Contagion in financial networks : a threat index

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Abstract

Financial institutions have developed numerous cross-liabilities that disseminate the shocks through the banking system. Under the occurrence of default, the repayments of the inter-bank claims must be jointly determined, thereby generating externalities across institutions and possibly the propagation of defaults and bankruptcy. This paper introduces a measure of the threat that a bank poses to the system, building on a simple model for the joint determination of the repayments of inter-bank claims as proposed by Eisenberg and Noe. Such a measure, called threat index, may be helpful to determine how to inject cash into banks so as to increase debt reimbursement, or to assess the contributions of individual institutions to the risk in the system. Although the threat index and the default level of a bank both re ect some form of weakness and are a ected by the liabilities network, the two indicators di er. As a result, injecting cash into the banks with the largest default level may not be optimal.

Keywords : contagion, systemic risk, financial linkages
JEL G01, G21, G28

1 Introduction

An intricate web of claims and obligations ties together the balance sheets of a wide variety of financial institutions, banks, hedge funds, and various intermediaries. Some argue that these inter-bank claims have played a large role in the dissemination of the financial crisis of 2007-2008. As such, inter-bank claims are an important concern for both bankers and regulators and there is a general call for addressing their role in the risk of the system, the famous 'systemic' risk. Following the recommendation made by the G20, the new framework proposed by the Basel committee (Basel III) plans to identify some 'systemically important financial institutions' from which higher standards will be required. Indeed regulation has so far typically been defined at the unit level, determined

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by the balance sheet of the bank under consideration. A prominent example is the Value at Risk indicator (VaR), which is based on a statistical assessment of a bank's payoffs independently of what is happening to other banks. Various proposals have been made to modify this measure to account for the contribution of a bank to systemic risk. But this contribution to systemic risk can be understood in a variety of ways, each one potentially leading to different assessments and cost evaluations. Tarashev, Borio, and Tsatsaronis (2010) distinguish between the role of a bank in spreading the losses and amplifying the default of other banks from that in participating in the systemic events, defined as those in which a large fraction of simultaneous defaults arise, say due to correlated portfolios. CoVar for example, proposed by Adrian and Brunnermeier (2008), is a measure similar to VaR but conditional on systemic events and falls in the second category.

My purpose is to propose a measure of the cost that a defaulting bank imposes on the debt repayments of all other banks in a simplified but explicit model of inter-bank liability structure. The measure is derived from an explicit criterion, linked to the overall repayments within the banking system and the loss incurred by the non-banking sector in case of bankruptcies. Such a measure may be helpful to determine how to inject cash into banks so as to optimally increase the reimbursement of debts, or to assess the contributions of individual institutions to the risk in the system. In particular the analysis shows that injecting cash into the banks that appear the weakest ones, in the sense of the largest proportion of liabilities not reimbursed, may be sub-optimal.

The model is a simplified description of a banking system due to Eisenberg and Noe (2000). Banks have claims on each other and the result of the activities of a bank excluding these inter-banking relationships is summarized by a single number, called here the ***net worth***.[1] Net worth can be negative, due to losses in derivative assets for example. Mutual inter-bank liabilities introduce linkages in defaults when they occur. The capacity of a bank to repay its inter-bank liabilities depends not only of its realized net worth but also on the capacity of its debtors, calling for a joint determination of the repayments. A simple 'clearing' mechanism on the proportions of the liabilities repaid by banks -repayment ratios- solves this loopback. In 'normal' times, liabilities are fully repaid and all repayment ratios are equal to unity. In case a default occurs and possibly propagates, the clearing mechanism determines the repayment ratio levels in a (almost) unique way. These levels form a kind of equilibrium in which each bank in default reimburses as much as it can given a limited liability constraint and priority rules on creditors. The clearing default ratio (the complement to one of its repayment ratio) reflects the weakness of a bank. However it is not the most appropriate indicator of the threat inflicted by the bank on the overall repayments to creditors within or outside the banking system.

The threat index of a bank measures the decrease in payments following a decrease in its net worth. A decrease in the net worth of a defaulting bank has an impact not only on its own repayment but also on those of its creditors which are in difficulty, and by propagation on the payments along

[1] Eisenberg and Noe restrict their analysis to non-negative net worth, referred to as cash-flows.

a chain of creditors. The chain may cycle, triggering further decrease in payments and possibly the bankruptcy of some banks. This paper therefore proposes the threat index to measure this overall impact. Although the threat index and the default ratio of a bank are both measures of its weakness and are affected by the liabilities network, they differ in general, and are in some precise sense dual to each other. A bank's default ratio is determined by the ability of its debtors to repay their debts; a bank's threat index on the other hand is determined by the impact its default inflicts on its creditors, and hence by the financial health of its creditors. The discrepancy between the two indices, default ratio and threat index (partially) depends on the asset-liability structure, in particular its asymmetry. The threat index is useful to determine a 'targeting policy' that injects an amount of cash into banks so as to improve effective payments and within the payment system and decrease the outside losses as much as possible. Cash should be injected into the defaulting banks with the largest threat index. As a result, due to the discrepancy between threat indices and default levels, injecting cash into the banks that appear the weakest, those with the largest default ratio, may be sub-optimal.

The results distinguish the situations in which defaults are limited to the inter-bank liabilities from those where some banks default on their obligations to the outside sector and go bankrupt. The first situation surely arises when the net worth values are all non-negative, as considered by Eisenberg and Noe (2000). Since each bank is a net creditor to the outside sector, default can only be due to the inter-bank liabilities, and the two requirements of limited liability and banks creditors' absolute priority characterize the clearing ratios. When some banks have a net liability towards the non-banking sector, a negative net worth, bankruptcy may be unavoidable. Bankruptcy arises if a bank which fully defaults on its obligations towards the financial sector nevertheless ends up with a debt to the non–banking sector. I extend the clearing mechanism so as to cope with bankruptcy by specifying a hierarchy between the creditors. with a priority to the creditors outside the financial sectors. Specifically a clearing mechanism is characterized by limited liability, banks creditors' priority over stockholders, and a bankruptcy condition that requires banks liabilities not to be repaid unless outside creditors are fully repaid. The determinants of the threat indices are shown to differ substantially between the two situations with or without bankruptcy.

The literature on financial contagion is growing. Most often, the status of a bank is binary, meaning that either a bank fulfills its obligations or totally fails, where failure is typically determined by a solvency constraint. In such a setup, the contagion risk of a bank is often defined as the expected number of failures (possibly weighted by their size) triggered by the single initial failure of this bank.

Empirical studies have examined the potential for contagion following a single bank's distress in real banking systems. The simulations are calibrated on real payment systems (see e.g. Furfine 2003 on Fedwire) or on inter-bank networks (e.g. Upper and Worms 2004, Elsinger et al. 2004, Degryse and Nguyen 2004 for Germany, Austria, Belgium respectively). These studies concluded that systemic risk was extremely limited, in the sense that the probability of a large number of failures

triggered by the single initial failure of a bank was almost null. A difficulty however is that data on bilateral exposures is limited, and the technics to 'fill' the missing data possibly lowers contagion.[2] Furthermore this makes difficult to assess the impact of the liability structure.

The main question on the liability structure is the trade-off between insurance and risk spreading generated by cross-liabilities. Indeed, inter-bank liabilities have two opposing effects on contagion. Increasing the number of links increases the opportunities for spreading liquidity shocks among counter-parties but also facilitates the channels through which default spreads. Theoretical works on the microeconomic foundations of banking networks examine this trade-off building on Diamond and Dybvig (1983) model in which banks are hit by liquidity shocks due to the needs of their consumers. Banks can use the inter-bank market or cross-deposits to respond to these shocks without costly liquidation. Allen and Gale (2000) provide a framework (limited to four banks) for how the degree of completeness of the inter-bank market may affect contagion risk, and conclude that a complete network is more stable to shocks than an incomplete one. In a model of spatially separated banks, Freixas, Parigi, and Rochet (2000) analyze the impact of credit lines and central bank policy on the stability of the economy subject to (local) consumers withdrawal needs possibly amplified by coordination problems. Clearly, our analysis differs since we take the liabilities as given. Gai and Kapadia (2008) analyze how the network structure affects the trade-off by using a random graph model in which links are formed randomly according to a given distribution. They find that financial systems exhibit a robust-yet-fragile tendency: while greater connectivity reduces the likelihood of widespread default, the impact on the financial system, should problems occur, can be on a significantly larger scale than hitherto.[3] The analysis however relies on an a priori symmetric network and measures systematic risk by the expected contagion size given an expected bank picked at random. Given the observed heterogeneity of financial institutions, in their size and in their connections, an important question is to assess the externalities initiated by a *given* bank. This is precisely the purpose of the threat index.

Finally, the paper relates to the large literature on the interactions and externalities channeled through a network of connections. Firstly, the threat index provides an assessment of a position in a network. As such it is related with some power indices introduced in the sociological literature to assign a value to positions. Indeed, in some special cases, the threat index coincides with a Katz-Bonacich index (1953 and 1987) computed on the endogenous subnetwork of the defaulting banks, hence can be seen as an extension of the power indices to a richer setting. Despite this

[2]Cifuentes, Ferrucci, and Shin (2005) introduce a further mechanism of contagion through asset sales amplified by regulatory solvency constraints and mark-to-market rules.

[3]This result however is driven by the assumption that a bank's total amount of assets and liabilities is kept fixed, irrespective of the network structure. In practice the amount of inter-bank assets and liabilities is related positively with the number of links. Intuitively, this correlation should hamper the risk-sharing benefit of forming links. Even if the relationship between asset and liability and the number of links is sufficiently strong, one may suspect that dense networks are vulnerable to contagion.

similarity, the approaches differ as the threat index is based on an explicit objective. Secondly, targeting policies have been investigated in alternative network models in which individual actions generate externalities channelled through a network. In a criminal network for example, the 'key player' to remove, the one whose arrest triggers the largest decrease in global criminal activity, may not be the one the more active (see Ballester, Calvó-Armengol, Zenou 2005). Similar insights hold in our model since the more threatening banks may not be the weakest.

The paper is organized as follows. Section 2 presents the model and the clearing mechanism assuming positive values for the banks' net worth. Section 3 introduces and analyzes the threat index. A targeting policy, which would inject cash in specific defaulting banks is analyzed, as well as a solidarity policy, which would force safe banks to increase their repayments beyond their nominal liabilities. Finally, some comparative static exercises on the impact of liabilities are performed. Section 4 extends the analysis to the situation where bankruptcy is unavoidable by allowing net worth to be negative. Section 5 gathers some proofs.

2 A contagion model

2.1 The framework

There are n financial institutions, called banks for simplicity. Denote $N = \{1, \ldots, n\}$. Banks draw some risky revenues from their activities with the non-financial sector and are linked through claims on each other. The various assets and liabilities on the non-financial sector are summarized by a single value for each bank, z_i for bank i. This value will be called the ***net worth***. The inter-bank assets and liabilities are described by a $n \times n$ matrix $\ell = (\ell_{ij})$ where ℓ_{ij} represents the magnitude of i's nominal debt obligation toward j, ℓ_{ii}=0. The total nominal liabilities of bank i are

$$\ell_i^* = \sum_j \ell_{ij}. \tag{1}$$

We are at an ex-post stage. The net worth represents the accounting values of all operations with the non-banking sector once the payoffs from previous investments are revealed. As a result, the net worth level can very well be negative due for example to portfolios in derivative assets.

Let us make a couple of remarks on the liability structure, sometimes referred to as a network. When dealing with a large number of banks, the pattern of their relationships is quite stable and specific, with some banks having regular and large relationship while others have none. In such a situation, the interpretation of financial interlinkages as a network, where banks are nodes and bilateral exposures are the links, is very compelling. The liability structure depends on the situation under consideration, in particular on the maturity of the debts. In payment systems, liabilities are often both ways, reflecting common clienteles for example. Not only both ℓ_{ij} and ℓ_{ji} can be simultaneously positive but they are likely to be both positive or both null. In long term arrangements,

some patterns are more directed, such as the ones described in the Austrian banking system, with a partial pyramidal structure. The liability structure is said to be ***complete*** if for each distinct i and j liability ℓ_{ij} is positive.

