{\tau}](x) = \left[\|x\| \, [x_1^2 + x_2^2 + 1]^{-1/2} \right] \mathrm{G}u(x)$.

2.3 Continuous symmetries

In Example 2.3, the mappings $\tau(x) = \langle x_1 + \pi, x_2 + \pi \rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi \rangle$ are both discrete symmetries of the relation $\succsim$ represented by $u(x) = 2[x_1 + x_2] + \sin[x_1 - x_2]$. There is, however, an important difference between these two symmetries. Suppose that we define a family of mappings $\sigma : X \times [0, 1) \to X$ by $\sigma(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 + 2\pi\alpha \rangle$. Since for each $\alpha \in [0, 1)$ we have that $\sigma(\cdot, \alpha)$ is a discrete symmetry of $\succsim$, the transformation τ (realized by $\alpha = 1/2$) and the identity mapping (realized by $\alpha = 0$) together belong to a one-parameter class of such symmetries. Contrastingly, if we define a new family $\bar{\sigma}(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 - 2\pi\alpha \rangle$ so as to include the transformation $\bar{\tau}$, then it is *not* the case that each $\bar{\sigma}(\cdot, \alpha)$ is a discrete symmetry of $\succsim$. (For example, $\bar{\sigma}(x, 1/4)$ is not a discrete symmetry.)

This notion of a one-parameter family of discrete symmetries that includes the identity mapping can be formalized as follows.

Definition 2.6. A C^2 function $\sigma : X \times [0, 1) \to X$ is a *continuous symmetry* of $\succsim$ if for each $\alpha \in [0, 1)$ the function $\sigma(\cdot, \alpha)$ is a discrete symmetry of $\succsim$ and moreover $\sigma(\cdot, 0)$ is the identity mapping.

Our illustrative example — with discrete symmetries τ and $\bar{\tau}$ and continuous symmetry σ — is depicted in Figure 2.

The nontrivial continuous symmetries of a preference relation impose structure on its utility representations. This fact is established by our first main result, which exhibits a system of partial differential equations in u implied by a given continuous symmetry.[11]

[11] Notation: We write σ_i^α for $\partial\sigma_i/\partial\alpha$, $\sigma_i^{k\alpha}$ for $\partial\sigma_i^\alpha/\partial x_k$, and σ^α for $\langle \sigma_i^\alpha \rangle_{i=1}^K$. A continuous symmetry σ is then deemed "trivial" if $\forall x \in X$ and $1 \leq i \leq K$ we have $\sigma_i^\alpha(x, 0) = 0$.

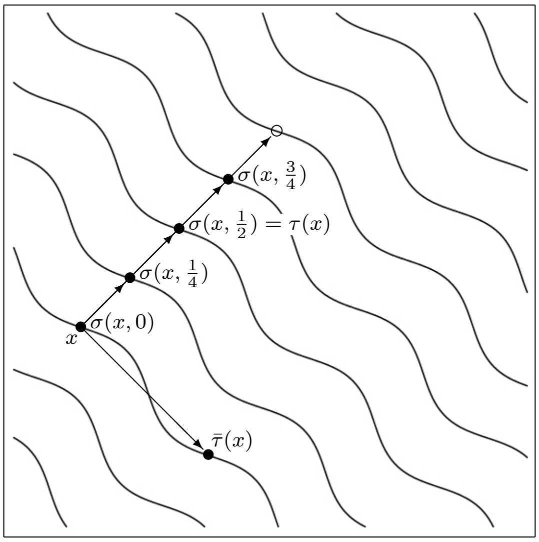

Figure 2: Discrete and continuous preference symmetries. The indifference curves shown are generated by preferences $\succsim$ with representation $u(x) = 2[x_1 + x_2] + \sin[x_1 - x_2]$. Both $\tau(x) = \langle x_1 + \pi, x_2 + \pi\rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi\rangle$ are discrete symmetries of $\succsim$; that is, transformations of the domain that preserve preference comparisons. Moreover, τ is a member of a one-parameter family of such transformations that includes the identity mapping — the continuous symmetry $\sigma(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 + 2\pi\alpha\rangle$ of $\succsim$.

Theorem 2.7. *If σ is a continuous symmetry of $\succsim$, then for each $1 \leq j < k \leq K$ and $\forall x \in X$ we have*

$$\sum_{i=1}^{K} \left[\frac{\partial}{\partial x_i} \frac{u^j(x)}{u^k(x)}\right] \sigma_i^{\alpha}(x, 0) = \sum_{i=1}^{K} \frac{u^i(x)}{u^k(x)} \left[\frac{u^j(x)}{u^k(x)} \sigma_i^{k\alpha}(x, 0) - \sigma_i^{j\alpha}(x, 0)\right]. \tag{5}$$

The proof leverages the fact that, as we vary α locally in the vicinity of $\alpha = 0$, the marginal rates of substitution (or, equivalently, the normalized gradient) of u must transform so as to maintain the preference symmetry. In particular, when the Jacobian $\mathrm{J}\sigma^{\alpha}(\cdot, 0)(x)$ of $\sigma^{\alpha}(x, 0)$ with respect to x is everywhere the $K \times K$ matrix of zeroes — a case we may refer to as that of a "uniform" σ — the RHS of Equation 5 vanishes and we obtain

$$\sum_{i=1}^{K} \left[\frac{\partial}{\partial x_i} \frac{u^j(x)}{u^k(x)}\right] \sigma_i^{\alpha}(x, 0) = 0. \tag{6}$$

This requires simply that as we raise α (infinitesimally) above 0 and thereby force $\sigma(\cdot, \alpha)$ away from the identity mapping, no change can be induced in any of the marginal rates of substitution.

Proof of Theorem 2.7. Let σ be a continuous symmetry of $\succsim$. For each $\alpha \in [0, 1)$ we then have that $\sigma(\cdot, \alpha)$ is a discrete symmetry of $\succsim$, so by Proposition 2.4 there exists a function $\rho(\cdot, \alpha) : X \to \Re_{++}$ such that $\forall x \in X$ we have $\mathrm{G}[u \circ \sigma(\cdot, \alpha)](x) = \rho(x, \alpha)\mathrm{G}u(x)$. Using the chain rule we can write this as $[\mathrm{J}\sigma(\cdot, \alpha)(x)]^{\top}[\mathrm{G}u \circ \sigma(x, \alpha)] = \rho(x, \alpha)\mathrm{G}u(x)$.[12]

[12]Notation: We write $[\cdot]^{\top}$ to indicate matrix transposition.

Differentiating with respect to α and writing $\mathrm{H}u(x)$ for the Hessian of u at x then gives the relation

$$[\mathrm{J}\sigma(\cdot,\alpha)(x)]^\top[\mathrm{H}u\circ\sigma(x,\alpha)]\sigma^\alpha(x,\alpha)+[\mathrm{J}\sigma^\alpha(\cdot,\alpha)(x)]^\top[\mathrm{G}u\circ\sigma(x,\alpha)]=\rho^\alpha(x,\alpha)\mathrm{G}u(x), \quad (7)$$

whereupon setting $\alpha = 0$ yields

$$\mathrm{H}u(x)\sigma^\alpha(x,0)+[\mathrm{J}\sigma^\alpha(\cdot,0)(x)]^\top\mathrm{G}u(x)=\rho^\alpha(x,0)\mathrm{G}u(x).^{13} \quad (8)$$

Using the j^{th} and k^{th} components of Equation 8 to eliminate $\rho^\alpha(x,0)$, we obtain

$$\frac{[\mathrm{G}u^j(x)]^\top\sigma^\alpha(x,0)+\sigma^{j\alpha}(x,0)^\top\mathrm{G}u(x)}{u^j(x)}=\frac{[\mathrm{G}u^k(x)]^\top\sigma^\alpha(x,0)+\sigma^{k\alpha}(x,0)^\top\mathrm{G}u(x)}{u^k(x)} \quad (9)$$

or, equivalently,

$$\sum_{i=1}^K[u^k(x)u^{ji}(x)-u^j(x)u^{ki}(x)]\sigma_i^\alpha(x,0)=\sum_{i=1}^K u^i(x)[u^j(x)\sigma_i^{k\alpha}(x,0)-u^k(x)\sigma_i^{j\alpha}(x,0)]. \quad (10)$$

And dividing both sides of Equation 10 by $[u^k(x)]^2$ then confirms Equation 5. □

Example 2.8. Let $X=\Re^2$. If the family of mappings σ defined by

$$\sigma(x,\alpha)=\langle x_1\cos\alpha-x_2\sin\alpha, x_1\sin\alpha+x_2\cos\alpha\rangle \quad (11)$$

is a continuous symmetry of $\precsim$, then u solves

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]x_2-\left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]x_1=\left[\frac{u^1(x)}{u^2(x)}\right]^2+1 \quad (12)$$

as a consequence of Theorem 2.7. (Here σ is a family of rotations around the origin.)

