REVEALED PREFERENCE TESTS OF THE COURNOT MODEL

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Abstract: The aim of this paper is to develop revealed preference tests for Cournot equilibrium. The tests are akin to the widely-used revealed preference tests for consumption, but have to take into account the special features of the oligopoly setting. These include the presence of strategic interaction, multiple and simultaneous changes to the underlying environment (shifts to both demand and cost functions), and the possibility of non-concave objective functions. The tests take the form of linear programs, the solutions to which also allow us to recover cost information on the firms. To check that these nonparametric tests are sufficiently discriminating to reject real data, we apply them to the market for crude oil.

Keywords: nonparametric test, observable restrictions, linear programming, multi-product Cournot oligopoly, collusion, crude oil market.

1. Introduction

In a seminal paper, Afriat (1967) identi ed the conditions that are necessary and su cient for a nite set of observations of price vectors and a demand bundles to be consistent with utility-maximizing behavior. A large literature on consumer behavior { both theoretical and empirical { has been built on Afriat's result, and an important reason for the success of this `revealed preference' approach is that the characterizing conditions take the form of a linear program that can be easily solved.

A natural extension of Afriat's theorem is to derive observable restrictions on outcomes in a general equilibrium setting. Brown and Matzkin (1996) consider a nite data set drawn from an exchange economy, where each observation consists of the aggregate endowment, an

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equilibrium price vector, and the nominal income distribution. They find the precise conditions under which these observations are consistent with Walrasian equilibria in an economy with utility-maximizing agents, where agents' endowments may change across observations while their utility functions are held fixed.

Questions analogous to that of Brown and Matzkin can be posed in other multi-agent contexts. Given the ubiquity of revealed preference methods in the study of consumer behavior, it is natural to explore a similar approach in the empirical study of oligopoly. This paper is an attempt at doing that. The oligopoly setting, however, presents challenges that are not present in the standard consumer problem: the presence of strategic interaction, multiple and simultaneous changes to the underlying environment (such as shifts to both demand and cost functions), and the possibility of non-concave objective functions, among others. Nonetheless, these difficulties can be overcome and we show that revealed preferences methods are useful for testing oligopoly behavior and for recovering certain firm-level characteristics (such as cost functions).[2]

Revealed preference tests of equilibrium in oligopoly models are useful for at least two reasons. The first is an advantage of revealed preference tests in general. Such tests are useful as "pre-tests" in the sense that they allow the researcher to test, with minimum ancillary assumptions, whether a given data set is compatible with the choice of model, before specific parameters are introduced and estimated. The second important advantage of such tests is peculiar to the oligopoly context. While the set of equilibria in a repeated game is large, if firms are not colluding, they ought to be playing the static Nash equilibrium at each time period. The ability to test – and thus potentially to *reject* – the static equilibrium, is thus useful to antitrust authorities as it provides evidence that firm interaction is taking a more complicated, and possibly collusive form. In addition, the revealed preference tests we develop have the advantage of not requiring detailed information on market demand and firms' production technologies, information which is often unavailable to antitrust authorities.

[2] The techniques developed in this paper can potentially be adapted to other models of oligopoly. One potential application is to models of differentiated product Bertrand oligopoly. Revealed preference tests of this model may have some of the features of the tests of differentiated product Cournot oligopoly we discuss in Section 7. We leave this for future work.

1.1. Testing the Cournot hypothesis and recovering cost functions

Consider a finite set of observations of an industry producing a single good; each observation consists of the price of the good and the output of each firm. We study whether there are any observable restrictions (i.e., restrictions on the data set) implied by the following *Cournot hypothesis*: that each observation of prices and quantities is a Cournot equilibrium, each firm has a convex cost function that does not vary across observations, and the data is generated by changes to a downward-sloping demand function.

In keeping with the revealed preference approach, we make no parametric assumptions about demand and cost functions and the object is to develop tests that are the most powerful possible given the data (in the sense of being both necessary and sufficient). Notice also that the observed data is extremely parsimonious; in particular, we do not assume that the observer has any knowledge of any variables that shift or twist the demand curves.