When a bank is indebted, $\ell_i^* > 0$, its relative liabilities (π_{ij}) describe the proportions of its liabilities across creditors: $\pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}$ Assuming each bank to be indebted, $\Pi = (\pi_{ij})$ is called the ***relative*** liability matrix. In what follows results are stated under this assumption for convenience of presentation but as will be clear, statements go through without it. Denoting by $dg(\ell^*)$ the diagonal $n \quad n$ matrix which has ℓ_i^* on the diagonal, ℓ and Π satisfy

$$\ell = dg(\ell^*)\Pi \text{ where } \pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}. \tag{2}$$

Consider now the possibility of default and contagion. Given the realized net worth levels and the mutual liabilities, z and ℓ, some banks may not be able to repay their debts and default may propagate due to the mutual liabilities. Default may be partial meaning that a bank in difficulty is not necessarily declared bankrupt if it pays a fraction of its liability. This fraction, between 0 and 1, is called ***repayment ratio*** or simply ratio, denoted by θ_i. Start by assuming all banks fully repay their debts. If each bank ends up with a non-negative value for its equity, $z_i + \sum_j \ell_{ji} \quad \ell_i^* \quad 0$, debts are indeed fully repaid. If instead the value for one bank is negative, this bank can only default. Due to the mutual liabilities, the capacity for a bank to repay its debts depends on the repayments made by its debtors, thereby introducing linkages between the repayment ratios of the banks and calling for their joint determination. The clearing mechanism introduced by Eisenberg and Noe (2000) provides such a determination when net worth levels are non-negative. I start with this case so as to introduce the threat indices build on the clearing ratio vectors. Section 4 extends the clearing mechanism to the case of negative net worth levels and the possibility of bankruptcy.

Notation. $\mathbb{I}$ denotes the $n \quad n$ identity matrix, $\mathbf{0}$ and $\mathbb{1}$ denote respectively a n-vector of 0 and 1s. Given a n-vector θ and S a subset of indices, θ_S denotes the vector obtained from θ by keeping the rows indexed by S. Similarly, $A_{S\times T}$ denotes the matrix obtained from a matrix A by keeping the rows indexed by i in S and the columns indexed by j in T. A^t denotes the transpose of A.

2.2 Clearing repayment ratio vectors

The net worth level of each bank is assumed to be positive, denoted as $z \quad 0$ where $z = (z_i)$. A ratio vector $\theta = (\theta_i)$ specifies the repayment ratio of each bank, where θ_i is between 0 and 1.

The clearing mechanism specifies two requirements on the ratio vector bearing on the net equity of the banks. Net equity refers to the accounting residual value that results from the realized operations of the bank with all other parties, outside or inside the banking sector. Formally, given

repayment vector[4] θ, i's ***net equity*** $e_i(\theta)$ is defined by

$$e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^*, \tag{3}$$

which is the sum of the net worth and the net payments within the banking system to i, composed of the payments received by i, $\sum_j \theta_j \ell_{ji}$, less those made by i, $\theta_i \ell_i^*$.

Given (z, ℓ), $z \geq 0$, a vector vector $\theta = (\theta_i)$ in $[0,1]^n$ is said to be a ***clearing (repayment) ratio vector*** if it satisfies the following two conditions for each i

(a) limited liability: net equity is non-negative

$$e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* \geq 0. \tag{4}$$

(b) absolute priority of creditors

$$\theta_i < 1 \Rightarrow e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* = 0. \tag{5}$$

Observe that if each equity is non-negative assuming full repayment, $e_i(\mathbb{1}) \geq 0$, $\mathbb{1}$ is a clearing ratio vector and no default arises. Limited liability states that stockholders are not required to repay more than the total net asset value in the bank (i.e. more than the net worth plus the repayments from other banks). Absolute priority requires debts to be repaid in full unless net equity is null. As a result, no bank fully defaults at a clearing ratio vector thanks to the positivity of net worth: θ_i is surely positive because otherwise its equity would be positive, in contradiction with (5).

Before proceeding, it is useful to consider values for the system as a whole. Since net equities are linear in θ and the payments within N cancel out, the following aggregation formula is obtained

$$\sum_{i \in N} e_i(\theta) = \sum_{i \in N} z_i. \tag{6}$$

According to this equation, the total of the net equity values over N does not depend on the liabilities and is equal to the aggregate net worth whatever ratio θ. Two implications follow. In 'normal' times, when no default arises, each equity is given by the nonnegative value $e_i(\mathbb{1})$. If some of these values are negative, the clearing mechanism selects a ratio vector different from $\mathbb{1}$ and operates a distribution of the total flow going into the banking system in such a way that each equity ends up to be non-negative (since $e_i(\theta) \geq 0$ at the clearing ratio vector). The second implication is that one bank at least fully repays its debt at a clearing ratio vector: since the net equity of some bank i must be positive, creditors' absolute priority requires this bank to fully honor its debt ($\theta_i = 1$).

Properties 1 and 2 on the existence and behavior of a clearing ratio vector have been proved by Eisenberg and Noe (2000). We recall them without proof (these properties extend in some form in the more general model where net worth values can be negative, as examined in Section 4).

[4] In a possible interpretation of the model as proposed in Shin (2008), the described quantities are the market values of the balance sheets in terms of the numeraire at the current date. In particular, z_i represents the market value of the bank's activities with the non-banking sector. The θ_i is then interpreted as the current price of one unit of debt of bank i, and assuming risk neutrality, a clearing ratio vector corresponds to an equilibrium

Property 1 : Existence and uniqueness *Under positive net worth levels, there is a unique clearing ratio vector.*

Property 1 relies on the fact that repayment ratios are complements: the higher the repayment ratios of other banks, the more a given bank is able to repay. Complementarities imply that there is a greatest and a least clearing ratio vector. It turns out that these two vectors coincide under the positivity of the net worth values no matter what the liability network, which ensures the uniqueness of a clearing ratio vector.[5]

Banks' status Given a clearing ratio vector, the banks are naturally partitioned into safe banks, which fully repay their debts and defaulting banks, which do not. I distinguish the banks at the boundary of the two cases, safe or default. These 'fragile' banks fully repay their debt and end up with a null net equity. Specifically, given clearing ratio vector θ, let us define S the set of safe banks, which fully repay their debts and end up with strictly positive net equity

$$S = \{i,\ \theta_i = 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} \quad \theta_i \ell_i^* > 0\},$$

F the set of fragile banks, which just break even under full debt repayment,

$$F = \{i,\ \theta_i = 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} \quad \theta_i \ell_i^* = 0\},$$

and D the set of defaulting banks:

$$D = \{i,\ \theta_i < 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} \quad \theta_i \ell_i^* = 0\}.$$

Typically there is no fragile bank, because a small perturbation of the parameters z or ℓ makes them either safe or defaulting.

Given the status of the banks, the clearing ratio vector is characterized by a system of linear inequalities. The ratio is of the form $(\theta_D, \mathbb{1}_{N-D})$ because the ratios of safe and fragile banks are equal to 1 by definition. For the defaulting banks, their equity is null so that a clearing ratio vector solves the system of linear inequalities :

$$\begin{aligned} \theta_i \ell_i^* \quad \sum_{j \in D} \theta_j \ell_{ji} \quad \sum_{j \notin D} \ell_{ji} \quad &= \quad z_i \text{ each } i \in D && (7) \\ \ell_i^* \quad \sum_{j \in D} \theta_j \ell_{ji} \quad \sum_{j \notin D} \ell_{ji} \quad & \quad\quad z_i \text{ each } i \text{ not in } D. && (8) \end{aligned}$$

(7) is a system of linear equations on D. It can be written in matrix form

$$(dg(\ell^*) \quad \ell)^t_{D \times D} \theta_D = \hat{z}_D \tag{9}$$

[5] Null values for net worth introduce the possibility of multiple clearing ratio vectors. However uniqueness is guaranteed provided the liability network is sufficiently connected. These difficulties will be examined in the more general model of section 4, so I prefer to keep the presentation simple in this section. Furthermore, the analysis readily extends to situations with multiple clearing ratio vectors by assuming the clearing mechanism to select the greatest clearing ratio vector.

where $\hat{z}_i$ for i in D is defined as the sum of the net worth and the assets on the safe banks: $\hat{z}_i = [z_i + \sum_{j \in S} \ell_{ji}]$. We will examine in more detail this system in next section, when we compare the default ratios with the threat indices. For the moment it is enough to say that (9) has a unique solution, which is positive. With a complete liabilities network the property straightforwardly follows from well known results on diagonal dominant matrices.[6] Thus, without further information on net worth values, any clearing ratio vector and banks' status are possible. To see this, observe that given repayment ratios for banks in D, θ_D, the net worth can be chosen so that $(\theta_D, \mathbb{1}_{N-D})$ is a clearing ratio vector: define the net worth for banks in D so that (7) is satisfied and the net worth for those in $N \quad D$ to be large enough so that these banks are indeed safe, i.e. (8) is met.[7] With an incomplete liability structure, a technical difficulty is that the linear system may not be invertible for a general subset of N. However, this is never the case for the subset of defaulting banks, as stated in Lemma 1 in Section 3.

Payment objective Complementarities in the ratios imply that a clearing ratio vector can be found by maximizing the total payments within the banking system[8] under some constraints.

Property 2 : Clearing ratio vector and total payments within the system *The clearing ratio vector solves the following program*

$$P \quad : \quad \max_{\theta} \sum_i \ell_i^* \theta_i$$

$$0 \quad \theta_i \quad 1 \text{ each } i \tag{10}$$

$$\theta_i \ell_i^* \quad \sum_j \theta_j \ell_{ji} \quad z_i \text{ each } i. \tag{11}$$

The constraints reflect those on debt contracts, (10), according to which a bank can only reimburse (the condition θ_i non-negative) and only up to its nominal liabilities (the condition θ_i less than 1) and those on equities, the limited liability condition (11). Observe that creditors' absolute priority is not a constraint. It is surely satisfied at the optimal solution thanks to complementarities because the objective is strictly increasing in each repayment ratio. The absolute priority condition can thus be seen as forcing the clearing repayment ratio vector to maximize the total payments within the banking system under the limited liability condition.

We now use Property 2 to define the threat indices, still maintaining the assumption of positive net worth levels.

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[6]Whatever strict subset D of N, the sum of row i of $\ell_{D \times D}$ is strictly smaller than ℓ_i^*. Matrix $(dg(\ell^*) - \ell)^t_{D \times D}$ is a Leontief diagonal dominant matrix, hence is invertible, with a positive inverse.

[7]The fact that (9) has a unique solution is not enough to prove the uniqueness of a clearing vector: The status of the banks could possibly differ at different clearing vectors. Uniqueness is obtained by relating the ratios of the defaulting banks to those of the safe banks, as performed in next section.

[8]The objective could be replaced by any function that is increasing in the θ_i.

3 The threat indices in case of default

The repayment ratio of a bank reflects its safety. From the system payment perspective however, the repayment ratio may not be the best indicator of the loss imposed by a bank. Specifically, let us assume that the objective of the payment system is defined by the total effective payments. Loss and benefit are thus in terms of these payments. The bank's threat index measures the loss induced by a decrease in the bank's net worth at the margin or, taking the opposite side, the index measures the benefit of rescuing the bank say in the form of additional capital. Of course, the threat index for a safe bank is null since a marginal decrease in its net worth has no impact on its repayment ratio, hence on the overall payments in the system. For the defaulting banks, the fact that the clearing ratio vector solves P, that is, maximizes the overall repayments under some linear constraints allows us to derive in a very simple way the threat indices.

We consider the value of the program P as the net worth levels vary. To make this dependence clear, let $V(z)$ denote the value of P associated to $z = (z_i)$ (or, more generally, denote $V(z, \ell)$ when considering variations in the liabilities as well).

The impact of a (marginal) increase in the net worth of a bank is assessed by considering the multiplier associated to the equity constraint (11). At points of differentiability of V, the envelope theorem applies:

$$\frac{\partial V}{\partial z_i} = \mu_i$$

where μ_i denotes the multiplier associated with constraint (11). Thus, the impact of a marginal increase of one unit in i's net worth is well defined as it increases the value V of the payments within the system by μ_i units. Alternatively the impact of a marginal decrease of one unit in i's net worth is to decrease the payments by μ_i units. This is why we call the multiplier μ_i a 'threat' index.