2.4 Symmetry fields

We have seen that if σ is a continuous symmetry of $\precsim$ then u must solve the system of PDEs in Equation 5. The converse could not possibly be true, since our derivation of this equation uses only local (i.e., $\alpha\to 0$) information about $\sigma(\cdot,\alpha)$ and so cannot guarantee a discrete symmetry for given $\alpha>0$. If, however, we take this local information as our starting point, constructing both the continuous symmetry and the associated system of PDEs from the "symmetry field" recording the direction and speed each point is to be transformed, then Theorem 2.7 can be made into a two-way result.

Given a vector field $S:X\to\Re^K$, let us write $\zeta[\![S]\!](x,\cdot)$ for the flow curve of S from a fixed starting point $x\in X$. This curve is defined by the initial conditions $\zeta[\![S]\!](x,0)=x$ together with the ordinary differential equations $\zeta[\![S]\!]^\alpha(x,\alpha)=S\circ\zeta[\![S]\!](x,\alpha)$. Note that we have also $\zeta[\![S]\!]^\alpha(x,0)=S\circ\zeta[\![S]\!](x,0)=S(x)$, which is to say that the family of mappings $\zeta[\![S]\!]:X\times[0,1)\to X$ transforms points locally according to the field S.

[13] Here we have used the fact that $\sigma(x,0)=x$ and so $\mathrm{J}\sigma(\cdot,0)(x)$ is the $K\times K$ identity matrix.

Definition 2.9. A vector field $S : X \to \Re^K$ is a *symmetry field* of $\succsim$ if the function $\zeta[\![S]\!]$ is a continuous symmetry of $\succsim$.

Our enhanced version of Theorem 2.7 can now be stated as follows.

Theorem 2.10. *A vector field $S : X \to \Re^K$ is a symmetry field of $\succsim$ if and only if for each $1 \leq j < k \leq K$ and $\forall x \in X$ we have*

$$\sum_{i=1}^{K} \left[\frac{\partial}{\partial x_i} \frac{u^j(x)}{u^k(x)} \right] S_i(x) = \sum_{i=1}^{K} \frac{u^i(x)}{u^k(x)} \left[\frac{u^j(x)}{u^k(x)} S_i^k(x) - S_i^j(x) \right]. \tag{13}$$

To show the sufficiency of Equation 13, we first unwind the proof of the earlier result as far as Equation 8 — whose counterpart below is Equation 17 — and then use this link between the field $S(\cdot) = \zeta[\![S]\!]^\alpha(\cdot, 0)$ and the representation u to show that S must be a symmetry field of $\succsim$, as asserted.

Proof of Theorem 2.10. The necessity of Equation 13 (for S to be a symmetry field of $\succsim$) follows immediately from Theorem 2.7. Conversely, if Equation 13 holds then

$$\sum_{i=1}^{K} [u^k(x)u^{ji}(x) - u^j(x)u^{ki}(x)] S_i(x) = \sum_{i=1}^{K} u^i(x)[u^j(x)S_i^k(x) - u^k(x)S_i^j(x)] \tag{14}$$

or, equivalently,

$$\frac{[\mathrm{G}u^j(x)]^\top S(x) + S^j(x)^\top \mathrm{G}u(x)}{u^j(x)} = \frac{[\mathrm{G}u^k(x)]^\top S(x) + S^k(x)^\top \mathrm{G}u(x)}{u^k(x)}. \tag{15}$$

And defining

$$\phi(x) = \frac{[\mathrm{G}u^K(x)]^\top S(x) + S^K(x)^\top \mathrm{G}u(x)}{u^K(x)}, \tag{16}$$

we can express Equation 15 (which holds for all $1 \leq j < k \leq K$) as the vector equality

$$\mathrm{H}u(x)S(x) + [\mathrm{J}S(x)]^\top \mathrm{G}u(x) = \phi(x)\mathrm{G}u(x). \tag{17}$$

Since Equation 17 holds everywhere on X, for given $y \in X$ and $\alpha \in [0, 1)$ it holds in particular at the point $\zeta[\![S]\!](y, \alpha)$, and we have

$$\begin{aligned} &[\mathrm{H}u \circ \zeta[\![S]\!](y, \alpha)][S \circ \zeta[\![S]\!](y, \alpha)] + [\mathrm{J}S \circ \zeta[\![S]\!](y, \alpha)]^\top [\mathrm{G}u \circ \zeta[\![S]\!](y, \alpha)] \cdots \\ &\qquad \cdots = [\phi \circ \zeta[\![S]\!](y, \alpha)][\mathrm{G}u \circ \zeta[\![S]\!](y, \alpha)]. \end{aligned} \tag{18}$$

Pre-multiplying by $[\mathrm{J}\zeta[\![S]\!](\cdot, \alpha)(y)]^\top$ and applying the chain rule, we can write this as

$$\begin{aligned} &[\mathrm{J}\zeta[\![S]\!](\cdot, \alpha)(y)]^\top [\mathrm{H}u \circ \zeta[\![S]\!](y, \alpha)][S \circ \zeta[\![S]\!](y, \alpha)] + \cdots \\ &\quad \cdots [\mathrm{J}[S \circ \zeta[\![S]\!](\cdot, \alpha)](y)]^\top [\mathrm{G}u \circ \zeta[\![S]\!](y, \alpha)] = [\phi \circ \zeta[\![S]\!](y, \alpha)]\mathrm{G}[u \circ \zeta[\![S]\!](\cdot, \alpha)](y). \end{aligned} \tag{19}$$

Substituting $\zeta[\![S]\!]^\alpha(y,\alpha)$ for $S \circ \zeta[\![S]\!](y,\alpha)$, the LHS of Equation 19 can be expressed as

$$\begin{aligned}&[\mathrm{J}\zeta[\![S]\!](\cdot,\alpha)(y)]^\top[\mathrm{H}u \circ \zeta[\![S]\!](y,\alpha)]\zeta[\![S]\!]^\alpha(y,\alpha) + [\mathrm{J}\zeta[\![S]\!]^\alpha(\cdot,\alpha)(y)]^\top[\mathrm{G}u \circ \zeta[\![S]\!](y,\alpha)]\cdots\\ &\cdots = \tfrac{\partial}{\partial\alpha}\left[[\mathrm{J}\zeta[\![S]\!](\cdot,\alpha)(y)]^\top[\mathrm{G}u \circ \zeta[\![S]\!](y,\alpha)]\right] = \tfrac{\partial}{\partial\alpha}\left[\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y)\right], \quad (20)\end{aligned}$$

and Equation 19 itself is then equivalent to

$$\tfrac{\partial}{\partial\alpha}\left[\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y)\right] = [\phi \circ \zeta[\![S]\!](y,\alpha)]\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y). \tag{21}$$

Now define a scalar-valued function $\rho : X \times [0,1) \to \Re$ for fixed $z \in X$ by $\rho(z,0) = 1$ and $\rho^\alpha(z,\alpha) = [\phi \circ \zeta[\![S]\!](z,\alpha)]\rho(z,\alpha)$, noting that then

$$\rho(z,\alpha) = \exp \int_0^\alpha [\phi \circ \zeta[\![S]\!](z,t)]\mathrm{d}t > 0. \tag{22}$$

Substituting $\rho^\alpha(y,\alpha)/\rho(y,\alpha)$ for $\phi \circ \zeta[\![S]\!](y,\alpha)$ in Equation 21, we obtain

$$\frac{\partial \mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y)}{\partial\alpha} = \frac{\rho^\alpha(y,\alpha)}{\rho(y,\alpha)}\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y), \tag{23}$$

or

$$\frac{\partial}{\partial\alpha}\left[\frac{\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y)}{\rho(y,\alpha)}\right] = 0. \tag{24}$$