In Section 2, we show that the Cournot hypothesis *does* impose observable restrictions on the data and the hypothesis is satisfied if, and only if, there is a solution to a particular linear program constructed from the data. Furthermore, the set of solutions to this linear program characterize the set of cost curves for each firm that are consistent with the data, so the observer not only tests for Cournot behavior, but also recovers cost information in the event that the data passes the test. This basic test can be extended in several ways, which we now discuss briefly.

(1) While convex cost functions are widely assumed, in some cases this assumption may not hold because of economies of scale or other reasons. We show in Section 5 that the possibility of non-convex cost curves does not remove observable restrictions on a data set, provided that each firm's cost function satisfies a `convincing' criterion, which, in essence, says that the firm's marginal cost curve is not `too wriggly' between the firm's observed output levels; this guarantees that information about a firm's marginal cost at observed output levels (inferred from its market share) conveys information about its marginal cost in the non-infinitesimal region *between* observed output levels.[3]

The possibility of non-convex costs means that the firm's profit maximization problem

[3] If a firm has increasing marginal cost, its marginal cost at an observed output level is a lower bound on marginal cost at all higher output levels, which is an instance of the convincing criterion being satisfied.

need not be quasiconcave. This makes our result quite unusual since most revealed preference models typically rely on convex analysis and so rely, in some form, on concavity or convexity assumptions.[4] Similarly, econometric analyses that recover model parameters (like firm conduct parameters, see below) through first-order conditions also rely on the quasi-concavity of the optimization problem; otherwise, there is no guarantee that observations satisfying the first-order conditions are in fact globally optimal with respect to the objective functions recovered from the econometric estimation.

(2) In the revealed preference analysis of consumer demand, utility functions are usually assumed to be unvarying because it is not always clear how these changes should be modeled. On the other hand, it is quite natural in our context to think of a firm's objective function changing through changes to its cost function (perhaps because of changes to input prices). We show in Section 6 that our tests of Cournot rationalizability can be naturally extended to incorporate data that is known to increase or reduce a firm's marginal cost function.

(3) In Section 4, we consider an oligopoly where firms produce multiple goods (and there is at least one good which is sold by more than one firm). These goods could be different in the usual sense, but another interpretation is that the different goods refer in fact to (materially) the same product sold in distinct markets, where they may fetch different prices. We develop a test for Cournot rationalizability that is applicable to this multi-product context.

(4) In Section 7, we show that our tests of Cournot rationalizability can be adapted to allow us to evaluate the statistical significance of violations of the revealed preference tests, following the approach suggested by Varian (1985).

(5) In the Supplement to this paper, we provide a test of the Cournot hypothesis in the case when the firms in the industry are producing differentiated goods. In contrast with the multi-product model of Section 4 (point 3 above), no two firms are selling exactly the same good. Nonetheless, it is possible to develop a revealed preference test provided we incorporate assumptions on how the demand function for one firm's output is related to the demand function for another firm's output.

(6) In certain settings, we may be interested in testing the hypothesis that a sub-group of firms in an industry is acting collusively, while playing a Cournot game with firms outside the group. The Supplement contains a revealed preference test for this hypothesis.

[4] An exception is Forges and Minelli's (2008) extension of Afriat's Theorem.

1.2. Estimating conduct parameters and testing for collusion

The principal objective of this paper is to devise a test for the Cournot model in which rationalizing cost functions can be chosen from a very large class and no assumptions are made on the evolution of demand across observations. For reasons that we will make clear, this is related to, but distinct from, testing the degree of competitiveness or collusion in an industry. There is, of course, a large empirical IO literature dealing with the latter issue, often through the measurement of conduct parameters (where a particular value of this parameter corresponds to Cournot behavior, and other values signify more or less competitive behavior).