The next proposition states that the function V is indeed differentiable at 'most' points and provides an expression for the multipliers. These properties are especially useful for deriving an optimal targeting policy. The goal of such a policy is to inject a given amount of cash so as to increase the payments within the banking system as much as possible, as investigated in Section 3.2.

Proposition 1 *The value function V of program P is concave in (z, ℓ). It is di erentiable with respect to z at each point (z, ℓ) for which there is no fragile bank. In that case the derivative vector $(\frac{\partial V}{\partial z_i})$ is the unique μ that is null on S and solves on D*

$$\ell_i^* \mu_i \quad \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*, \text{ for each } i \text{ in } D. \tag{12}$$

The multiplier μ_i is called i's threat index.

The proposition is proved by applying well known results on linear programming and duality. Since there are fragile banks only for a degenerate set of values for (z, ℓ), the multipliers are unique and the function V is differentiable almost everywhere. Standard complementarity relationships between

primal and dual variables apply to the repayment ratios (the solutions to the primal) and the threat indices (the solutions to the dual). This explains why the threat index of a safe bank is null. Otherwise, for a bank with a repayment ratio θ_i strictly smaller than 1, the multiplier is non-null and even larger than 1. Similar results obtain for the differentiability of V with respect to liabilities, as investigated in Section 3.3.

Comparing ratios and threat indices A comparison of the determinants of the repayment ratios and the threat indices of the defaulting banks makes clear that they may differ and be not necessarily aligned. Assuming no fragile banks, they satisfy $\theta_i = 0, \mu_i = 1$ on S and respectively (7) and (12) or:

$$\theta_i \ell_i^* \quad \sum_{j \in D} \theta_j \ell_{ji} = z_i + \sum_{j \notin D} \ell_{ji}, \text{ and } \ell_i^* \mu_i \quad \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*, \text{ for each } i \text{ in } D. \tag{13}$$

Whereas the distress of a bank as measured by its repayment ratio depends on the distress of its debtors (through the ℓ_{ji}), the threat the bank imposes on the payment system depends on the threat of its creditors (through the ℓ_{ij}). Thus the impact of the liability structure differs except under strong symmetry of the liabilities/loans. Also, the repayment ratios are affected by the precise values taken by the net worth whereas the indices depend on these values only through the status of the banks (the interpretation of the threat indices will explain why this is the case).

Let us illustrate how to compute the ratios and indices in a very simple example. Consider a single debt chain in which each bank i has a liability towards $i + 1$. The clearing ratio vector is easily computed recursively starting from bank 1. Since bank 1 has no claims on other banks, its repayment ratio is determined by its net worth as the minimum of 1 and z_1/ℓ_1^*. Knowing θ_1, θ_2 is determined in the same way and so on. This computation reveals the defaulting banks. Let us compute the threat indices assuming 1 defaults. Let $1, .., j$ be the connected set of defaulting banks (i.e. $j + 1$ does not default). It is easy to see that the index of a bank in $1, .., j$ is simply equal to 1 plus the number of its defaulting creditors, direct or indirect, in the interval. (The index μ_j is 1 since its creditor does not default, and (12) simplifies into $\mu_i = 1 + \mu_{i+1}$ for $i < j$.) Hence the indices are decreasing along $1, ..., j$. There is no reason for the ratios to follow the same order.[9]

This chain example is a particular case of the pyramidal structure in which computation[10] can be recursive as well, as examined in section 3.1.

[9]More generally, in this chain example, the set of defaulting banks is formed with disjoint intervals (and bank 1 does not necessarily default). The indices are easily computed in each interval in the same way as above, hence decrease along each interval. A bank that has the largest index is one which has the largest number of connected defaulting (direct or indirect) creditors.

[10]With a directed circle, the structure of defaulting banks also consists of disjoint intervals and the threat indices are computed similarly. However the computation of the clearing vector is less easy because it is not recursive, starting from the top of the chain.

We now analyze more closely the linear systems (13). They write in matrix form

$$(dg(\ell^*) - \ell)^t_{D\times D}\theta_D = \hat{z}_D \text{ and } (dg(\ell^*) - \ell)_{D\times D}\mu_D = \ell^*_D. \tag{14}$$

The matrices are invertible and their inverse are positive, as stated in the following lemma. Recall that Π is defined as the relative liabilities matrix.

Lemma 1 *Let $z \geq \mathbf{0}$. Let D be the default set at a clearing ratio vector θ. Then the matrix $(dg(\ell^*) - \ell)_{D\times D}$ and its transpose are invertible and their inverse have all their elements positive. The same properties hold for $(\mathbb{I} - \Pi)_{D\times D}$.*

Recall the relation (2) : $dg(\ell^*) - \ell = dg(\ell^*)(\mathbb{I} - \Pi)$. Taking the restrictions of these matrices[11] on $D \times D$, the properties stated in Lemma 1 are therefore equivalent for the two matrices $(dg(\ell^*) - \ell)_{D\times D}$, $(\mathbb{I} - \Pi)_{D\times D}$ (and their transpose). Let us argue in term of the relative liabilities. For a complete liability structure, the property straightforwardly follows from well known results on positive matrices because whatever strict subset T of N, the total of each row of Π is strictly smaller than 1. In particular, the inverse of $(\mathbb{I} - \Pi)_{T\times T}$ is given by a converging infinite sum:

$$(\mathbb{I} - \Pi)^{-1}_{T\times T} = \mathbb{I}_{T\times T} + \Pi_{T\times T} + \Pi^{(2)}_{T\times T} + ... + \Pi^{(p)}_{T\times T} + ... \tag{15}$$

For an incomplete liability structure, this expression extends to a subset T for which each bank's equity is null under some positive ratio vector (see the proof in Section 5). Call such a set a null equity set. A default set is a null equity set. The argument is based on the fact that a null equity set has an outside creditor (flows are going in, so some flow must go out). Using the fact that each subset of a null equity set is itself a null equity set, hence has an outside creditor,[12] the proof shows that for some p the p-power $\Pi^{(p)}_{T\times T}$ has all its rows totals smaller than 1. This implies that the infinite sum on the right hand side of (15) converges; the positive limit is the inverse to $(\mathbb{I} - \Pi)_{T\times T}$. Expression (15) is useful to interpret the indices.

Interpreting the threat indices For the interpretation, let us work with the formulation in terms of the relative liability structure. This facilitates the comparison with some 'centrality' indices introduced in the network literature such as the Katz/Bonacich index (1953) and (1987). Dividing equation (12) by ℓ^*_i, the threat indices among the defaulting banks satisfy:

$$\mu_i = 1 + \sum_{j\in D} \pi_{ij}\mu_j, \text{ for each } i \text{ in } D \text{ or in matrix form } (\mathbb{I} - \Pi)_{D\times D}\mu_D = \mathbb{1}_D. \tag{16}$$

[11] Lemma 1 extends to the situation in which some banks are not indebted because each defaulting bank is surely indebted, and their relative liabilities are well defined.

[12] Without an outside creditor for each subset, invertibility may fail, as in the following example

$$\Pi_{T\times T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3 has outside creditors, thus $T = \{1, 2, 3\}$ has an outside creditor but $\{1, 2\}$ has not. Since the vector $x = (1, 1, 0)$ satisfies $x = x\Pi_{T\times T}$ the matrix $(\mathbb{I} - \Pi)_{T\times T}$ is not invertible.

Only the relative liabilities *within* the set D determines the indices. The threat index of a distressed bank i is equal to 1 plus the weighted sum of the threat indices of its creditors weighted by the amount of the obligations of i to them. Such a relationship has a flavor of a Katz/Bonacich index within the network restricted to the defaulting banks.[13]

To interpret the indices, we consider the impact of a modification in the net worth of a bank. An increase in the net worth can be interpreted as an injection of cash. The analysis allows us to understand the boundary case where there are fragile banks.

We first assume that there is no fragile bank, so that V is differentiable at the standing point (z, ℓ) with a derivative with respect to z_i given by i's threat index $\frac{\partial V}{\partial z_i} = \mu_i$. Thus the marginal impact marginal on the total payments V of a marginal increase in the net worth δ units in z_i is given by

$$\frac{\partial V}{\partial z_i}\delta = \mu_i \delta.$$

A safe bank, which already fulfills its obligations, keeps the additional cash: there is no impact on payments ($\mu_i = 0$). A defaulting bank instead uses the additional cash (at least partially) for reimbursing its debts. Its creditors in default in turn use (part of) the additional cash to repay their debts, and so on. The initial additional cash thus triggers a sequence of additional reimbursements along the creditors which are themselves in default. For δ small enough, each amount that is received by a defaulting bank is entirely used to repay its debts. This leads to the following computation.

The additional cash received by the defaulting bank i, which is entirely used for reimbursement, generates a first increase of δ units in the payments. This explains why μ_i is larger than 1. Each i's creditor j receives the share π_{ij} of δ, entirely used for reimbursement by those in default, thereby generating a first 'indirect' additional payment in the system equal to $(\sum_{j \in D} \pi_{ij})\delta$. The sum term is the i th element of $\Pi_{D\times D}\mathbb{1}_D$. By the same argument, each of the $\pi_{ij}\delta$ units received by the defaulting i's creditor j generates $\sum_{k\in D} \pi_{jk}$ extra units of payments. So, summing over all defaulting creditors of i, the 'second' indirect additional increase equals $\sum_{j\in D}(\sum_{k\in D} \pi_{jk})\pi_{ij}\delta$, or exchanging the order of summation, $\sum_{k\in D}[\sum_{j\in D} \pi_{ij}\pi_{jk}]\delta$. Since the element in square brackets is the (i, k) element of the matrix $\Pi^{(2)}_{D\times D} = \Pi_{D\times D} \quad \Pi_{D\times D}$, the 'second' indirect impact is δ times the i th component of $\Pi^{(2)}_{D\times D}\mathbb{1}_D$. Iterating, the additional indirect impact along a path of p banks, each one

[13] The threat index coincides with a Katz/Bonacich index in some specific cases. Let G be the incidence matrix of the liabilities network, which has 1 if ℓ_{ij} is positive and 0 otherwise. Assume a liability matrix in which all the positive ℓ_{ij} are equal to an identical level, and furthermore let each bank have the same number of creditors, say p, hence the same total liabilities. The matrix Π is proportional to the incidence matrix of the liabilities network : $\Pi = \frac{1}{p}G$. Then given D

$$\mu_D = \left(\mathbb{I} - \frac{1}{p}G_{D\times D}\right)^{-1} \mu_D.$$

Up to the constant 1, the index coincides with the Bonacich index for the 'attenuation' parameter $1/p$ in the liabilities network within D. The attenuation parameter defines the importance of indirect links. Contrary to the sociology framework however, it is here determined, defined by the reciprocal of the number of total creditors of a bank: the more creditors, the less the influence of indirect creditors.

defaulting and debtor to its successor, is δ times the i th component of $\Pi^{(p)}_{D\times D}\mathbb{1}_D$. Summing all indirect impacts gives the value of $\mu_i\delta$ as the i-th component of the infinite sum $\delta\sum_{p\geq 0}\Pi^{(p)}_{D\times D}+$ ...$)\mathbb{1}_D$. Considering all defaulting banks, we thus obtain

$$\mu_D = (\mathbb{I}_{D\times D} + \Pi_{D\times D} + \Pi^{(2)}_{D\times D} + ... + \Pi^{(p)}_{D\times D} + ...)\mathbb{1}_D$$

or $(\mathbb{I} \quad \Pi)_{D\times D}\mu_D = \mathbb{1}_D$, the equation (16).

The above argument explains why the indices are determined by the ***relative*** liability structure ***within*** the set D only. As long as the banks status do not change, which holds true for small enough δ, a cash injection triggers automatic increases in the payments entirely determined by the liability shares of the recipient defaulting banks, and do not depend upon other banks' liabilities or net worth levels.