Integrating and using $\rho(y,0) = 1$ then yields $\mathrm{G}[u \circ \zeta[\![S]\!](\cdot,\alpha)](y) = \rho(y,\alpha)\mathrm{G}u(y)$, so by Proposition 2.4 we have that each $\zeta[\![S]\!](\cdot,\alpha)$ is a discrete symmetry of $\succsim$. And finally, since $\zeta[\![S]\!](\cdot,0)$ is the identity map we can conclude that $\zeta[\![S]\!]$ is a continuous symmetry and S a symmetry field of $\succsim$. □

Example 2.11. Let $X = \Re^2$ and $S(x) = \langle x_1 - x_2, x_1 + x_2\rangle$. Then the family of mappings $\zeta[\![S]\!]$ defined by

$$\zeta[\![S]\!](x,\alpha) = e^\alpha \langle x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha\rangle \tag{25}$$

satisfies both $\zeta[\![S]\!](x,0) = x$ and $\zeta[\![S]\!]^\alpha(x,\alpha) = S \circ \zeta[\![S]\!](x,\alpha)$, and thus $\zeta[\![S]\!](x,\cdot)$ is the flow curve of S from the point x. Note that the family σ defined by

$$\sigma(x,\alpha) = [1+\alpha] \langle x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha\rangle \tag{26}$$

also satisfies both $\sigma(x,0) = x$ and $\sigma^\alpha(x,0) = S \circ \sigma(x,0) = S(x)$, but for $\alpha > 0$ does *not* satisfy $\sigma^\alpha(x,\alpha) = S \circ \sigma(x,\alpha)$. This illustrates how two distinct families of transformations can lead to the same S and thus impose the same restrictions on u via Equation 13, but at most one of the two will be consistent with the flow patterns of the field.

When a continuous symmetry traces out the flow curves of a field, the two are equivalent ways of expressing the behavior of $\succsim$. Continuous symmetries are the more general concept, since every field leads to such a symmetry (by definition) but not vice versa. However, symmetry fields provide a more concise description of the properties in question, and moreover possess a natural algebraic structure.

Proposition 2.12. *The set of symmetry fields of $\succsim$ comprise a convex cone in the space $[\Re^K]^X$ of vector fields over X.*

This can be shown by composing the continuous symmetries $\zeta[\![S]\!]$ and $\zeta[\![T]\!]$ associated with fields S and T, and then applying our two main results in succession.

Proof of Proposition 2.12. Given $S, T : X \to \Re^K$ and $a_1, a_2 \in \Re_+$, suppose that S and T are both symmetry fields of $\succsim$. We shall show that the vector field $a_1 S(\cdot) + a_2 T(\cdot)$ is also a symmetry field of $\succsim$.

Since S and T are symmetry fields, $\zeta[\![S]\!]$ and $\zeta[\![T]\!]$ are continuous symmetries of $\succsim$. Consider the function $\sigma : X \times [0,1) \to X$ defined by $\sigma(x,\alpha) = \zeta[\![T]\!](\zeta[\![S]\!](x, a_1\alpha), a_2\alpha)$. Note first that

$$\sigma(x,0) = \zeta[\![T]\!](\zeta[\![S]\!](x,0),0) = \zeta[\![T]\!](x,0) = x. \tag{27}$$

Given $x, y \in X$ and $\alpha \in [0,1)$ we have also

$$\begin{aligned}\sigma(x,\alpha) \succsim \sigma(y,\alpha) &\iff \zeta[\![T]\!](\zeta[\![S]\!](x,a_1\alpha),a_2\alpha) \succsim \zeta[\![T]\!](\zeta[\![S]\!](y,a_1\alpha),a_2\alpha)\\ &\iff \zeta[\![S]\!](x,a_1\alpha) \succsim \zeta[\![S]\!](y,a_1\alpha)\\ &\iff x \succsim y,\end{aligned} \tag{28}$$

using the definition of σ and the fact that $\zeta[\![S]\!](\cdot, a_1\alpha)$ and $\zeta[\![T]\!](\cdot, a_2\alpha)$ are both discrete symmetries of $\succsim$. It follows that σ is a continuous symmetry of $\succsim$, so Equation 5 holds by Theorem 2.7. Moreover, we can compute

$$\begin{aligned}\sigma^\alpha(x,0) &= \tfrac{\partial}{\partial\alpha}\, \zeta[\![T]\!](\zeta[\![S]\!](x,a_1\alpha),a_2\alpha)\big|_{\alpha=0}\\ &= a_1 \mathrm{J}[\zeta[\![T]\!](\cdot,0)](\zeta[\![S]\!](x,0))\zeta[\![S]\!]^\alpha(x,0) + a_2\zeta[\![T]\!]^\alpha(\zeta[\![S]\!](x,0),0)\\ &= a_1 \mathrm{J}[\zeta[\![T]\!](\cdot,0)](x)\zeta[\![S]\!]^\alpha(x,0) + a_2\zeta[\![T]\!]^\alpha(x,0)\\ &= a_1 S(x) + a_2 T(x),\end{aligned} \tag{29}$$

since $\zeta[\![S]\!](x,0) = \zeta[\![T]\!](x,0) = x$, $\zeta[\![S]\!]^\alpha(x,0) = S(x)$, and $\zeta[\![T]\!]^\alpha(x,0) = T(x)$. But then by Theorem 2.10 the vector field $a_1 S(\cdot) + a_2 T(\cdot)$ is a symmetry field of $\succsim$. □

3 Applications

3.1 Univariate separability

As a first application of the theory outlined in Section 2, we now characterize all utility functions that are additively separable in one variable (taken to be x_1).

Proposition 3.1. *Let $X = \Re^K_{++}$. Given a C^2 function $g : \Re_{++} \to \Re$ with g'*

Observe the structure of this result: The function g controlling how x_1 affects utility is taken as given, and the relevant symmetry field is expressed in terms of this function. In contrast, the function h of the variables $x_{\neg 1}$ and the transformation f remain unknown — their existence merely being asserted by the proposition.

An immediate consequence of this characterization of univariate separability in general is a version of the standard result for additive utility in particular.

Corollary 3.2. *Let $X = \Re^K_{++}$. Then the vector field $S(x) = \vec{\imath}_1$ is a symmetry field of $\succsim$ if and only if there exist C^1 functions $h : \Re^{K-1}_{++} \to \Re$ with $\mathrm{G}h \gg 0$ and $f : v[X] \to \Re$ with $f' > 0$, where v is defined by $v(x) = x_1 + h(x_{\neg 1})$, such that $u = f \circ v$.*

Given $p > 0$, Proposition 3.1 also links the functional form $v(x) = x_1^p + h(x_{\neg 1})$ to the symmetry field $S(x) = [p^{-1} x_1^{1-p}]\vec{\imath}_1$, the form $v(x) = e^{px_1} + h(x_{\neg 1})$ to $S(x) = [p^{-1} e^{-px_1}]\vec{\imath}_1$, and so on. Moreover, by integrating these fields to obtain the corresponding flow curves, we can express our results in terms of continuous symmetries. For example, the assertion in Corollary 3.2 that $S(x) = \vec{\imath}_1$ is a symmetry field of $\succsim$ amounts to the quasilinearity assumption that the family of transformations $\sigma(x, \alpha) = \langle x_1 + \alpha, x_{\neg 1}\rangle$ is a continuous symmetry of $\succsim$, and likewise for other specifications of g.

To sketch the argument for Proposition 3.1, specialize Equation 13 to the symmetry field in (i) as

$$\frac{\partial}{\partial x_1} \frac{u^1(x)}{u^K(x)} = \frac{u^1(x)}{u^K(x)} \frac{g''(x_1)}{g'(x_1)} \tag{30}$$

and, for each $2 \leq j < K$,

$$\frac{\partial}{\partial x_1} \frac{u^j(x)}{u^K(x)} = 0.\text{[16]} \tag{31}$$

Integrating Equations 30–31 with respect to x_1 leads to expressions for the marginal rates of substitution generated by u, and it can be shown that for a well-chosen h these rates are shared by $v(x) = g(x_1) + h(x_{\neg 1})$. Applying Proposition 2.1, we then have that there exists an f such that $u = f \circ v$, as desired. And for the converse implication we need only check that Equation 13 holds when $S(x) = [g'(x_1)^{-1}]\vec{\imath}_1$ and u has the form in (ii).