We show that it is *not* possible (via revealed preference arguments) to estimate or even to bound a firm's conduct parameter if the observer only has access to a data set consisting of clearing prices and firm-level outputs. This observation is consistent with the results of Bresnahan (1982) and Lau (1982), who show that identifying the conduct parameter of a firm requires an observer to know how demand changes with some observable variables and also that the changes be generated by at least a two-variable family; hence empirical IO studies typically estimate the demand function alongside estimating the conduct parameters. Our tests of Cournot rationalizability avoid having to do this by exploiting the fact that, while the conduct parameter of a particular firm cannot be determined without greater information on demand, the Cournot equilibrium implies that *conduct parameters are equal amongst firms* and equality can be tested without information on demand.

There is a natural way to extend our methods to obtain bounds on conduct parameters (and hence test for collusion) provided more information is incorporated. In particular, we show that if the observer has information that leads to bounds on the price elasticity of demand at each observation, then it is possible to test whether firms' conduct parameters fall within any given interval. Tests of this sort are potentially useful to anti-trust authorities. These issues are formally addressed in Section 3.

1.3. Application to the market for crude oil

A possible reason why there have been few attempts to derive revealed preference tests for game-theoretic models and models with externalities in general is that the presence of externalities often means that the set of permissible preferences for each agent in the model

could be very large, which leads to very weak testable restrictions. In particular, in principle it is not clear that the tests of the Cournot model we have constructed have the power to reject real data. In Section 8, we apply our tests to the oil-producing countries both within and outside of OPEC. We test for Cournot rationalizability with convex cost functions and also with cost functions obeying the convincing criterion. The former hypothesis is clearly rejected by the data. With the latter the outcome is more mixed, but it is clear that this test is also discriminating.

1.4. Related literature

Brown and Matzkin's result has been extended to take into account, for example, financial markets (Kubler, 2003), random preferences (Carvajal, 2004), and externalities (Deb (2009) and Carvajal (2010)). Forges and Minelli (2008) extends Afriat's Theorem to constraint sets which need not be classical budget sets and need not even be convex; the authors point out that their results can be applied to games in which each player's constraint set is dependent on the actions of other players. A Cournot game belongs to this class, since the output decisions of other firms affect each firm's residual demand curve, which can be thought of as a constraint over which the firm chooses its output and the market price. However, unlike in the Forges and Minelli's setup, the market demand function (and hence each player's residual demand curve) is not fully specified as part of the data in our context. Furthermore, each player's objective function is a profit function, which has a specific functional form.

The testable implications of equilibrium behavior in abstract games have been investigated by Sprumont (2000) and Ray and Zhou (2001). These papers differ from ours in two ways. Firstly, in their work, payoff functions remain fixed and the variability in the data arises from each player choosing actions from different subsets of his available strategies. On the other hand, our paper is most naturally understood as one in which the payoff functions are changing across observations (because of changes to demand). A second difference is that they assume that observations are *complete*. In other words, the map from the available strategies (of each player) to strategy choices is fully specified; the conditions under which this maps strategy sets to Nash equilibria are then identified. Any restrictions found in such a context must remain necessary when this map is only partially known (in the sense that

one knows the outcomes at some but not all strategy subsets), but they may no longer be sufficient (see, for example, Section 4.1 in Sprumont (2000)). The problem we consider is analogous to the case where only part of this map is known, since we observe industry outcomes for some but not all possible demand functions. Nonetheless, we can find observable restrictions that are not just necessary, but also sufficient, for Cournot rationalizability.

2. Cournot Rationalizability

An industry consists of I firms producing a homogeneous good; we denote the set of firms by $\mathcal{I} = \{1, 2, \ldots, I\}$. Consider an experiment in which T observations are made of this industry. We index the observations by $t \in \mathcal{T} = \{1, 2, \ldots, T\}$. For each t, the industry price P_t and the output of each firm $(Q_{i,t})_{i\in\mathcal{I}}$ are observed, so the data set can be written as $\{P_t, (Q_{i,t})_{i\in\mathcal{I}}\}_{t\in\mathcal{T}}$. We require $Q_{i,t} > 0$ for all (i, t), and denote the aggregate output of the industry at observation t by $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$.