The argument extends to the situation where there are some fragile banks by distinguishing an increase in the net worth of a fragile bank from a decrease. Increasing the net worth in a fragile bank has no impact on the payments because it already repays its debt. If instead the net worth is decreased, its ratio is necessarily lowered and the same argument as above allows us to compute the impact on the payments (other fragile banks (if any) are treated as defaulting banks because each one can only receive less reimbursements). This explains why the value function V is not differentiable. Furthermore, for fragile bank i

$$\frac{\partial V}{\partial z_i^+} = 0, \frac{\partial V}{\partial z_i^-} \quad 1,$$

in which the expressions denote respectively the right and left derivatives of V. V may not be differentiable with respect to the net worth of a defaulting bank either. Let us consider a defaulting bank with a chain of defaulting creditors leading to a fragile bank. An increase in the net worth of the defaulting bank makes the fragile bank become safe while a decrease makes it defaulting, generating a further decrease in payments. The extremal values for the threat indices are obtained by applying expression (12) as if the default set was either D [F (for the maximal values) or D (for the minimal ones).

3.1 Examples

Pyramidal network In a pyramidal network, liabilities go in the same direction without loop, from the top or from the bottom. Consider first the situation where these liabilities are directed from the top, as represented in Figure 1, in which an arrow from i to j represents a positive liability of i towards j. This describes a situation in which chains of intermediaries collect funds for the bank at the top. Each bank collects funds from the institutions directly below it or from outside the banking system for the banks without 'subordinate'. Each bank except the bank at the top lends these funds to its unique superior.

Let the level of a bank be defined as the number of links between this bank and the top.

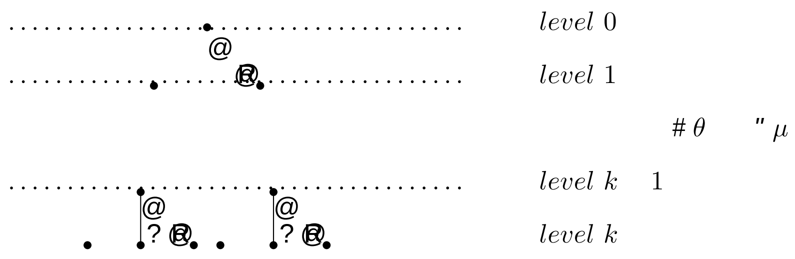

Figure 1: Pyramidal network

The clearing ratio vector is easily computed recursively starting from the top. The top bank, say bank 1, has no claims on other banks so its repayment ratio is determined by its net worth as the minimum of 1 and z_1/ℓ_1^*. The repayment ratio of the banks at level 1 can now be computed since they receive payments only from bank 1 and these are known. The computation proceeds: at step $s+1$, the repayment ratios of the banks at level s are determined since the repayments of all their debtors are known (i's ratio is the minimum of 1 and $(z_i + \sum_{j\in D}\theta_j \ell_{ji})/\ell_i^*$). The clearing vector is obtained after k steps and the status of the banks are determined.

Knowing the status of the banks, the threat indices are computed recursively in a similar way, starting from the bottom instead of the top. As we know, the index is not uniquely defined if there are fragile banks. So assume first no fragile banks. Set the indices of all safe banks to zero. The threat index of a bank at level k is either equal to 0 (the bank is safe) or to 1 (the bank is in default but has no creditor). At each further step, the threat index of a bank only depends on those of its creditors, which have been determined at the previous step. Hence the indices are computed recursively for the defaulting banks using expression (12): $\ell_i^*\mu_i \quad \sum_{j\in D}\ell_{ij}\mu_j = \ell_i^*$. If there are fragile banks, the computation can be performed by either considering these fragile banks as defaulting banks, so as to obtain the maximal threat indices, or by considering them as safe banks, so as to obtain the minimal indices.

It can be easily checked that the orders given by the default ratios and the threat indices may differ.

Similar recursive computations can be performed in the reverse situation in which all the liabilities point toward the top. This may represent a 'conglomerate' in which at each level a unit has lent funds to its direct subordinates. In this situation, the top has a null threat index. This situation can be qualified as less prone to contagion than the previous one because a single default cannot touch all banks.

Log-fitting model There is a lack of data on bilateral inter-bank exposures. The log-fitting method is most often used to estimate the missing data given the available information on some total exposures. The method is justified if the missing data are independent conditional on the

current information. Explicit formula are obtained in the following simple situation.

Let the total amount of liabilities l_i^* and loans $l_{*i} = \sum_{j \in N} l_{ji}$ be known for each bank. Thus the sums in each row and each column of matrix ℓ are known. Without any specific information on bilateral exposures, the estimated proportions of i's liabilities are independent of i, thus equal to the overall proportions of the loans. The bilateral exposures are estimated by

$$\pi_{ij} = \frac{\ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \frac{\ell_i^* \ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \ell_i^* \pi_{ij}.$$

Since the values π_{ij} are independent of i, the expression (16) for the threat indices of defaulting banks, $\mu_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j$, implies that index μ_i is independent of i on D. Straightforward computation gives

$$\mu_i = 1 + \frac{\sum_{j \in D} \ell_{*j}}{\sum_{j \in S} \ell_{*j}} \text{ for each } i \in D. \tag{17}$$

According to this expression, the log-fitting model does not discriminate the banks within each status class. The common value for the threat index of defaulting banks is up to 1 equal to the loans distributed by the defaulting banks relative to those distributed by the safe banks.

3.2 Targeting policy

A targeting policy aims to inject a given amount of cash so as to increase the effective payments V within the banking system as much as possible. We first consider a policy that injects cash in the banking system.

Cash injection The targeting policy is easily derived from Proposition 1.

Proposition 2 *A marginal injection of cash that optimally increases V is targeted towards the defaulting banks with the largest value for the threat index μ_i. These targeted banks are kept identical as long as the status of the banks remain unchanged.*

PROOF The first assertion straightforwardly follows from Proposition 1. The function V is concave and $\frac{\partial V}{\partial z} = \mu$ at each z for which no bank is fragile. Since the marginal impact of cash injection towards i increases the value V of the payments within the system by μ_i units, cash should be allocated to the i with largest μ_i. As for the last statement, recall that μ is constant for a given set D and no fragile banks. Thus, as long as cash injection does not modify the banks' status, the targeted banks can be kept unchanged. ∎

The policy is especially simple since there is no need to modify the targets while injecting cash as long as no defaulting bank is transformed into a fragile one. Recall that the orders given by the default ratios and the threat indices may differ. Thus the targets may not be the banks with the largest default ratios.

As already seen, the threat indices appear not to be related to size. What matters is the *share* of the liabilities of a defaulting bank towards its creditors that are in default. Accordingly, injecting cash at the margin in a large bank or a small bank may be more or less profitable. However, one may suspect that a large bank with large liabilities generates more default among its creditors when it defaults. If true, its index is likely to be large, conditional on its default. Also, injecting a large amount of cash and following the targeting policy requires to adjust the targeted banks when some defaulting banks become safe. Arguing along the same lines as above, it is plausible that a small bank becomes safe before a large one. A full analysis of these questions requires to specify how the net worth and the liabilities are likely to be related, and would be more meaningful in a stochastic setting.

The next proposition states the sub-modularity of the total payments V. We use the standard notation: z_{-i} denotes the net worth levels for banks other than i; given i's net worth z_i, (z_i, z_{-i}) denotes the net worth levels for all banks; as ℓ is fixed, the argument ℓ is omitted.

Proposition 3 *The value function V is sub-modular in the net worth levels:*

$$V(z_i', z_{-i}') \quad V(z_i', z_{-i}) \quad V(z_i, z_{-i}') \quad V(z_i, z_{-i})$$

$$z_i' \quad z_i, z_{-i}' \quad z_{-i}.$$

According to sub-modularity, the benefit of injecting cash in a given bank is larger the less net worth in each other bank, that is the more fragile the system is.

$< \mathit{Comment} >$

Solidarity policy Alternative policies based on different tools than cash injection can be contemplated. One tool is to force banks to pay more than their liabilities, which amounts to increase the upper-bound of 1 on the repayment ratio. Such a tool is effective only on the safe banks, which are the only ones to be able to increase their payment. Hence the targeted banks differ from those under capital injection. The associated policies can be qualified as 'solidarity policies' since they involve transfers from safe to defaulting banks. An optimal solidarity policy depends on the precise constraints one wants to impose on these transfers. Whatever these constraints are, one needs to assess the impact on the payments V of increasing the upper-bound on the repayment ratio, which is given by the multiplier associated to the constraint $\theta_i \quad 1$. To facilitate comparison with the threat index and work with the same unit, we write the constraint $\theta_i \quad 1$ as $\theta_i \ell_i^* \quad \ell_i^*$. We call the associated multiplier λ_i the *solidarity index*. For the same reasons as for the threat indices, solidarity indices are uniquely defined when there are no fragile banks, so we assume F to be empty. Increasing the upper-bound on the ratio of a defaulting bank has no effect: λ is null on D. For the safe banks, assuming no fragile banks, the values of λ on S are given by

$$\ell_i^* \lambda_i = \sum_{j \in D} \ell_{ij} \mu_j + \ell_i^*, i \text{ in } S \text{ or } \lambda_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j. \tag{18}$$

The solidarity index of a safe bank is easy to interpret. Increasing the upper-bound on the ratio of a safe bank say by one percent has a direct effect of increasing the payment by one percent of ℓ_i^*, and an indirect effect on the banks that receive these payments. This indirect effect is similar to an increase in the net worth of each creditor in the proportion given by the relative liabilities. This explains expression (18) (accounting for the fact that the threat indices μ_j are null outside D).

Remark The threat and solidarity indices, which are the multipliers of the constraints of the program P, can be given the standard interpretation of shadow prices of resources. Here the resources of the payment system are given by the net worth and liabilities of the banks. Let the system be able to buy one unit of extra net worth for bank i at price μ_i and increase i's liabilities by one unit at price λ_i. The objective of the dual is to find prices for the resources that minimize the overall value of the resources subject to the constraint that increasing the ratio of a bank is not beneficial (from the view point of the system). Here increasing i's ratio costs $(\lambda_i + \mu_i)\ell_i^*$ per 'unit' with a benefit equal to 1 (direct benefit) plus $\sum_j \ell_{ji}\mu_j$ (indirect benefit). As shown in the proof section this gives the dual of P:

$$\mathsf{D} \quad : \min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j \quad 0 \text{ each } i.$$

The expressions for μ and λ follow by oberving that either λ_i or μ_i is null depending on the bank's status (except for fragile banks).

3.3 Comparative statics in liabilities

There is a concern about the impact of large cross-liabilities on the stability of the system. This section analyzes this impact on the effective payments V.

Consider first an increase in the liability between two banks, ℓ_{ij} of i to j. The increase in the effective payments has to be compared with the increase in nominal liabilities. The impact on the creditor is akin to an additional unit of net worth, the amount of which depends on how much the debtor can pay. Thus the status of both banks matter to determine the impact on the payments. Indeed thanks to the envelope theorem, a marginal increase in ℓ_{ij} generates a marginal increase in payments given by

$$\frac{\partial V}{\partial \ell_{ij}} = \theta_i[1 \quad \mu_i + \mu_j]. \tag{19}$$

The interpretation is as follows. To simplify discussion let the 'unit' of increase be small enough so that its impact is equal to the marginal impact. If both banks are safe, payments increase by 1 unit as expected. To understand other cases, let us distinguish between the status of bank i.

If i is safe but j defaults, the payments increase by $1 + \mu_j$: there is an additional payment of 1 unit by i to j and this unit has the same impact on defaulting j as an additional unit of net worth,

hence an additional increase by μ_j.

If i defaults, the impact here is more subtle. As i already exhausts its repayment capacity, an additional liability has no impact on its overall repayment, but there is a change in the composition of the reimbursements. Specifically the impact can be decomposed into two parts, a direct effect and a composition effect. First, i sends an additional amount θ_i to bank j, and this has a direct effect equal to $\theta_i(1+\mu_j)$ arguing as above. Second, as bank i is constrained, it has θ_i less units to repay its original debts, as if its net worth was diminished by θ_i: the composition effect is $-\theta_i\mu_i$. The sum of the two effects is $\theta_i[1+\mu_j-\mu_i]$, which is (19).