Full proofs of Proposition 3.1 and all other results in Section 3 are provided in the Appendix.

3.2 Multivariate separability

We proceed now to characterize functional forms for utility exhibiting additive separability in all variables simultaneously.

Proposition 3.3. *Let $X = \Re^K_{++}$. Given C^2 functions $g_k : \Re_{++} \to \Re$ for $1 \leq k \leq K$ with each $g_k' > 0$, the following statements are equivalent:*

- **(i)** *For each $1 \leq k \leq K$ the vector field $S[k](x) = [g_k'(x_k)^{-1}]\vec{\imath}_k$ is a symmetry field of $\succsim$.*
- **(ii)** *There exist $\lambda \in \Re^K_{++}$ and a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where $v : X \to \Re$ is defined by $v(x) = \sum_k \lambda_k g_k(x_k)$, such that $u = f \circ v$.*

[16] Cf. Samuelson [12, pp. 176–177], who for $K = 2$ obtains a version of Equation 30.

Here each function g_k is given, with only the coefficients λ_k and the transformation f remaining unknown.[17]

Consequences of the above multivariate separability result include characterizations of Cobb-Douglas and (other) CES utility functions.

Corollary 3.4. *Let $X = \Re^K_{++}$. Then for each $1 \leq k \leq K$ the vector field $S[k](x) = [x_k]\vec{\imath}_k$ is a symmetry field of $\succsim$ if and only if there exist $\lambda \in \Re^K_{++}$ and a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where v is defined by $v(x) = \sum_k \lambda_k \log x_k$, such that $u = f \circ v$.*

Corollary 3.5. *Let $X = \Re^K_{++}$ and fix $p > 0$. Then for each $1 \leq k \leq K$ the vector field $S[k](x) = [p^{-1}x_k^{1-p}]\vec{\imath}_k$ is a symmetry field of $\succsim$ if and only if there exist $\lambda \in \Re^K_{++}$ and a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where v is defined by $v(x) = \sum_k \lambda_k x_k^p$, such that $u = f \circ v$.*

And setting $p = 1$ in Corollary 3.5 then links the linear specification $v(x) = \lambda^\top x$ to the collection of symmetry fields $S[k](x) = \vec{\imath}_k$ for each $1 \leq k \leq K$.

Recalling Proposition 2.12 ("the set of symmetry fields of $\succsim$ comprise a convex cone"), this last result implies that preferences over $\Re^K_{++}$ representable by a linear utility function possess, for each $a \in \Re^K_+$, the symmetry field $\sum_k a_k S[k](x) = \sum_k a_k \vec{\imath}_k = a$. Equivalently, each family of transformations $\sigma(x, \alpha) = x + \alpha a$ is a continuous symmetry of $\succsim$, and this is precisely the class of all (increasing) continuous translations.[18]

3.3 Joint separability

Now let $K = 3$ for simplicity and consider again Proposition 3.1. Given a function g, this result tells us that $T(x) = [g'(x_3)^{-1}]\vec{\imath}_3$ being a symmetry field of $\succsim$ is both necessary and sufficient for the existence of functions h and f such that $u(x) = f(g(x_3) + h(x_{\neg 3}))$. But suppose that we also wish h to have a particular functional form, rather than remaining unspecified. What additional restrictions does this place on the preference relation? Our next result answers this question.

Proposition 3.6. *Let $X = \Re^3_{++}$. Given a pair of C^2 functions $g : \Re_{++} \to \Re$ with $g' > 0$ and $h : \Re^2_{++} \to \Re$ with $\mathrm{G}h \gg 0$, the following statements are equivalent:*

- **(i)** *The vector fields $S(x) = \langle h^1(x_{\neg 3})^{-1}, -h^2(x_{\neg 3})^{-1}, 0\rangle$ and $T(x) = [g'(x_3)^{-1}]\vec{\imath}_3$ are both symmetry fields of $\succsim$.*
- **(ii)** *There exists a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where $v : X \to \Re$ is defined by $v(x) = g(x_3) + h(x_{\neg 3})$, such that $u = f \circ v$.*

[17] In contrast, the multivariate separability results of Debreu [1, pp. 20–25], Fishburn [4, pp. 346–349], and Leontief [7] involve unknown g_k functions.

[18] Alternatively, linear utility on $\Re^K_{++}$ can be characterized via the class of symmetry fields $S(x) = x - b$ for $b \in \Re^K_{++}$, with corresponding continuous symmetries $\sigma(x, \alpha) = e^\alpha x + [1 - e^\alpha]b$ (i.e., the continuous dilations centered at each b). The latter characterization in effect restates the expected utility theorem, with the required symmetry condition recognizable as the traditional mixture-independence axiom. And related "certainty-independence" axioms (of the sort introduced by Gilboa and Schmeidler [6] and also used, e.g., by Ghirardato et al. [5] and Maccheroni et al. [8]) can similarly be thought of as continuous symmetries.

As an example of how this characterization can be used, imagine that we have adopted the utility function $v(x) = x_1x_2 + x_3$ and would like to identify the implicit assumptions on $\succsim$. Here $g(x_3) = x_3$ and $h(x_{\neg 3}) = x_1x_2$, so from (i) we have that $S(x) = \langle x_2^{-1}, -x_1^{-1}, 0\rangle$ and $T(x) = \vec{\imath}_3$ are both symmetry fields of $\succsim$. Using Proposition 2.12, for each $a \in \Re^2_+$ we then have that $a_1S(x) + a_2T(x) = \langle a_1/x_2, -a_1/x_1, a_2\rangle$ is a symmetry field of $\succsim$. And these fields are easily integrated to obtain their flow curves, the continuous symmetries $\sigma(x, \alpha) = \langle x_1e^{[\alpha a_1]/[x_1x_2]}, x_2e^{-[\alpha a_1]/[x_1x_2]}, x_3 + \alpha a_2\rangle$. Thus we can conclude that the above utility specification is equivalent to the assumption that for each $\bar{a} \in \Re^2_+$ we have $x \succsim y$ if and only if $\langle x_1e^{\bar{a}_1/[x_1x_2]}, x_2e^{-\bar{a}_1/[x_1x_2]}, x_3 + \bar{a}_2\rangle \succsim \langle y_1e^{\bar{a}_1/[y_1y_2]}, y_2e^{-\bar{a}_1/[y_1y_2]}, y_3 + \bar{a}_2\rangle$.[19]

Proposition 3.6 provides a good context in which to observe the intuition behind the symmetry fields associated with additively separable utility functions. Given (ii), we can compute the local change in the structured utility representation v induced by movement through the field S as

$$\mathrm{G}v(x)^\top S(x) = \langle h^1(x_{\neg 3}), h^2(x_{\neg 3}), g'(x_3)\rangle^\top \langle h^1(x_{\neg 3})^{-1}, -h^2(x_{\neg 3})^{-1}, 0\rangle = 0, \tag{32}$$

and similarly

$$\mathrm{G}v(x)^\top T(x) = \langle h^1(x_{\neg 3}), h^2(x_{\neg 3}), g'(x_3)\rangle^\top \langle 0, 0, g'(x_3)^{-1}\rangle = 1. \tag{33}$$

Joint separability of x_1 and x_2 imposes a preference symmetry that holds utility constant by moving along the level curves of the function h governing these variables' contribution. At the same time, univariate separability of x_3 imposes a symmetry that increases utility at a rate of unity for the representation v (and otherwise at a rate $f' \circ v(x)$ determined by the distortion of v effected by f).[20]

3.4 Homogeneity and related functional forms

As our last set of applications we develop results related to homogeneity of degree one, limiting attention to the case of $K = 2$.

The basic homogeneity characterization appears as follows.[21]

Proposition 3.7. *Let $X = \Re^2_{++}$. The following statements are equivalent:*

- **(i)** *The vector field $S(x) = x$ is a symmetry field of $\succsim$.*
- **(ii)** *There exist C^1 functions $v : X \to \Re$ and $f : v[X] \to \Re$, where v is homogeneous of degree one and $f' > 0$, such that $u = f \circ v$.*

Here $\mathrm{G}v(x)^\top S(x) = \mathrm{G}v(x)^\top x = v(x)$ by Euler's (homogeneous function) theorem, so the local *relative* change in the structured representation v induced by movement through the field S equals unity. And the continuous symmetry associated with this field is of course $\sigma(x, \alpha) = e^\alpha x$.