We say that the date set is Cournot rationalizable if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations, and with the demand and cost functions obeying certain regularity properties. By a *cost function* of firm i, we mean a strictly increasing function $C_i : \mathbb{R}_+ \to \mathbb{R}$ satisfying $C_i(0) = 0$. The market inverse demand function $\mathcal{P}_t : \mathbb{R}_+ \to \mathbb{R}$ (for each t) is said to be *downward sloping* if it is differentiable at all $q > 0$, with $\mathcal{P}_t'(q) < 0$. Formally, the data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is *Cournot rationalizable* if there exist cost functions C_i for each firm i and downward sloping demand functions $\mathcal{P}_t$ for each observation t such that

(i) $\mathcal{P}_t(Q_t) = P_t$; and

(ii) $Q_{i,t} \in \text{argmax}_{q_i \geq 0}\{q_i \mathcal{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - C_i(q_i)\}$.

Condition (i) says that the inverse demand function must agree with the observed data at each t. Condition (ii) says that, at each observation t, firm i's observed output level $Q_{i,t}$ maximizes its profit given the output of the other firms.[5]

A standard assumption made in theoretical and econometric work is that cost functions

[5] Note that in any Cournot rationalizable data set, the observed prices P_t must be strictly positive, because we assume that observed output is nonzero and firms' costs are strictly increasing in output.

are convex, which helps to make the optimization problem tractable, and is not an implausible assumption in many settings. Our main goal in this section is to characterize the data sets that are Cournot rationalizable with convex cost functions. It is not obvious that such a condition imposes any restrictions on the data, so we first demonstrate that it does.

Suppose that $\{[P_t, (Q_{i,t})_{i \in I}]\}_{t \in T}$ is rationalized by demand functions $\{P_t\}_{t \in T}$ and cost functions $\{C_i\}_{i \in I}$, and denote by $C_i'(Q_{i,t})$ the set of subgradients of C_i at $Q_{i,t}$. At observation t, firm i chooses q_i to maximize its profit given the output of the other firms; at its optimal choice, $Q_{i,t}$, the first-order conditions say that there is $\delta_{i,t} \in C_i'(Q_{i,t})$ such that

$$Q_{i,t}P_t'(Q_t) + P_t(Q_t) - \delta_{i,t} = 0.$$

Using Condition (i) above, it follows that the array $\{\delta_{i,t}\}_{i \in I}$ must obey the following condition, which we shall refer to as the *common ratio property*: for every $t \in T$,

$$\frac{P_t - \delta_{1,t}}{Q_{1,t}} = \frac{P_t - \delta_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{Q_{I,t}} > 0. \tag{1}$$

This holds because the first-order condition guarantees that $(P_t - \delta_{i,t})/Q_{i,t} = -P_t'(Q_t)$, and the latter is positive and independent of i. With the common ratio property we could recover information about a firm's marginal cost without directly observing it; when combined with the convexity of the cost function, it allows us to conclude that certain observations are not rationalizable, as we show in the following examples.

Example 1. Suppose that at observation t, firm i produces 20 and firm j produces 15. At another observation t', firm i produces 15 and firm j produces 16. We claim that these observations are not Cournot rationalizable with convex cost functions. Suppose, to the contrary, that they are. In that case, by the common ratio property, observation t tells us that there are $\delta_{i,t} \in C_i'(20)$ and $\delta_{j,t} \in C_j'(15)$ such that $\delta_{i,t} < \delta_{j,t}$. At observation t', firm i produces 15, which is less than its output at t; since C_i is convex, $C_i'(15) \leq \delta_{i,t}$ (by this we mean that $\delta_{i,t}$ is weakly greater than every element in $C_i'(15)$). Similarly, the convexity of C_j guarantees that $C_j'(16) \geq \delta_{j,t}$ since firm j's output at t' is higher than its output at t. Putting these together, we obtain $C_i'(15) \leq \delta_{i,t} < \delta_{j,t} \leq C_j'(16)$, but this violates the common ratio property since it means that at observation t', firm j has larger output *and* higher marginal cost compared to i.