As for the net increase -the increase in the payments diminished of that in nominal liabilities- we see that it is positive for i safe and j in default, negative in the opposite situation, and ambiguous when both banks are defaulting.

Let us consider now an identical increase in the joint liabilities of two banks, which leaves unchanged their net liabilities: both ℓ_{ij} and ℓ_{ji} are increased by an identical amount. The marginal overall net increase in the payments is

$$(\theta_i+\theta_j)+[\mu_j-\mu_i](\theta_i-\theta_j)-2.$$

When both banks are safe, the net increase is null as expected since the increase in cross payments cancel out. When one bank is safe, net payments never decrease: For i safe, $\theta_i=1$ and $\mu_i=0$, the net increase equals $(1-\theta_j)(\mu_j-1)$, which is non-negative since each term in the product is. The intuition is that increasing each liability calls for more transfers across banks, which results in safe banks paying more per unit of additional liability than those in default. When both banks are defaulting however, the impact on payments is ambiguous because of the recomposition effect we have identified previously. With similar repayment ratios or similar threat indices, the impact is indeed negative: the direct effects identified above, which increase repayments between the two banks, cancel out and we are left with the negative composition effect, according to which other banks get less.

Finally, let us consider an equal increase in all liabilities, which would follow for example from a softening of regulation constraints. The net increase in the payoffs is, adding up the impact of all pairs and rearranging[14]

$$-(n-1)\sum_i(1-\theta_i)+n\sum_i(1-\theta_i)(\mu_i-\overline{\mu}) \tag{20}$$

where $\overline{\mu}=\frac{1}{n}\sum_i\mu_i$ is the average value of the μ_i.

Let us illustrate in the log-fitting model (section 3.1). Recall that the multipliers of the defaulting banks are all identical. Denote by μ_d the common value and ℓ_{*j} the amount of loans distributed by

[14]For a given i, the sum of the net increase $\theta_i[1-\mu_i+\mu_j]-1$ over j distinct of i is $(n-1)(\theta_i-1)-\theta_i[(n-1)\mu_i-\sum_{j\neq i}\mu_j]$ which can be written as $(n-1)(\theta_i-1)+n\theta_i[\mu_i-\overline{\mu}]$. Summing over i, we obtain the net increase $-(n-1)\sum_i(1-\theta_i)-n\sum_i\theta_i(\mu_i-\overline{\mu})$ and it suffices to add the null term $n\sum_i(\mu_i-\overline{\mu})$ to obtain (20).

j. Simple computation[15] yields that the net increase is positive if

$$\mu_d \quad \frac{n \quad 1}{n \quad d} \text{ or } \frac{\sum_{j\in D} \ell_{*j}}{d} \quad \frac{d \quad 1}{d} \frac{\sum_{j\in S} \ell_{*j}}{n \quad d} \tag{21}$$

where d and $n \quad d$ are respectively the number of defaulting and safe banks. The inequality requires the average loan per defaulting bank to be larger than the average loan per safe bank by the factor $\frac{d-1}{d}$. As seen earlier, increasing liabilities has a positive impact on a pair formed with a safe and a defaulting banks and has a negative one on a pair with defaulting banks (because their threat indices are equal). Under (21) the threat is large enough so that the positive impact dominates.

With an identical amount of loans per bank, the ℓ_{*j} are equal across j, the condition (21) is surely met: increasing liabilities is beneficial. Such a situation corresponds to a priori similar institutions, which are engaged into symmetrical inter-bank relationships. Due to shocks in their activities, they may end up in an asymmetrical situation, with some of them defaulting. However, independently of the realized net worth levels, and the subsequent status for the firms, more links are better for net reimbursements. Thus there is a benefit 'ex post', which implies an 'insurance' benefit ex ante, taking the expectation over all values of the net worth.

4 Bankruptcy

So far, we have considered a situation in which banks default but can still reimburse part of their debts. Thanks to the positive value derived from the non-banking sector (the positivity of the z_i), the net equity of each bank can be made positive for some positive repayment ratios. This section considers the possibility for a bank to have a net liability to the non-banking sector, which is represented by a negative net worth. In such a situation, even if a bank fully defaults on its banks' liabilities, it may still leave some debt to the outside sector. We call such a situation bankruptcy.

Bankruptcy is bound to arise for a low enough realization of the net worth. To see this, consider an extreme case in which a bank cannot repay its liability to the outside banking sector even if other banks fully repay their liabilities, the more favorable scenario. This arises for bank i if $z_i + \sum_j \ell_{ji}$ is negative. Whatever repayment ratio, bank i's cannot fulfill its obligations and the bank must go bankrupt.

4.1 Clearing ratio vectors

We first extend the notion of clearing ratio vector to allow for the possibility of bankruptcy. A negative net worth corresponds to a liability towards creditors outside the financial system. So we have two types of creditors, outside or inside the banking system. The bankruptcy condition that

[15] From (17) we have $\mu_d = 1+\frac{\sum_{j\in D}\ell_{*j}}{\sum_{j\in S}\ell_{*j}}$. Factoring out by $\sum_i(1-\theta_i)$, expression (20) is positive if $-(n-1)+n(\mu_d-\overline{\mu})$ is positive. The average value $\overline{\mu}$ is $\frac{d}{n}\mu_d$ so we obtain $\mu_d \geq \frac{n-1}{n-d}$, or $\frac{\sum_{j\in D}\ell_{*j}}{\sum_{j\in S}\ell_{*j}} \geq \frac{d-1}{n-d}$, and finally (21).

we introduce now specifies a hierarchy among the creditors, specifically the priority of the creditors outside the banking sector over those inside it.

It is convenient to introduce the ***net asset value*** of a bank. The net asset value is defined as the accounting sum of the net worth and the loans repayments by other banks.[16] Thus, the net equity is given by the asset value minus the repayments made by the bank. Formally, denoting bank i's net asset at θ by $a_i(\theta)$, we have

$$a_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} \text{ and } e_i(\theta) = a_i(\theta) \quad \theta_i \ell_i^*. \tag{22}$$

The net asset value can be negative, but only if net worth z_i is negative. When the net asset value is negative, the bank is declared bankrupt. In that case, the ***bankruptcy*** condition introduced in the next definition requires that the bank repays nothing to its creditors within the financial sector. Thus, as explained below, under the possibility of bankruptcy, the two rules of absolute priority of creditors and bankruptcy specify a hierarchy among the creditors. First a bank defaults on its liabilities to the financial system. Second, only if full default on its interbank liabilities is not sufficient for the bank to fulfill its obligations to the outside sector, the bank is declared bankrupt and defaults on its outside creditors.

The two rules of limited liability and absolute priority of creditors are rewritten slightly because a negative equity and a null ratio are no longer excluded. As explained after the definition, they are equivalent to the conditions defined in the previous section when net worth is positive. Observe that the net asset value of a bank depends only on the ratios of other banks. The conditions that must be satisfied by a bank depend on this net asset value, especially whether it is positive or not.

Definition 1 *Given* (z, ℓ)*, a vector* $\theta = (\theta_i)$ *in* $[0, 1]^n$ *is said to be a clearing ratio vector if it satis es for each i*

(a) limited liability

$$a_i(\theta) > 0 \textit{ implies } e_i(\theta) \quad 0 \tag{23}$$

(b) absolute priority of creditors over stockholders

$$a_i(\theta) > 0 \textit{ implies } \theta_i > 0 \textit{ and } e_i(\theta) = 0 \textit{ if } \theta_i < 1 \tag{24}$$

(c) bankruptcy (absolute priority of outside creditors)

$$a_i(\theta) \quad 0 \textit{ implies } \theta_i = 0 \tag{25}$$

Limited liability and absolute priority apply to a bank with positive net asset value, as is surely the case if its net worth is positive, whereas the bankruptcy condition applies to a bank with a non-positive net asset value.

[16] There is an asymmetry in our treatment of assets and liabilities within and outside the banking sector. First we do not consider the possibility of partial default to the outside sector. Second we work on net worth, allowing compensation between assets and liabilities outside the financial sector.

Limited liability requires that the payments to the banks' creditors never exceed the (positive) net asset value. When net worth is positive, the asset value is surely positive, and (23) is equivalent to the non-negativity of net equity. The absolute priority of banks creditors over stockholders is interpreted as previously as requiring a bank to repay its liabilities as much as it can: bank i defaults towards the banking sector only if its asset value is less than its liabilities. The *bankruptcy* condition requires that a bank repays nothing to other banks if its net asset value is negative.

The two rules of absolute priority of creditors and bankruptcy distinguish situations with positive asset value and possible default within the banking sector from those with negative asset value and default towards both the banking and the outside sectors. They specify a hierarchy among the creditors. To make this point clear, consider a bank that has a net liability towards the non-financial sector, i.e., z_i is negative. Two cases arise according to the sign of the net asset value.

When the net asset value is positive thanks to the repayment of its loans to the financial sector, $\sum_j \theta_j \ell_{ji} \quad z_i$, the liabilities to the outside sector are repaid. The remainder, which amounts to the net asset value, is used for reimbursing the bank's liabilities to other banks, and the left-over (possibly null) is $e_i(\theta)$, the net equity for the stockholders.

When the net asset value is non-positive, $\sum_j \theta_j \ell_{ji} \quad z_i$, bankruptcy arises; bank's creditors within the financial system get nothing and those outside the financial system seize the repayments $\sum_j \theta_j \ell_{ji}$, which is less than their liabilities $\quad z_i$. Hence the outside creditors incur a loss that amounts to $\quad z_i \quad \sum_j \theta_j \ell_{ji}$, the opposite of net asset value (assuming no bankruptcy cost). Of course this loss may be passed on partially to some others institutions -insurance scheme, taxpayers- if the bank is bailed out or to stockholders in case of recapitalization. The creditors in the financial sector receive nothing.

Observe that by the limited liability condition, the net equity of a bank can be negative only if its asset value is negative. In that case the bankruptcy condition requires the repayment to be null: if negative, the net equity coincides with the net asset values.

Existence, complementarity

Proposition 4 *There is a greatest clearing ratio vector. The values of net equities are the same at the clearing ratio vectors (if there are several). If aggregate net worth* $\sum_i z_i$

(a) is positive then one bank at least fully repays its debts, $\theta_i = 1$ *for some* i

(b) is negative then one bank at least is bankrupt, $\theta_i = 0$ *for some* i.

The existence of a clearing ratio vector and the fact that there is a greatest one rely on a key monotony property of the asset values: $a_i(\theta)$ is independent of θ_i and the $a_j(\theta)$ for $j \neq i$ are non-decreasing in θ_i. So increasing i's ratio does not affect its asset value and can only increase other banks' asset values. The existence of a clearing ratio vector follows by considering a 'feasible' set. A vector θ in $[0,1]^n$ is said to be *feasible* if it satisfies $\theta_i \ell_i^* \quad \max(a_i(\theta), 0)$ for each bank i. In words,

a ratio vector is feasible if each bank i reimburses its liabilities up to its asset value if positive. The limited liability and the bankruptcy constraints are thus satisfied. But the absolute priority rule does not hold in general: for a positive $a_i(\theta)$, feasibility only requires $\theta_i \ell_i^*$ $a_i(\theta)$ so that the ratio may be lower than 1 and at the same time equity strictly positive: the ratio is 'too low'. However, thanks to the key monotony property on asset values, the absolute priority condition is satisfied for all banks at a maximal element of the feasible set,[17] which readily implies that this element is a clearing repayment ratio vector.

The monotony property on asset values also implies that feasible ratio vectors are complements, in the sense that taking the maximum component by component of two feasible vectors yields a feasible vector.[18] It follows that there is a greatest feasible vector, which is a clearing repayment ratio vector.

The final statements use that aggregate net equity is equal to aggregate net

The fact that the clearing ratios are uniquely defined for the banks in B and S 'propagates' to all other banks if the banks are sufficiently connected -the irreducibility assumption- or if there are all linked to a universal creditor. touches all defaulting banks.