As usual, proving necessity of the symmetry field S for a homogeneous representation amounts to verifying Equation 13. And sufficiency can be shown as a consequence of our next, more general characterization.

[19] Note that here α has been subsumed into $\bar{a}$, which suffices to parameterize the symmetries of $\succsim$.
[20] Cf. Figure 2, where the translation $\bar{\tau}$ holds utility constant while τ increases utility.
[21] Proposition 3.7 holds as stated for arbitrary K, though we prove here only the two-dimensional case.

Proposition 3.8. *Let $X = \Re^2_{++}$. Given C^2 functions $g_k : \Re_{++} \to \Re$ for $k = 1, 2$ with each $g_k' > 0$, the following statements are equivalent:*

(i) *The vector field $S(x) = \langle g_1'(x_1)^{-1}, g_2'(x_2)^{-1}\rangle$ is a symmetry field of $\succsim$.*

(ii) *There exist an $h : \Re \to \Re_{++}$ and a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where $v : X \to \Re$ is defined by $v(x) = g_1(x_1) + \int_1^{g_2(x_2)-g_1(x_1)} \frac{\mathrm{d}t}{h(t)+1}$, such that $u = f \circ v$.*

This result encompasses preferences generating marginal rates of substitution of the form

$$\frac{u^1(x)}{u^2(x)} = \frac{g_1'(x_1)}{g_2'(x_2)} h(g_2(x_2) - g_1(x_1)), \tag{34}$$

with the homothetic case arising when $g_k(x_k) = \log x_k$. Alternatively, when $g_k(x_k) = b_k x_k$ we have that the marginal rate of substitution depends only on $b_2 x_2 - b_1 x_1$, and likewise for other specifications of the g_k functions.[22]

Our final characterization concerns preferences that admit both homogeneous and additively separable representations, as in (iii) below. We establish that these are precisely the preferences that admit CES utility functions, as in (ii), and describe their symmetry fields in (i).

Proposition 3.9. *Let $X = \Re^2_{++}$. The following statements are equivalent:*

(i) *For some C^2 functions $s_1, s_2 : \Re_{++} \to \Re$ with each $s_k > 0$ and $x_k s_k'(x_k) \le s_k(x_k)$ for all $x_k \in \Re_{++}$, the vector fields $S[1](x) = \langle s_1(x_1), 0\rangle$, $S[2](x) = \langle 0, s_2(x_2)\rangle$, and $T(x) = x$ are all symmetry fields of $\succsim$.*

(ii) *There exist $p \ge 0$, $a_1, a_2 > 0$, and a C^1 function $f : v[X] \to \Re$ with $f' > 0$, where $v : X \to \Re$ is defined by*

$$v(x) = \left\{ \begin{array}{ll} a_1 \log x_1 + a_2 \log x_2 & \textit{for } p = 0 \\ a_1 x_1^p + a_2 x_2^p & \textit{for } p > 0 \end{array} \right\}, \tag{35}$$

such that $u = f \circ v$.

(iii) *There exist both*

[**a**] *C^1 functions $w : X \to \Re$ and $\theta : w[X] \to \Re$, where w is homogeneous of degree one and $\theta' > 0$, such that $u = \theta \circ w$; and*

[**b**] *C^2 functions $g_1, g_2 : \Re_{++} \to \Re$ with each $g_k' > 0$ and $-x_k g_k''(x_k) \le g_k'(x_k)$ for all $x_k \in \Re_{++}$, and a C^1 function $\bar{\theta} : \bar{w}[X] \to \Re$ with $\bar{\theta}' > 0$, where $\bar{w} : X \to \Re$ is defined by $\bar{w}(x) = g_1(x_1) + g_2(x_2)$, such that $u = \bar{\theta} \circ \bar{w}$.*

Here, in contrast to Proposition 3.3, the g_k functions are not taken as given, though their form is deduced in (ii). The requirements $x_k s_k'(x_k) \le s_k(x_k)$ in (i) and $-x_k g_k''(x_k) \le g_k'(x_k)$ in (iii) correspond to $p \ge 0$ in (ii), which is needed to ensure that $\mathrm{G}u \gg 0$. And note in conclusion that the equivalence of (ii) and (iii) provides an example of how our methods may yield results that can be stated without reference to the symmetry notions on which the present theory is based.

[22] Observe that Proposition 3.8 relates to the class of symmetry fields $aS(x) = \langle a g_1'(x_1)^{-1}, a g_2'(x_2)^{-1}\rangle$ for $a \ge 0$, whereas Proposition 3.3 relates to the class $a_1 S[1](x) + a_2 S[2](x) = \langle a_1 g_1'(x_1)^{-1}, a_2 g_2'(x_2)^{-1}\rangle$ for $a_1, a_2 \ge 0$, which is more general.

A Appendix

Here we provide proofs of the results stated in Section 3.

Proof of Proposition 3.1. If (i) holds, then $\forall x \in X$ we have both

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^K(x)}\right]\frac{1}{g'(x_1)} = \frac{u^1(x)}{u^K(x)}\frac{g''(x_1)}{g'(x_1)^2} \tag{36}$$

and, for each $2 \leq j < K$,

$$\left[\frac{\partial}{\partial x_1}\frac{u^j(x)}{u^K(x)}\right]\frac{1}{g'(x_1)} = 0 \tag{37}$$

by Theorem 2.10. Simplifying and integrating each equation with respect to x_1, we obtain

$$\frac{u^1(x)}{u^K(x)} = g'(x_1)\eta_1(x_{\neg 1}) \tag{38}$$

and, for each $2 \leq j < K$,

$$\frac{u^j(x)}{u^K(x)} = \eta_j(x_{\neg 1}), \tag{39}$$

where $\log \eta_1(x_{\neg 1})$ and $\eta_j(x_{\neg 1})$ are integration constants. Now let $\eta_K(x_{\neg 1}) = 1$ everywhere and define a vector field $H : \Re^{K-1}_{++} \to \Re^{K-1}_{++}$ by $H(x_{\neg 1}) = \langle \eta_k(x_{\neg 1})/\eta_1(x_{\neg 1})\rangle_{k=2}^K$. For each $2 \leq j < k \leq K$ and $\forall x \in X$ we then have

$$\begin{aligned} H_j^k(x_{\neg 1}) &= \frac{\partial}{\partial x_k}\frac{g'(x_1)u^j(x)}{u^1(x)} = \frac{g'(x_1)[u^1(x)u^{jk}(x) - u^j(x)u^{1k}(x)]}{u^1(x)^2}\cdots \\ \cdots &= \frac{g'(x_1)[u^1(x)u^{jk}(x) - u^k(x)u^{1j}(x)]}{u^1(x)^2} = \frac{\partial}{\partial x_j}\frac{g'(x_1)u^k(x)}{u^1(x)} = H_k^j(x_{\neg 1}), \end{aligned} \tag{40}$$

where the third equality in the chain follows from the fact that

$$\frac{\partial}{\partial x_1}\frac{u^j(x)}{u^k(x)} = \frac{\partial}{\partial x_1}\frac{\eta_j(x_{\neg 1})}{\eta_k(x_{\neg 1})} = 0 \tag{41}$$

and hence $u^j(x)u^{1k}(x) = u^k(x)u^{1j}(x)$. Equation 40 establishes that the vector field H is conservative and hence admits a C^1 potential function $h : \Re^{K-1}_{++} \to \Re$ with $\mathrm{G}h = H \gg 0$. We then have both

$$\frac{v^1(x)}{v^K(x)} = \frac{g'(x_1)}{H_K(x_{\neg 1})} = g'(x_1)\eta_1(x_{\neg 1}) = \frac{u^1(x)}{u^K(x)} \tag{42}$$

and, for each $2 \leq j < K$,

$$\frac{v^j(x)}{v^K(x)} = \frac{H_j(x_{\neg 1})}{H_K(x_{\neg 1})} = \eta_j(x_{\neg 1}) = \frac{u^j(x)}{u^K(x)}. \tag{43}$$

And by Proposition 2.1 we can conclude that there exists a C^1 function $f : v[X] \to \Re$ such that $f' > 0$ and $u = f \circ v$, so that (ii) holds.