Notice that Example 1 does not even rely on price information, so the mere observation of firm-level outputs can, in principle, contradict the Cournot hypothesis. In this example, the firms change ranks - the larger firm becomes smaller in another observation and also the outputs of the two firms are not moving co-monotonically. The next example is one in which the firms do not switch ranks and output movements are co-monotonic, but it is still not Cournot rationalizable.

Example 2. Consider the following observations of two firms i and j:
(i) at observation t, $P_t = 10$, $Q_{i,t} = 50$ and $Q_{j,t} = 100$;
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 60$ and $Q_{j,t'} = 110$.
We claim that these observations are not Cournot rationalizable with convex cost functions. Indeed, if they are, then there is $\delta_{i,t} \in C_i'(Q_{i,t})$ and $\delta_{j,t} \in C_j'(Q_{j,t})$ such that

$$\delta_{i,t} = P_t - (P_t - \delta_{j,t})\frac{Q_{i,t}}{Q_{j,t}} \geq P_t\left(1 - \frac{Q_{i,t}}{Q_{j,t}}\right), \tag{2}$$

where the equality follows from the common ratio property, and the inequality from the assumption that marginal cost is positive. Substituting in the numbers given, we obtain $\delta_{i,t} \geq 5$, where $\delta_{i,t} \in C_i'(50)$. Since firm i has increasing marginal costs, $\delta_{i,t'} \geq 5$ for any $\delta_{i,t'} \in C_i'(Q_{i,t'})$, but this cannot be since it exceeds $P_{t'} = 4$.

The next theorem is the main result of this section and shows that a set of observations is Cournot rationalizable with convex cost functions if, and only if, there is a solution to a certain linear program constructed from the data.

Theorem 1. *The following statements on* $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ *are equivalent:*
[A] *The set of observations is Cournot rationalizable with convex cost functions.*
[B] *There exists a set of positive numbers* $\{\delta_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ *satisfying the common ratio property and such that, for each* i, $\{\delta_{i,t}\}_{t \in \mathcal{T}}$ *is increasing with* $Q_{i,t}$ *(in the sense that* $\delta_{i,t'} \geq \delta_{i,t}$ *whenever* $Q_{i,t'} > Q_{i,t}$*).*

It is worth emphasizing that Theorem 1 is useful even in situations where the output of one or more firms is missing from the data set. This is because if all of the firms in an industry are playing a Cournot game, then any subset of firms whose outputs *are* observed must also be playing a Cournot game (with the residual demand function as their `market'

demand function), and the latter hypothesis can be tested using the theorem.[6]

Our proof of Theorem 1 uses two lemmas; the first one provides an explicit construction of the demand curve needed to rationalize the data at any observation t, while the second lemma provides a way of constructing a cost curve for each firm obeying stipulated conditions on marginal cost.

LEMMA 1. *Suppose that, at some observation t, there are positive scalars $\{\delta_{i,t}\}_{i \in I}$ such that equation (1) is satisfied and that there are convex cost functions C_i with $\delta_{i,t} \in C_i'(Q_{i,t})$. Then, there exists a downward-sloping demand function P_t such that $P_t(Q_t) = P_t$ and, with each firm i having the cost function C_i, $\{Q_{i,t}\}_{i \in I}$ constitutes a Cournot equilibrium.*