Let us give an intuition for the proof, which is detailed in Section 5. Let θ^+ be the greatest clearing ratio vector, and θ another clearing ratio vector, and T be the set on which they differ. As argued above, the bankrupt and the safe banks, respectively those with negative and positive equity values, coincide at any clearing ratio vector, and have the same ratios, respectively 0 or 1. Thus the ratios can differ only for the banks with null equity: T is a null equity set. T is not the whole set N because there is surely a safe or a bankrupt bank under the assumption of a non-null aggregate net worth, as seen in Proposition 4.

The key point is to show that T must have no outside creditor. By definition, each bank in T receives the same amount from outside banks under the two clearing ratio vectors. So thanks to the property that net equities are independent of the clearing vector, their repayment to banks outside T must be identical under the two vectors. But, because repayments can only be larger under θ^+ than under θ, this implies that T has no creditors outside T.

We show that this is impossible under either irreducibility or the presence of a universal creditor. The irreducibility of ℓ precisely requires any strict subset to have an outside creditor (note that we use here that T is not the whole set N). The universal creditor has always a ratio of 1 since it has no liability, so it is not in T, and furthermore it is an outside creditor of T.

The conditions of either irreducibility or the presence of a universal creditor are only sufficient but not necessary for uniqueness. A pyramidal structure as considered in Section 3.1 for example do not satisfy them but one easily checks that the clearing ratio is unique, computed recursively as explained before whatever the sign of the net worth levels (with the only modification of setting a ratio to zero for a bank with a negative net asset value).

The irreducibility of ℓ requires the existence of a path of liabilities between any one bank to any other bank. A related but weaker notion of connectedness is to require the existence of a path of loans-liabilities between any two banks. In graph terminology, irreducibility considers directed path in the liabilities network whereas connectedness considers non-directed paths. The following example has a connected network with multiple clearing vectors.

The system has four banks. Banks 1 and 2 have a negative net worth and banks 3 and 4 a positive one: $z = (\ \ 1, \ \ 1, 1.5, 1.5)$. The liabilities matrix is:

$$\ell = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

All nominal liabilities are of the same amount 1. Bank 1 has a liability to bank 2 and vice versa. Bank 3 has a liability to banks 1 and 4, and bank 4 to 2 and 3. We show that any ratio vector in

which 3 and 4 fully repay their debts and 1 and 2 have equal ratios is a clearing vector. Let θ with $\theta_1 = \theta_2, \theta_3 = \theta_4 = 1$. Bank 1's equity, $e_1(\theta) = -1 + \theta_2 + \theta_3 - \theta_1$, is null so bank 1 is allowed to default; bank 3's equity, $e_3(\theta) = 1.5 + \theta_4 - 2\theta_3 = 0.5$, is positive so bank 3 is safe (and similarly for banks 2 and 4 by symmetry).

Observe that the liability matrix is reducible: $\{1, 2\}$ have no liabilities towards 3 and 4. But $\{1, 2\}$ has 3 and 4 as debtors and the liabilities-loans graph is connected. At a clearing vector, banks 1 and 2 may appear as almost bankrupt or almost safe as their ratio may take any value between 0 and 1 (but whatever value their equity is null).

4.2 Defining threat indices

As seen earlier, bankruptcy is unavoidable if aggregate net worth is negative. More generally, bankruptcy is unavoidable when, whatever the ratio vector, one bank at least has negative net equity. In that case no ratio vector satisfies the constraints of the program P. As a result, the clearing ratio vector is surely not a solution to P. This section shows how to modify the program in such a way that the clearing vector appears to maximize a meaningful objective. The threat indices are then defined as measuring the incremental benefit in the objective due to an additional unit of cash in the banks.

By definition, a negative net equity represents a loss. This loss is borne by the creditors if the bank is declared bankrupt, or by the stockholders and whatever entity called to help such as creditors or taxpayers if the bank is bailed out. Under some conditions, a clearing repayment ratio vector solves a program in which these losses are accounted for. Specifically, given a cost c per unit of loss, the objective of the program is given by the payments flow net of the cost of the losses. The program, parameterized by the cost c, writes

$$Q_c \quad : \quad \max_{(\theta,\delta)} \sum_i \theta_i \ell_i^* - c\Big(\sum_i \delta_i\Big)$$

$$0 \le \theta_i \le 1,\ 0 \le \delta_i \ \text{each } i \tag{26}$$

$$\theta_i \ell_i^* \le \sum_j \theta_j \ell_{ji} + z_i + \delta_i \ \text{each } i \tag{27}$$

The inequality (27) can also be written using net equity as $\delta_i + e_i(\theta) \ge 0$. At a solution, the inequality is surely binding for a bank i with a positive δ_i. Thus, a positive δ_i is equal to the opposite of the negative net equity, which, as explained above, represents a loss to bank i's creditors or alternatively an additional amount of cash in the bank.

Observe that the constraints of the two programs P and Q_c differ. In particular, P is not the program Q_0 obtained for c null. (In Q_0, the equity constraint can be relaxed at no cost; ratios equal to 1 are feasible and maximize the flows). On the contrary, when net worth levels are positive, the solutions to P coincide with those to Q_c for large enough c, as stated in the next proposition. The intuition is that in that case bankruptcy is avoidable, and that it is optimal to avoid it when the

cost associated to bankruptcy losses is large enough. More generally, under some conditions, even when bankruptcy must occur, the clearing repayment ratio vector solves Q_c for c large enough.

Proposition 6 *Let the aggregate net worth z be positive, $\sum_{i \in N} z_i > 0$, and each liability ℓ_{ij} be positive, i and j distinct. The (unique) clearing repayment ratio vector solves Q_c for c large enough.*[19] *The threat indices, defined as the multipliers associated to the equity constraint (27), satisfy*

$$\mu_k = c \text{ for } k \text{ with } \delta_k > 0 \tag{28}$$

$$\mu_i - \sum_{j \in D} \pi_{ij} \mu_j = 1 + c \sum_{k \in B} \pi_{ik} \text{ for } i \text{ in } D. \tag{29}$$

The key point in the proof is to show that it is not optimal to inject cash into a bank and to have it repay at the same time, that is δ_i and θ_i cannot be both positive at an optimal solution. If it was the case, i's net equity would be negative (because δ_i is positive) but the bank would repay part of its liabilities, so that the ratio would not satisfy the bankruptcy condition, hence would not be a clearing ratio vector. This key point is proved under the additional assumption of a positive aggregate net worth.

Without an additional assumption, the solution to Q_c may not be a clearing repayment ratio vector even for large c. Let for example each bank have a negative net equity under full repayment: $e_i(\mathbf{1}) = z_i + \sum_j \ell_{ji} - \ell_i^* < 0$. The optimal solution is to have banks fully repay their debts and to inject in each of them exactly the amount necessary to do so: $\delta_i = -e_i(\mathbf{1})$. The payment flows within the financial system is clearly maximal. Furthermore the aggregate injection is minimal. To see this, note that summing (27) over all the banks requires $\sum_i (\delta_i + z_i) \geq 0$ at a feasible solution. Since $\sum_i z_i = \sum_i e_i(\mathbf{1})$, feasibility requires $\sum_i \delta_i \geq -\sum_i e_i(\mathbf{1})$ hence the aggregate injection is minimal for $\delta_i = -e_i(\mathbf{1})$.

The influence of c can be explained as follows. Arguing as in the previous section, a bank's threat index measures the incremental benefit in the objective due to an additional unit of cash in the bank. As c becomes large, the objective becomes dominated by the cost associated with capital injection. This explains why the behavior of the threat indices largely depends whether there are bankrupt banks or not. Without bankrupt banks, an empty set B, expression (29) coincides with (12) found in the previous section. Threat indices reflect the impact of the banks' net worth on the payment system and do not depend on c because there is no capital injection. With bankrupt banks instead, although the solution to Q_c stays constant for large enough c, the threat indices adjust to the value of c. Specifically, a bank's threat index is increasingly driven by the impact of its net worth on the losses of the bankrupt banks. Working with the threat indices per unit of cost, $\hat{\mu}_i = \mu_i / c$, we obtain at the limit for c increasingly large

$$\hat{\mu}_i - \sum_{j \in D} \pi_{ij} \hat{\mu}_j = \sum_{k \in B} \pi_{ik} \text{ for each } i \text{ in } D. \tag{30}$$

[19] It suffices that $c\pi_{ij} = c\ell_{ij}/\ell_i^*$ be larger than 1 for any pair i, j of distinct banks.

This expression can be interpreted as in the previous section by considering the additional flow of repayments that an increase in the net worth of a defaulting bank induces. With bankruptcy, what matters is not only the payments flowing along the defaulting banks but also how much of this flow reaches the bankrupt banks because this allows to diminish capital injection. For c extremely large, the cost of capital injection becomes predominant and the threat index measures the payments reaching the bankrupt banks, as we now show.

To show this, observe that the right hand side of (30) is the proportion of i's liabilities towards bankrupt banks. The vector of relative liabilities of banks in D to banks in B, $\sum_{k \in B} \pi_{ik}$ for i in D, can be written as $\Pi_{D \times B} \mathbb{1}_B$. Thus equation (30) in matrix form is $(\mathbb{I} \quad \Pi)_{D \times D} \hat{\mu}_D = \Pi_{D \times B} \mathbb{1}_B$ which gives the following expression for $\hat{\mu}_D$:

$$\hat{\mu}_D = \Pi_{D \times B} \mathbb{1}_B + \Pi_{D \times D} \Pi_{D \times B} \mathbb{1}_B + \Pi^{(2)}_{D \times D} \Pi_{D \times B} \mathbb{1}_B ... + \Pi^{(p)}_{D \times D} \Pi_{D \times B} \mathbb{1}_B ... \tag{31}$$

Each term in the sum corresponds to the amounts received by bankrupt banks following an increase in the net worth values of defaulting banks, either directly (for the first term) or indirectly through a chain of p defaulting banks (for the $p+1$-th term). Let defaulting bank i receive an additional unit of cash. The unit is entirely used for reimbursement. Each bankrupt bank k receives π_{ik}, thereby generating a direct total flow into bankrupt banks equal to $\sum_{k \in B} \pi_{ik}$. This term is the i-th component of the vector $\Pi_{D \times B} \mathbb{1}_B$, the first element in the sum on the right hand side of (31). Non bankrupt banks also receive additional payment, π_{ij} for j, and for those which are defaulting, they will pass this to their creditors: defaulting j pays an amount of $\pi_{ij} \sum_{k \in B} \pi_{jk}$ to the bankrupt banks. Hence there is a total of $\sum_{j \in D} \pi_{ij} \sum_{k \in B} \pi_{jk}$ reaching the bankrupt banks through an intermediary defaulting bank. This term is equal to the i-th component of $\Pi_{D \times D} \Pi_{D \times B} \mathbb{1}_B$, the second element in the sum on the right hand side of (31). Iterating, the amount received by the bankrupt banks after flowing through a chain of p defaulting banks is the i-th component of $\Pi^{(p)}_{D \times D} \Pi_{D \times B} \mathbb{1}_B$. Finally, the total amount received by bankrupt banks is obtained by summing over all p, which gives the right hand side of (31).

An alternative interpretation of expression (31) is in stochastic term. Interpret Π as a transition matrix in which element π_{ij} is the probability of reaching j from i (by definition the sum $\sum_j \pi_{ij}$ is equal to 1). In this interpretation, the element i, j of the matrix $\Pi^{(p)}_{D \times D}$ is the probability of reaching j from i in p steps while staying all along in D, and the i-th component of the vector $\pi_{D \times B} \mathbb{1}_B = (\sum_{k \in B} \pi_{ik})_{i \in D}$ is the probability of reaching in one step an element of B from i. Thus the i-th element of $\Pi^{(p)}_{D \times D} \Pi_{D \times B} \mathbb{1}_B$, which is $\sum_j \pi^{(p)}_{D \times D}(i, j)(\sum_{k \in B} \pi_{ik})$, is the probability of reaching a bankrupt bank for the first time in $p+1$ steps starting from i and staying all along in D, i.e., never reaching a safe nor a bankrupt bank. Such an interpretation of μ could be useful because it allows to rely on standard probability techniques.