Conversely, if (ii) holds then for each $1 \leq j < k \leq K$ and $\forall x \in X$ we have

$$\begin{aligned}
\sum_{i=1}^{K} \left[\frac{\partial}{\partial x_i} \frac{u^j(x)}{u^k(x)} \right] S_i(x) &= \left[\frac{\partial}{\partial x_1} \frac{u^j(x)}{u^k(x)} \right] S_1(x) \\
&= \left\{ \begin{array}{ll} g''(x_1)[g'(x_1)h^k(x_{\neg 1})]^{-1} & \text{for } j = 1 \\ 0 & \text{for } j \neq 1 \end{array} \right\} \\
&= \frac{u^1(x)}{u^k(x)} \left[-S_1^j(x) \right] \\
&= \sum_{i=1}^{K} \frac{u^i(x)}{u^k(x)} \left[\frac{u^j(x)}{u^k(x)} S_i^k(x) - S_i^j(x) \right]. \qquad (44)
\end{aligned}$$

Hence S is a symmetry field of $\succsim$ by Theorem 2.10, and (i) holds. □

Proof of Proposition 3.3. If (i) holds, then for each $1 \leq j < K$ and $\forall x \in X$ we have

$$\left[\frac{\partial}{\partial x_j} \frac{u^j(x)}{u^K(x)} \right] \frac{1}{g_j'(x_j)} = \frac{u^j(x)}{u^K(x)} \frac{g_j''(x_j)}{g_j'(x_j)^2}, \qquad (45)$$

$$\left[\frac{\partial}{\partial x_K} \frac{u^j(x)}{u^K(x)} \right] \frac{1}{g_K'(x_K)} = -\frac{u^j(x)}{u^K(x)} \frac{g_K''(x_K)}{g_K'(x_K)^2}, \qquad (46)$$

and, for each $k \neq j, K$,

$$\left[\frac{\partial}{\partial x_k} \frac{u^j(x)}{u^K(x)} \right] \frac{1}{g_k'(x_k)} = 0 \qquad (47)$$

by Theorem 2.10. Simplifying and integrating each equation with respect to the relevant variable now yields

$$\frac{u^j(x)}{u^K(x)} = \left\{ \begin{array}{ll} g_j'(x_j)\eta_j(x_{\neg j}) & \\ [g_K'(x_K)]^{-1}\eta_K(x_{\neg K}) & \\ \eta_k(x_{\neg k}) & \text{for } k \neq j, K \end{array} \right\} \qquad (48)$$

where $\log \eta_j(x_{\neg j})$, $\log \eta_K(x_{\neg K})$, and $\eta_k(x_{\neg k})$ are integration constants. But then for each $1 \leq j < K$ there exists a $\lambda_j > 0$ such that $\forall x \in X$ we have

$$\frac{g_K'(x_K)}{g_j'(x_j)} \frac{u^j(x)}{u^K(x)} = \lambda_j. \qquad (49)$$

Letting $\lambda_K = 1 > 0$, for each $1 \leq j < K$ we then have

$$\frac{v^j(x)}{v^K(x)} = \frac{\lambda_j g_j'(x_j)}{\lambda_K g_K'(x_K)} = \frac{u^j(x)}{u^K(x)}. \qquad (50)$$

And by Proposition 2.1 we can conclude that there exists a C^1 function $f : v[X] \to \Re$ such that $f' > 0$ and $u = f \circ v$, so that (ii) holds.

Conversely, if (ii) holds then for each $1 \leq k \leq K$ and $1 \leq j < l \leq K$ and $\forall x \in X$ we have

$$\begin{aligned}
\sum_{i=1}^{K} \left[\frac{\partial}{\partial x_i} \frac{u^j(x)}{u^l(x)}\right] S[k]_i(x) &= \left[\frac{\partial}{\partial x_k} \frac{u^j(x)}{u^l(x)}\right] S[k]_k(x) \\
&= \left\{ \begin{array}{ll} \lambda_k g_k''(x_k)[\lambda_l g_l'(x_l) g_k'(x_k)]^{-1} & \text{for } k = j \\ -\lambda_j g_j'(x_j) g_k''(x_k)[\lambda_k g_k'(x_k)^3]^{-1} & \text{for } k = l \\ 0 & \text{for } k \neq j, l \end{array} \right\} \\
&= \frac{u^k(x)}{u^l(x)} \left[\frac{u^j(x)}{u^l(x)} S[k]^l_k(x) - S[k]^j_k(x)\right] \\
&= \sum_{i=1}^{K} \frac{u^i(x)}{u^l(x)} \left[\frac{u^j(x)}{u^l(x)} S[k]^l_i(x) - S[k]^j_i(x)\right]. \qquad (51)
\end{aligned}$$

Hence each $S[k]$ is a symmetry field of $\succsim$ by Theorem 2.10, and (i) holds. □

Proof of Proposition 3.6. If (i) holds, then T is a symmetry field of $\succsim$ and so for each $j = 1, 2$ and $\forall x \in X$ we have

$$\frac{u^j(x)}{u^3(x)} = \frac{\eta_j(x_{\neg 3})}{g'(x_3)} \qquad (52)$$

by analogy with Equation 38. (Here each $\log \eta_j(x_{\neg 3})$ is an integration constant.) Hence

$$0 = \frac{\partial}{\partial x_3} \frac{\eta_1(x_{\neg 3})}{\eta_2(x_{\neg 3})} = \frac{\partial}{\partial x_3} \frac{u^1(x)}{u^2(x)} = \frac{u^2(x) u^{13}(x) - u^1(x) u^{23}(x)}{u^2(x)^2}, \qquad (53)$$

and so $u^2(x)u^{13}(x) = u^1(x)u^{23}(x)$.

Turning to the symmetry field S, for each $j = 1, 2$ and $\forall x \in X$ we have

$$\left[\frac{\partial}{\partial x_1} \frac{u^j(x)}{u^3(x)}\right] \frac{1}{h^1(x_{\neg 3})} - \left[\frac{\partial}{\partial x_2} \frac{u^j(x)}{u^3(x)}\right] \frac{1}{h^2(x_{\neg 3})} = \frac{u^1(x)}{u^3(x)} \frac{h^{1j}(x_{\neg 3})}{h^1(x_{\neg 3})^2} - \frac{u^2(x)}{u^3(x)} \frac{h^{2j}(x_{\neg 3})}{h^2(x_{\neg 3})^2} \qquad (54)$$

by Theorem 2.10. Equation 54 can be rewritten as the system

$$h^2(x_{\neg 3}) \frac{\partial}{\partial x_1} \left[\frac{u^1(x)}{u^3(x)} \frac{1}{h^1(x_{\neg 3})} - \frac{u^2(x)}{u^3(x)} \frac{1}{h^2(x_{\neg 3})}\right] = \left[\frac{\partial}{\partial x_2} \frac{u^1(x)}{u^3(x)}\right] - \left[\frac{\partial}{\partial x_1} \frac{u^2(x)}{u^3(x)}\right], \qquad (55)$$

$$h^1(x_{\neg 3}) \frac{\partial}{\partial x_2} \left[\frac{u^1(x)}{u^3(x)} \frac{1}{h^1(x_{\neg 3})} - \frac{u^2(x)}{u^3(x)} \frac{1}{h^2(x_{\neg 3})}\right] = \left[\frac{\partial}{\partial x_2} \frac{u^1(x)}{u^3(x)}\right] - \left[\frac{\partial}{\partial x_1} \frac{u^2(x)}{u^3(x)}\right], \qquad (56)$$

and since $u^2(x)u^{13}(x) = u^1(x)u^{23}(x)$ it is clear that the common RHS of Equations 55–56 is equal to 0. But then for some $\mu : \Re_{++} \to \Re$ we have

$$\frac{u^1(x)}{u^3(x)} \frac{1}{h^1(x_{\neg 3})} - \frac{u^2(x)}{u^3(x)} \frac{1}{h^2(x_{\neg 3})} = \mu(x_3), \qquad (57)$$

so that substituting from Equation 52 yields

$$\frac{\eta_1(x_{\neg 3})}{h^1(x_{\neg 3})} - \frac{\eta_2(x_{\neg 3})}{h^2(x_{\neg 3})} = g'(x_3)\mu(x_3). \qquad (58)$$