4.3 Concluding remarks

We have defined and investigated a threat index in an explicit model of interbank liabilities. The analysis relies on a clearing mechanism for these liabilities in which default may be limited within the banking system or may extend to the outside creditors in case of bankruptcy. The analysis is at the ex post stage. Given the liabilities and the realized values for the banks' net worth, the threat index for a bank computes how cash injection in that bank would modify the (weighted) repayments of the debts of all banks to their creditors both within and outside the banking system. The threat indices may substantially differ from the default levels. As a result, injecting cash into the banks that appear the weakest ones, those with the largest default ratio, may be sub-optimal.

The threat index reflects an externality imposed by a defaulting bank on the debt repayments of all other banks. While the default level of a bank depends on its assets and the safety of its debtors, its threat index depends on its liabilities and the safety of its creditors. A bank thus may not assess properly the externality it will impose on the system if it defaults when it decides on its interbank relationships, since it is concerned with the safety of its debtors and not with that of its creditors. This raises the issue of the adequate regulatory tools. This should be addressed in an ex ante perspective in which the liabilities and the investment decisions, which generate future net worth levels, are chosen.

5 Proofs

Proof of Lemma 1 It suffices to prove the result for one of the matrices. Consider $\mathbb{I} \quad \Pi$. By construction $\sum_{j \in N} \pi_{ij} = 1$ so we have $\sum_{j \in T} \pi_{ij} \quad 1$ for each i in T. If this inequality holds strictly for each i in T, the result follows from well known results on productive matrix (alternatively matrix $\mathbb{I} \quad \Pi$ is dominant diagonal). If the inequality does not hold strictly for each row, we show that an iterate of the matrix has all its sum totals smaller than 1: $\Pi^{(p)}_{T\times T} \mathbb{1}_T \quad \mathbb{1}_T$. The result then follows: Multiplying this inequality by $\Pi_{T\times T}$ yields $\Pi^{(p+1)}_{T\times T} \mathbb{1}_T \quad \Pi_{T\times T} \mathbb{1}_T \quad \mathbb{1}_T$, hence by induction the inequality $\Pi^{(p')}_{T\times T} \mathbb{1}_T \quad \mathbb{1}_T$. holds for all p' larger than p'. This implies that the sum $\mathbb{I}_{T\times T} + \Pi_{T\times T} + \Pi^{(2)}_{T\times T} + ... + \Pi^{(p)}_{T\times T} + ...$ converges. The limit is the inverse of matrix $I \quad \Pi_T$ and is positive.

To show that the inequality $\Pi^{(p)}_{T\times T} \mathbb{1}_T \quad \mathbb{1}_T$ holds for some p, it suffices to show that for each i in T there is some q for which the strict inequality $\sum_{j \in T} \pi^{(q)}_{ij} < 1$: if the inequality holds for q it holds for all q' larger than q (as we have seen above) and p can be taken to be the maximum of the q that work for each i in T.

We argue by contradiction: there is i in T for which the equality $\sum_{j \in T} \Pi^{(q)}_{T\times T}(i,j) = 1$ holds for each q where $\Pi^{(q)}_{T\times T}(i,j)$ denotes the element i, j of the matrix $\Pi^{(q)}_{T\times T}$. Interpret Π as a transition matrix in which element π_{ij} is the probability of reaching j from i. The element $\Pi^{(q)}(i,j)$ is the probability of reaching j from i in q steps and $\Pi^{(q)}_{T\times T}(i,j)$ is the probability of reaching j from i in

q steps while staying all along in T. Thus the equality $\sum_{j \in T} \Pi^{(q)}_{T \times T}(i,j) = 1$ for each q implies that all the paths from i to j are included in T. Let C be composed with all the elements that can be reached from i. By construction, C has no outside creditor. Furthermore, from the argument above, C is included in T. Hence C is a null equity set (as a subset of the null equity set T) and Property 2 gives the desired contradiction: C must have an outside creditor. ∎

Proof of Proposition 1 The program P has a finite solution: the feasible set is non-empty (because it contains $\theta = \mathbf{0}$ since z is positive) and is compact. From well known results on linear programming, the multipliers of the constraints are the solutions to the dual program of P, and furthermore, the values of the primal and dual coincide.

We first explicit the dual. Recall program P:

$$\begin{aligned} \mathsf{P} \quad &: \quad \max_{\theta} \sum_i \ell_i^* \theta_i \\ 0 \quad &\theta_i \quad 1 \text{ each } i \quad (10) \\ \theta_i \ell_i^* \quad \sum_j \theta_j \ell_{ji} \quad &z_i \text{ each } i \quad (11) \end{aligned}$$

Program P is of the form: maximize $\ell^*.\theta$ under $A\theta \quad b$, $\theta \quad 0$ where A is the $2n \quad n$ matrix and b is the $2n$-vector

$$A = \begin{pmatrix} dg(\ell^*) & \ell^t \\ & dg(\ell^*) \end{pmatrix}, \; b = \begin{pmatrix} z \\ \ell \end{pmatrix}$$

Recall that the dual of $\max \ell^*.\theta$ under $A\theta \quad b$ and $\theta \quad 0$ is $\min b.\gamma$ under $A^t\gamma \quad \ell^*$, and $\gamma \quad 0$.

We show that the dual program of P is

$$\begin{aligned} \mathsf{D} \quad &: \min_{(\lambda,\mu) \geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^* \\ (\lambda_i + \mu_i \quad 1)\ell_i^* \quad &\sum_j \ell_{ij}\mu_j \quad 0 \text{ each } i. \end{aligned} \tag{32}$$

and furthermore that the constraints of the dual (32)

Apply the duality theorem to P. Write the $2n$-vector γ as $\begin{pmatrix} \mu \\ \lambda \end{pmatrix}$. The objective of the dual $b.\gamma$ is $\sum_i z_i \mu_i + \sum_i \lambda_i \ell_i^*$ and the constraints

$$\begin{pmatrix} dg(\ell^*) & \ell & dg(\ell^*) \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \quad \ell^*$$

Spelling out the i-th constraint of the dual yields

$$\ell_i^* \mu_i \quad \sum_j \ell_{ij}\mu_j + \ell_i^* \lambda_i \quad \ell_i^*$$

which is (32). These constraints are binding:

$$(\lambda_i + \mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j = 0 \text{ each } i. \tag{33}$$

for each i. By contradiction suppose $(\lambda_i + \mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j > 0$ for some i. If $\lambda_i > 0$, λ_i can be decreased without affecting the other constraints and the objective is decreased. If $\lambda_i = 0$, we have $\mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j > 0$ so μ_i must be strictly positive. Decreasing μ_i is feasible because constraint (32) is not binding for i by assumption and a decrease in μ_i relaxes the constraints for the banks distinct from i. But a decrease in μ_i results in a decrease in the objective, a contradiction again.

Now, let S be the set of safe banks, for which (11) is strict. By the slackness conditions, $\mu_i = 0$ for i in S. Equation (33) immediately gives that the solidarity indices satisfy (18). Let us assume that there are no fragile banks. All banks that are not in S are in D with a repayment ratio strictly smaller than 1. By the slackness conditions, their solidarity indices λ_i are null. Using $\mu_i = 0$ for i in S and $\lambda_i = 0$ for i in D, equations (33) write as $\mu_i \ell_i^* \quad \sum_{j \in D} \ell_{ij}\mu_j = \ell_i^*$: this proves (12). The fact that the system (12) has a unique solution, which is furthermore positive, follows from Lemma 1. ■

Proof of Proposition 3 The sub-modularity of V with respect to z can be proved iteratively by increasing the values of each component of z_i: it suffices to show that given a pair i, j and z_{-ij} a payoff vector for banks other than i and j, V satisfies

$$V(z_i', z_j', z_{-ij}) \quad V(z_i, z_j', z_{-ij}) \quad V(z_i', z_j, z_{-ij}) \quad V(z_i, z_j, z_{-ij}) \text{ for } z_i' \quad z_i, z_j' \quad z_j.$$

For a differentiable function V, the sub-modularity is satisfied if the partial derivative $\frac{\partial V}{\partial z_i}$ is non-increasing in z_j: This follows from the following expression:

$$V(z_i', z_j, z_{-ij}) \quad V(z_i, z_j, z_{-ij}) = \int_{z_i}^{z_i'} \frac{\partial V}{\partial z_i}(t, z_j, z_{-ij})dt. \tag{34}$$

Here V is not differentiable at all points but the partial derivative of V exists almost everywhere, $\frac{\partial V}{\partial z_i}(t, z_j, z_{-ij})$ is given by the unique multiplier μ_i at (t, z_j, z_{-ij}) when there is no fragile bank, which is the case but for a finite number of points as t runs in the interval (z_i, z_i'). Since V is continuous, the integral expression (34) is still valid. V is thus sub-modular if for each i the multiplier μ_i at (t, z_j, z_{-ij}) is a non-decreasing function of z_j, j distinct from i for fixed t and fixed z_{-ij} (actually μ_i is also decreasing in z_i from the concavity of V)

Let us start at z_j and increase it progressively up to z_j'. We use a $'$ to denote values corresponding to z_j'.

Consider D the set of defaulting banks at z. Assume not fragile bank. The clearing ratio vector θ' is given by (9), $(dg(\ell^*) \quad \ell)^t_{D\times D}\theta'_D = \hat{z}'_D$, as long as each θ'_i for i in D is strictly less than 1. θ' is a non-decreasing function of the z_j because the inverse of $(dg(\ell^*) \quad \ell)^t_{D\times D}$ is a positive matrix by Lemma 1. Hence $\theta' \quad \theta$ (for the banks not in D their ratios is constant equal to 1). Since D remains the set of defaulting banks, the index μ' satisfies the same (invertible) system (12): μ' is equal to μ. As z_j is increased further, some clearing ratios for i in D may hit the upper bound 1, making them fragile. Increasing by any small amount z_j, these banks become safe and the set D is reduced to some D': $D' \quad D$.

The multiplier μ is null outside D and μ' is null outside D' so it suffices to show that $\mu'_{D'} \quad \mu_{D'}$ to prove $\mu' \quad \mu$. The vectors μ_D and $\mu'_{D'}$ are given respectively by $(dg(\ell^*) \quad \ell)_{D\times D}\mu_D = \ell^*_D$ and $(dg(\ell^*) \quad \ell)_{D'\times D'}\mu'_{D'} = \ell^*_{D'}$. The matrix $(dg(\ell^*) \quad \ell)_{D\times D}$ restricted to the rows indexed by elements in D' is $(dg(\ell^*) \quad \ell)_{D'\times D} = (dg(\ell^*) \quad \ell)_{D'\times D'}, \ \ell_{D'\times(D-D')})$. Thus the equation $(dg(\ell^*) \quad \ell)_{D\times D}\mu_D = \ell^*_D$ restricted to the rows indexed by elements in D' writes

$$(dg(\ell^*) \quad \ell)_{D'\times D'}\mu_{D'} \quad \ell_{D'\times(D-D')}\mu_D = \ell^*_{D'}.$$

The above equality implies $(dg(\ell^*) \quad \ell)_{D'\times D'}\mu_{D'} \quad \ell^*_{D'}$. Hence $\mu_{D'} \quad (dg(\ell^*) \quad \ell)^{-1}_{D'\times D'}\ell^*_{D'}$ because the inverse of $M_{D'\times D'}$ exists and is non-negative (thanks to Lemma 1). By definition $(dg(\ell^*) \quad \ell)^{-1}_{D'\times D'}\ell^*_{D'}$ is equal to $\mu'_{D'}$ so we obtain $\mu'_{D'} \quad \mu_{D'}$, the desired result. ∎

Proof of Proposition 4 The existence of a clearing ratio vector relies on the feasible set introduced in the text: A vector θ in $[0,1]^n$ is said to be ***feasible*** if it satisfies $\theta_i\ell^*_i \quad \max(a_i(\theta), 0)$ for each bank i. The limited liability and the bankruptcy constraints are satisfied. To prove that a maximal element θ satisfies the absolute priority rule condition, we only need to consider a bank with a positive asset. Because increasing i's ratio does not affect its asset value and can only increase other banks' asset values, if i's ratio is less than 1, it must be that $\theta_i\ell^*_i$ is equal to $\max(a_i(\theta), 0)$. Since the value $a_i(\theta)$ is positive, the equality writes $\theta_i\ell^*_i = a_i(\theta)$. So either θ_i is equal to 1 or equity is null: the creditor's absolute priority rule is satisfied.

We first prove that the net equity of each bank is the same at each clearing ratio vector, in case of multiplicity. For that, it suffices to compare the values of net equity with those for the greatest clearing repayment ratio vector. Let θ^+ be the greatest clearing repayment ratio vector, and θ be another clearing repayment ratio vector. Recall that $e_i(\theta^+) = a_i(\theta^+) \quad \theta^+_i\ell^*_i$ and $a_i(\theta^+) = z_i + \sum_j \theta^+_j\ell_{ji}$ denote respectively i's equity and asset values at vector θ^+, and similarly $e_i(\theta)$ and $a_i(\theta)$ at θ. Note that $a_i(\theta^+) \quad a_i(\theta)$ each i since $\theta^+ \quad \theta$. To prove the equalities $e_i(\theta^+) = e_i(\theta)$ for each i, it is enough to show

$$e_i(\theta^+) \quad e_i(\theta) \text{ for each } i \tag{35}$$

thanks to the aggregation formula: Since the sum of the net equity values over all banks is equal to the sum of their net worth values whatever repayment ratio vector, summing all inequalities (35) over i implies that the right and left hand sides are equal, hence each inequality is binding: net equity values are the same under both repayment ratio vectors.

To prove (35), we consider various cases depending on the values of θ_i and θ^+_i. (The first two cases are the same as in Eisenberg and Noe 2000).

For $\theta_i = 1$, θ^+_i is also equal to 1, thus inequality $e_i(\theta^+) \quad e_i(\theta)$ follows from the inequality $a_i(\theta^+) \quad a_i(\theta)$ and $\theta^+_i = \theta_i$.

For $0 < \theta_i < 1$, i's net equity under θ is null; since surely $0 < \theta^+_i$, i's net equity under θ^+ can only be non-negative, so $e_i(\theta^+) \quad e_i(\theta) = 0$.

For $\theta_i = 0$, $a_i(\theta) = e_i(\theta)$, and $a_i(\theta)$ cannot be positive by absolute priority. Hence $0 \quad e_i(\theta)$. If $\theta_i^+ > 0$, $e_i(\theta^+) \quad 0 \quad e_i(\theta)$; if $\theta_i^+ = 0$, net equities are given by the net asset values, hence again $e_i(\theta^+) \quad e_i(\theta)$.

Let us now consider the possibility of multiple clearing vectors and prove the uniqueness under the assumptions stated in the proposition. Recall that there is a greatest clearing ratio θ^+. Let θ be another clearing repayment ratio vector. To prove uniqueness, it is sufficient to show that the non-negative vector $\theta^+ \quad \theta$ is null. Let T be the set for which $\theta_i^+ \quad \theta_i > 0$. Because net equities are identical across clearing repayment ratio vectors, bankrupt banks (with a negative equity value) and safe banks (with a positive equity value) coincide at any clearing ratio vector. Hence $\theta_i^+ \quad \theta_i > 0$ can only hold for banks with null net equity: T is a null equity set. Furthermore, under the assumption that aggregate net worth is not null, there are surely some bankrupt banks (if the flow is negative) or safe banks (if the flow is negative): T is not the whole set N. We show that no bank in T has a creditor outside T.

The vector of banks' net equities is given by $e(\theta) = z \quad (dg(\ell^*) \quad \ell)^t\theta$ at a repayment ratio vector θ. Since net equities are identical across clearing repayment ratio vectors, we have $(dg(\ell^*) \quad \ell)^t(\theta^+ \quad \theta) = 0$. The aggregation formula over the null equity set T implies $e_i(\theta^+) = e_i(\theta^+) = 0$, which yields

$$\sum_{j \notin T, i \in T} (\theta_j^+ \quad \theta_j)\ell_{ji} = \sum_{j \notin T, i \in T} (\theta_i^+ \quad \theta_i)\ell_{ij}$$

By definition, the repayment ratios differ on T and only on T: $\theta_i^+ \quad \theta_i$ is null for i in $N \quad T$ and positive for i in T. So the left hand side is null, and the nullity of right hand side requires $\ell_{ij} = 0$ for i in T and j not in T, i.e., no bank in T has a creditor outside T.

Uniqueness follows in the following cases.

(1) The liability matrix ℓ is irreducible. In that case any subset T has an outside creditor.[20]

(2) There is a universal creditor. The universal creditor has surely a positive equity hence is an outside creditor of a null equity set. ∎

Proof of Proposition 6 Write the program Q_c as

$$\begin{aligned} Q_c \quad : \quad & \max_{\theta, \delta \geq 0} [\sum_i \ell_i^* \theta_i] \quad c[\sum_i \delta_i] \\ \theta_i \ell_i^* \quad \sum_j \theta_j \ell_{ji} \quad & z_i + \delta_i \text{ each } i \quad (27) \\ & \theta_i \ell_i^* \quad \ell_i^* \text{ each } i. \end{aligned}$$

In matrix form the objective is $l^*.\theta \quad c\mathbb{1}.\delta$ and the constraints write as $A \begin{pmatrix} \theta \\ \delta \end{pmatrix} \quad b$ where A is the

[20] An alternative proof, less elaborate, is to apply Perron-Frobinius theorem. The identity $(dg(\ell^*) - \ell)^t(\theta^+ - \theta) = 0$ that the vector $\theta^+ - \theta$ is a non-negative eigenvector of $dg(\ell^*) - \ell)^t$ associated to the largest eigenvalue 1. Applying Perron-Frobinius theorem to the irreducible matrix, $\theta^+ - \theta$ is a strictly positive vector, in contradiction with T a strict subset of N.

$2n \quad 2n$ matrix and b is the $2n$-vector

$$A = \begin{pmatrix} dg(\ell^*) & \ell^t & I \\ & dg(\ell^*) & 0 \end{pmatrix}, \; b = \begin{pmatrix} z \\ \ell \end{pmatrix}.$$

Applying the duality theorem as in the proof of Proposition 1, the dual is

$$\mathsf{D}_c \quad : \min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j \quad 0 \text{ each } i \tag{36}$$

$$\mu_i \quad c \text{ each } i. \tag{37}$$

From well known results, if (θ, δ) and (λ, μ) are feasible respectively for the primal and the dual, and satisfy the complementary slackness conditions, each one is a solution respectively for the primal and the dual. The slackness conditions are (using notation $\delta_i + e_i(\theta) \quad 0$ for nequality (27))

$$\begin{cases} \theta_i = 0 & \text{or} & (\lambda_i + \mu_i \quad 1)\ell_i^* \quad \sum_j \ell_{ij}\mu_j = 0 \\ \theta_i = 1 & \text{or} & \lambda_i = 0 \\ \delta_i = 0 & \text{or} & \mu_i = c \\ \delta_i + e_i(\theta) = 0 & \text{or} & \mu_i = 0 \end{cases}$$

Consider a solution (θ, δ) to Q_c. We prove that, for c large enough, θ is a clearing repayment ratio vector under the assumptions stated in the proposition, i.e., the positivity of all interbank liabilities of ℓ and the positivity of aggregate net worth. Define

$$S = fi, \delta_i = 0 \text{ and } e_i(\theta) > 0g, \; B = fi, \delta_i > 0g \text{ and } D = N \quad B \; [\; S.$$

For a bank in B, i's constraint (27) must be binding, $\delta_i + e_i(\theta) = 0$, because otherwise the objective of Q_c could be increased. Thus i's net equity is strictly negative.

Note that D is the set of banks i for which δ_i is null (because i is not in B) and net equity $e_i(\theta)$ is null (because otherwise i would be in S): $D = fi, \delta_i = 0 \text{ and } e_i(\theta) = 0g$.

From this, we have that $e_i(\theta)$ is non-positive for each bank in B or in D. Under the assumption of a positive aggregate net worth, this implies that S is non-empty, thanks to the aggregation formula (which holds whatever ratio).

We check that θ is a clearing ratio vector with S as the set of safe banks and B as the set of bankrupt banks.

S is the set of banks for which injection is null and equity is strictly positive. We need to check that the ratio of each bank i in S is 1. This is clearly true: otherwise the ratio can be increased while still satisfying all constraints, thereby increasing the objective

A bank in D has $e_i(\theta) = 0$, so the clearing conditions are satisfied for it whatever its ratio.

B is the set of banks for which injection is positive. We have to prove that i's ratio is null. In other words we need to exclude the situation where the bank repays some debts and receives some cash, i.e. both $\theta_i > 0$ and $\delta_i > 0$. By contradiction, if $\theta_i > 0$, then $\lambda_i + \mu_i \quad 1 \quad \sum_j \pi_{ij}\mu_j = 0$ (by the first slackness conditions and working on relative liabilities). $\delta_i > 0$ implies $\mu_i = c$ by the third slackness condition. Hence $c \quad 1 \quad \sum_j \pi_{ij}\mu_j$ because λ_i is non-negative. Let $\underline{\pi}$ be the minimum value of the off-diagonal elements π_{ij}. We show that $\sum_j \pi_{ij}\mu_j \quad c(1 \quad \underline{\pi})$. First each μ_j is not greater than c; second there is surely a bank k that is in S hence for which μ_k is null; using $\pi_{ik} \quad \underline{\pi}$ we obtain $\sum_j \pi_{ij}\mu_j \quad c(1 \quad \underline{\pi})$. Hence

$$c \quad 1 \quad c(1 \quad \underline{\pi})$$

which is impossible for c large enough. This ends the proof. ■

References

Adrian T. and M.K. Brunnermeier (2008) CoVaR, Sta Reports 348, Federal Reserve Bank of New York.

Allen, F. and D. Gale (2000) \Financial contagion", Journal of Political Economy, 108, 1-33.

Ballester C., A. Calvo-Armengol, and Y. Zenou (2006) "Who's Who in Networks. Wanted: The Key Player," Econometrica, vol. 74(5), pages 1403-1417.

Bonacich P. (1987) Power and centrality: a family of measures, American Journal of Sociology, 1987 - Vol 92, pp. 1170{1182.

Cifuentes R., G. Ferrucci and H. S. Shin (2004) Liquidity Risk and Contagion, Journal of the European Economic Association, 3 (2-3) 556-566.

Degryse H. and G. Nguyen (2004) Interbank Exposures: An Empirical Analysis of Systemic Risk in the Belgian Banking System, National Bank of Belgium, Working Paper.

Diamond D.W and Dybvig P.H (1983). "Bank runs, deposit insurance, and liquidity". Journal of Political Economy 91 (3): 401{419.

Eisenberg L. and T.H. Noe (2001) Systemic Risk in Financial Systems, Management Science, Vol. 47, No. 2, pp. 236-249

Elsinger H., A. Lehar, and M. Summer (2004) Risk assessment for banking systems, working paper, University of Vienna.

Freixas X., B. Parigi and J.-C. Rochet (2000) Systemic Risk, Interbank Relations, and Liquidity Provision by the Central Bank, Journal of Money, Credit and Banking, Vol. 32, No. 3.

Fur ne C.H. (2003) Interbank Exposures: Quantifying the Risk of Contagion, Journal of Money, Credit and Banking

Gai P. and S.Kapadia (2008) Contagion in Financial Networks, Discussion Paper 2008-1231, Bank of England.

Katz L. (1953) A new status index derived from sociometric analysis. Psychometrika, 18:39{43.

Shin H. S. (2008) Risk and liquidity in a system context, Journal of Financial Intermediation Volume 17, Issue 3, July 2008, Pages 315-329.

Tarashev, N.A., C. E. Borio, and , K. Tsatsaronis (2010) Attributing Systemic Risk to Individual Institutions, BIS Working Paper No. 308. Available at SSRN: http://ssrn.com/abstract=1631761

Upper C. and A. Worms (2004) Estimating bilateral exposures in the German interbank market: Is there a danger of contagion? European Economic Review 48 (2004) 827 { 849.