And it follows that there exists a constant η^* such that

$$\frac{\eta_1(x_{\neg 3})}{h^1(x_{\neg 3})} - \frac{\eta_2(x_{\neg 3})}{h^2(x_{\neg 3})} = \eta^*. \tag{59}$$

The remaining implication of Theorem 2.10 for the symmetry field S takes the form

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]\frac{1}{h^1(x_{\neg 3})} - \left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]\frac{1}{h^2(x_{\neg 3})}\cdots$$

$$\cdots = \frac{u^1(x)}{u^2(x)}\left[-\frac{u^1(x)}{u^2(x)}\frac{h^{12}(x_{\neg 3})}{h^1(x_{\neg 3})^2} + \frac{h^{11}(x_{\neg 3})}{h^1(x_{\neg 3})^2}\right] + \frac{u^1(x)}{u^2(x)}\frac{h^{22}(x_{\neg 3})}{h^2(x_{\neg 3})^2} - \frac{h^{12}(x_{\neg 3})}{h^2(x_{\neg 3})^2} \tag{60}$$

or, more simply,

$$\frac{\partial}{\partial x_1}\left[\frac{u^1(x)}{u^2(x)}\frac{1}{h^1(x_{\neg 3})} - \frac{1}{h^2(x_{\neg 3})}\right] = \left[\frac{u^1(x)}{u^2(x)}\right]^2 \frac{\partial}{\partial x_2}\left[\frac{1}{h^1(x_{\neg 3})} - \frac{u^2(x)}{u^1(x)}\frac{1}{h^2(x_{\neg 3})}\right]. \tag{61}$$

Since $u^1(x)/u^2(x) = \eta_1(x_{\neg 3})/\eta_2(x_{\neg 3})$ by Equation 52, we then have

$$\frac{\partial}{\partial x_1}\left[\frac{\eta_1(x_{\neg 3})}{\eta_2(x_{\neg 3})}\frac{1}{h^1(x_{\neg 3})} - \frac{1}{h^2(x_{\neg 3})}\right] = \left[\frac{\eta_1(x_{\neg 3})}{\eta_2(x_{\neg 3})}\right]^2 \frac{\partial}{\partial x_2}\left[\frac{1}{h^1(x_{\neg 3})} - \frac{\eta_2(x_{\neg 3})}{\eta_1(x_{\neg 3})}\frac{1}{h^2(x_{\neg 3})}\right], \tag{62}$$

whereupon substitution from Equation 59 yields

$$\frac{\partial}{\partial x_1}\left[\frac{\eta^*}{\eta_2(x_{\neg 3})}\right] = \left[\frac{\eta_1(x_{\neg 3})}{\eta_2(x_{\neg 3})}\right]^2 \frac{\partial}{\partial x_2}\left[\frac{\eta^*}{\eta_1(x_{\neg 3})}\right]. \tag{63}$$

But this implies that $\eta_1^2(x_{\neg 3}) = \eta_2^1(x_{\neg 3})$, so that the vector field $H : \Re^2_{++} \to \Re^2_{++}$ defined by $H(x_{\neg 3}) = \langle \eta_1(x_{\neg 3}), \eta_2(x_{\neg 3})\rangle$ is conservative and hence admits a C^1 potential function $h : \Re^2_{++} \to \Re$ with $\mathrm{G}h = H \gg 0$. For each $j = 1, 2$ we then have

$$\frac{v^j(x)}{v^3(x)} = \frac{h^j(x_{\neg 3})}{g'(x_3)} = \frac{\eta_j(x_{\neg 3})}{g'(x_3)} = \frac{u^j(x)}{u^3(x)}. \tag{64}$$

And by Proposition 2.1 we can conclude that there exists a C^1 function $f : v[X] \to \Re$ such that $f' > 0$ and $u = f \circ v$, so that (ii) holds.

Conversely, if (ii) holds then $\forall x \in X$ we have

$$\begin{aligned}
\sum_{i=1}^{K}\left[\frac{\partial}{\partial x_i}\frac{u^1(x)}{u^2(x)}\right] S_i(x) &= \left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right] S_1(x) + \left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right] S_2(x) \\
&= \frac{h^2(x_{\neg 3})h^{11}(x_{\neg 3}) - h^1(x_{\neg 3})h^{12}(x_{\neg 3})}{h^1(x_{\neg 3})h^2(x_{\neg 3})^2}\cdots \\
&\quad \cdots - \frac{h^2(x_{\neg 3})h^{12}(x_{\neg 3}) - h^1(x_{\neg 3})h^{22}(x_{\neg 3})}{h^2(x_{\neg 3})^3} \\
&= \frac{u^1(x)}{u^2(x)}\left[\frac{u^1(x)}{u^2(x)}S_1^2(x) - S_1^1(x)\right] + \frac{u^1(x)}{u^2(x)}S_2^2(x) - S_2^1(x) \\
&= \sum_{i=1}^{K}\frac{u^i(x)}{u^2(x)}\left[\frac{u^1(x)}{u^2(x)}S_i^2(x) - S_i^1(x)\right]
\end{aligned} \tag{65}$$

and, for each $j = 1, 2$,

$$\sum_{i=1}^{K}\left[\frac{\partial}{\partial x_i}\frac{u^j(x)}{u^3(x)}\right]S_i(x) = \left[\frac{\partial}{\partial x_1}\frac{u^j(x)}{u^3(x)}\right]S_1(x) + \left[\frac{\partial}{\partial x_2}\frac{u^j(x)}{u^3(x)}\right]S_2(x)$$
$$= \frac{1}{g'(x_3)}\left[\frac{h^{1j}(x_{\neg 3})}{h^1(x_{\neg 3})} - \frac{h^{2j}(x_{\neg 3})}{h^2(x_{\neg 3})}\right]$$
$$= \frac{u^1(x)}{u^3(x)}\left[-S_1^j(x)\right] + \frac{u^2(x)}{u^3(x)}\left[-S_2^j(x)\right]$$
$$= \sum_{i=1}^{K}\frac{u^i(x)}{u^3(x)}\left[\frac{u^j(x)}{u^3(x)}S_i^3(x) - S_i^j(x)\right]. \tag{66}$$

Hence S is a symmetry field of $\succsim$ by Theorem 2.10. Moreover, T is a symmetry field of $\succsim$ by Proposition 3.1, and so (i) holds. □

Proof of Proposition 3.7. If (i) holds, then by Proposition 3.8 there exist an $h : \Re \to \Re_{++}$ and a C^1 function $\bar{f} : w[X] \to \Re$ with $\bar{f}' > 0$, where $w : X \to \Re$ is defined by

$$w(x) = \log x_1 + \int_1^{\log[x_2/x_1]}\frac{\mathrm{d}t}{h(t)+1}, \tag{67}$$

such that $u = \bar{f} \circ w$. Letting $v = \exp w$ and $f = \bar{f} \circ \log(\cdot)$, we have that v is homogeneous of degree one, $f' > 0$, and $u = \bar{f} \circ w = f \circ \exp\log v = f \circ v$, so that (ii) holds.

Conversely, if (ii) holds then there exists a C^1 function $\mu : \Re_{++} \to \Re$ such that $\forall x \in X$ we have $v(x) = x_2\mu(x_1/x_2)$. Thus

$$\frac{u^1(x)}{u^2(x)} = \frac{[f' \circ v(x)]v^1(x)}{[f' \circ v(x)]v^2(x)} = \frac{\mu'(x_1/x_2)}{\mu(x_1/x_2) - [x_1/x_2]\mu'(x_1/x_2)}, \tag{68}$$

and since this ratio depends on x only via x_1/x_2 it follows that

$$\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)} = -\frac{x_1}{x_2}\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]. \tag{69}$$

We then have

$$\sum_{i=1}^{K}\left[\frac{\partial}{\partial x_i}\frac{u^1(x)}{u^2(x)}\right]S_i(x) = \left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]S_1(x) + \left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]S_2(x)$$
$$= \left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right][x_1 - x_1]$$
$$= \frac{u^1(x)}{u^2(x)}[1-1]$$
$$= \sum_{i=1}^{K}\frac{u^i(x)}{u^2(x)}\left[\frac{u^1(x)}{u^2(x)}S_i^2(x) - S_i^1(x)\right]. \tag{70}$$

Hence S is a symmetry field of $\succsim$ by Theorem 2.10, and (i) holds. □

Proof of Proposition 3.8. If (i) holds, then $\forall x \in X$ we have

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]\frac{1}{g_1'(x_1)}+\left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]\frac{1}{g_2'(x_2)}=\frac{u^1(x)}{u^2(x)}\left[\frac{g_1''(x_1)}{g_1'(x_1)^2}-\frac{g_2''(x_2)}{g_2'(x_2)^2}\right] \tag{71}$$

by Theorem 2.10. Writing $\mu(x)=\log[u^1(x)/u^2(x)]$ and expressing Equation 71 as

$$\frac{\mu^1(x)}{g_1'(x_1)}+\frac{\mu^2(x)}{g_2'(x_2)}=\frac{g_1''(x_1)}{g_1'(x_1)^2}-\frac{g_2''(x_2)}{g_2'(x_2)^2}, \tag{72}$$

a particular solution of this PDE is $\mu(x)=\log[g_1'(x_1)/g_2'(x_2)]$. Furthermore, the general solution of the homogenous equation

$$\frac{\bar{\mu}^1(x)}{g_1'(x_1)}+\frac{\bar{\mu}^2(x)}{g_2'(x_2)}=0 \tag{73}$$

is $\bar{\mu}(x)=\log h(g_2(x_2)-g_1(x_1))$, where $h:\Re\to\Re_{++}$ is an arbitrary function. Hence the general solution of Equation 72 is $\mu(x)=\log[g_1'(x_1)/g_2'(x_2)]+\log h(g_2(x_2)-g_1(x_1))$, and it follows that Equation 71 is equivalent to Equation 34 above. We then have

$$\frac{v^1(x)}{v^2(x)}=\frac{g_1'(x_1)}{g_2'(x_2)}h(g_2(x_2)-g_1(x_1))=\frac{u^1(x)}{u^2(x)}. \tag{74}$$

And by Proposition 2.1 we can conclude that there exists a C^1 function $f: v[X]\to\Re$ such that $f'>0$ and $u=f\circ v$, so that (ii) holds.

Conversely, if (ii) holds then $\forall x\in X$ we have

$$\begin{aligned}\sum_{i=1}^{K}\left[\frac{\partial}{\partial x_i}\frac{u^1(x)}{u^2(x)}\right]S_i(x)&=\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]S_1(x)+\left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]S_2(x)\\&=\frac{g_1'(x_1)}{g_2'(x_2)}\left[\frac{g_1''(x_1)}{g_1'(x_1)^2}-\frac{g_2''(x_2)}{g_2'(x_2)^2}\right]h(g_2(x_2)-g_1(x_1))\\&=\frac{u^1(x)}{u^2(x)}\left[S_2^2(x)-S_1^1(x)\right]\\&=\sum_{i=1}^{K}\frac{u^i(x)}{u^2(x)}\left[\frac{u^1(x)}{u^2(x)}S_i^2(x)-S_i^1(x)\right].\end{aligned} \tag{75}$$

Hence S is a symmetry field of $\succsim$ by Theorem 2.10, and (i) holds. □

Proof of Proposition 3.9. If (i) holds, then there exists a $\lambda>0$ such that

$$\frac{u^1(x)}{u^2(x)}=\lambda\frac{s_2(x_2)}{s_1(x_1)} \tag{76}$$

by analogy with Equation 49. Turning to the symmetry field T, $\forall x\in X$ we have

$$\left[\frac{\partial}{\partial x_1}\frac{u^1(x)}{u^2(x)}\right]x_1+\left[\frac{\partial}{\partial x_2}\frac{u^1(x)}{u^2(x)}\right]x_2=0 \tag{77}$$

by Theorem 2.10, and combining Equations 76–77 yields

$$-\lambda\frac{s_2(x_2)s_1'(x_1)}{s_1(x_1)^2}x_1+\lambda\frac{s_2'(x_2)}{s_1(x_1)}x_2=0. \tag{78}$$

There must then exist a constant p such that

$$\frac{x_1 s_1'(x_1)}{s_1(x_1)}=1-p=\frac{x_2 s_2'(x_2)}{s_2(x_2)}, \tag{79}$$

with $p\geq 0$ since $x_1 s_1'(x_1)\leq s_1(x_1)$. For $k=1,2$ it follows that $s_k(x_k)=\eta_k x_k^{1-p}$, where each $\eta_k>0$ is an integration constant, and from Equation 76 we then have

$$\frac{u^1(x)}{u^2(x)}=\frac{\lambda\eta_2}{\eta_1}\left[\frac{x_1}{x_2}\right]^{p-1}. \tag{80}$$

Letting $a_1=\lambda\eta_2/\eta_1>0$ and $a_2=1$, we find that

$$\frac{v^1(x)}{v^2(x)}=\left\{\begin{array}{ll}[a_1/a_2][x_1/x_2]^{-1} & \text{for } p=0\\ [a_1/a_2][x_1/x_2]^{p-1} & \text{for } p>0\end{array}\right\}=\frac{\lambda\eta_2}{\eta_1}\left[\frac{x_1}{x_2}\right]^{p-1}=\frac{u^1(x)}{u^2(x)}. \tag{81}$$

And by Proposition 2.1 we can conclude that there exists a C^1 function $f:v[X]\to\Re$ such that $f'>0$ and $u=f\circ v$, so that (ii) holds.

If (ii) holds, then we can let

$$w(x)=\left\{\begin{array}{ll}[x_1^{a_1}x_2^{a_2}]^{\frac{1}{a_1+a_2}} & \text{for } p=0\\ [a_1x_1^p+a_2x_2^p]^{1/p} & \text{for } p>0\end{array}\right\} \tag{82}$$

and

$$\theta(\xi)=\left\{\begin{array}{ll}f(\log\xi^{a_1+a_2}) & \text{for } p=0\\ f(\xi^p) & \text{for } p>0\end{array}\right\}, \tag{83}$$

whereupon w is homogeneous of degree one, $\theta'>0$, and $\forall x\in X$ we have

$$u(x)=f\circ v(x)=\left\{\begin{array}{ll}f(\log w(x)^{a_1+a_2}) & \text{for } p=0\\ f(w(x)^p) & \text{for } p>0\end{array}\right\}=\theta\circ w(x), \tag{84}$$

establishing [a]. Likewise, we can let $\bar{\theta}=f$ and each

$$g_k(x_k)=\left\{\begin{array}{ll}a_k\log x_k & \text{for } p=0\\ a_kx_k^p & \text{for } p>0\end{array}\right\}, \tag{85}$$

whereupon $\bar{\theta}'>0$, $g_k'>0$,

$$-x_kg_k''(x_k)=\left\{\begin{array}{ll}a_k/x_k & \text{for } p=0\\ [1-p]pa_kx_k^{p-1} & \text{for } p>0\end{array}\right\}=[1-p]g_k'(x_k)\leq g_k'(x_k), \tag{86}$$

and $\forall x \in X$ we have

$$\begin{aligned} u(x) &= f \circ v(x) \\ &= \left\{ \begin{array}{ll} f(a_1 \log x_1 + a_2 \log x_2) & \text{for } p = 0 \\ f(a_1 x_1^p + a_2 x_2^p) & \text{for } p > 0 \end{array} \right\} \\ &= \bar{\theta}(g_1(x_1) + g_2(x_2)) \\ &= \bar{\theta} \circ \bar{w}(x), \end{aligned} \tag{87}$$

establishing [b]. Hence (iii) holds.

If (iii) holds then letting each $s_k(x_k) = g_k'(x_k)^{-1} > 0$ we have

$$x_k s_k'(x_k) = \frac{-x_k g_k''(x_k)}{g_k'(x_k)^2} \leq \frac{1}{g_k'(x_k)} = s_k(x_k) \tag{88}$$

since $-x_k g_k''(x_k) \leq g_k'(x_k)$, and each $S[k]$ is a symmetry field of $\succsim$ by Proposition 3.3. Clearly T is also a symmetry field of $\succsim$ by Proposition 3.7, and hence (i) holds. $\square$

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