Proof: We define P_t by $P_t(Q) = a_t - b_t Q$, where $b_t = [P_t - \delta_{i,t}]/Q_{i,t}$ – notice that this is well-defined because of (1) – and choosing a_t such that $P_t(Q_t) = P_t$. Firm i's decision is to choose $q_i \geq 0$ to maximize $\Pi_{i,t}(q_i) = q_i P_t(q_i + \sum_{j \neq i} Q_{j,t}) - C_i(q_i)$. This function is concave, so an output level is optimal if and only if it obeys the first-order condition. Since $\delta_{i,t} \in C_i'(Q_{i,t})$ and since $P_t'(Q_t) = -b_t$, a supergradient[7] of $\Pi_{i,t}$ at $Q_{i,t}$ is

$$Q_{i,t} P_t'(Q_t) + P_t(Q_t) - \delta_{i,t} = -Q_{i,t} \frac{[P_t - \delta_{i,t}]}{Q_{i,t}} + P_t - \delta_{i,t} = 0.$$

So we have shown that $Q_{i,t}$ is profit-maximizing for firm i at observation t. *Q.E.D.*

LEMMA 2. *Suppose that for some firm i, there are positive scalars $\{\delta_{i,t}\}_{t \in T}$ that are increasing with $Q_{i,t}$. Then there exists a convex cost function C_i such that $\delta_{i,t} \in C_i'(Q_{i,t})$.*

Proof: Define $\hat{Q} = \{q_i \in \mathbb{R}_+ : q_i = Q_{i,t}$ for some observation $t\}$; $\hat{Q}$ consists of those output levels actually chosen by firm i at some observation. Since $\{\delta_{i,t}\}_{t \in T}$ are increasing with $Q_{i,t}$ it is possible to construct a strictly positive and increasing function $m_i : \mathbb{R}_+ \to \mathbb{R}$ with the following properties: (a) for any output $\hat{q} \in \hat{Q}$, set $m_i(\hat{q}) = \max\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$; (b) for any $\hat{q} \in \hat{Q}$, $\lim_{q \to \hat{q}^-} m_i(q) = \min\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$; and (c) m_i is continuous at all $q \notin \hat{Q}$. The function m_i is piecewise continuous with a discontinuity at $\hat{q} \in \hat{Q}$ if, and only if, the set $\{\delta_{i,t} : Q_{i,t} = \hat{q}\}$ is non-singleton. Define $C_i : \mathbb{R} \to \mathbb{R}$ by

$$C_i(q) = \int_0^q m_i(s)\, ds. \tag{3}$$

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This function is strictly increasing because m_i is strictly positive and it is convex because m_i is increasing. Lastly, (a) and (b) guarantee that $\delta_{i,t} \in C_i'(Q_{i,t})$. *Q.E.D.*

Proof of Theorem 1: To see that [A] implies [B], suppose that the data is rationalized with demand functions $\{P_t\}_{t\in T}$ and cost functions $\{C_i\}_{i\in I}$. We have already shown that the first-order condition guarantees the existence of $\delta_{i,t} \in C_i'(Q_{i,t})$ obeying the common ratio property (1). Since C_i is convex, $\{\delta_{i,t}\}_{t\in T}$ is increasing with $Q_{i,t}$.

The fact that [B] implies [A] is an immediate consequence of Lemmas 1 and 2. *Q.E.D.*

The linear program in statement [B] of Theorem 1 provides more than a test for Cournot rationalizability: its solutions completely identify the family of convex cost functions that are compatible with Cournot rationalizability. So when a data set is rationalizable, this gives us an easy way of recovering cost information from the data. The next result follows immediately from Lemmas 1 and 2.

COROLLARY 1. *Suppose* $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ *is Cournot rationalizable with convex cost functions. Then the convex cost functions* $\{C_i\}_{i\in I}$ *may serve as rationalizing functions if, and only if, there is* $\delta_{i,t} \in C_i'(Q_{i,t})$ *such that* $\{\delta_{i,t}\}_{(i,t)\in I\times T}$ *satisfies* [B] *(in Theorem 1).*

Sometimes it is convenient to consider rationalizations where each firm's cost functions are differentiable (so kinks on the cost curves are not allowed). This can be characterized by strengthening the condition imposed on $\{\delta_{i,t}\}_{t\in T}$ in Theorem 1; we say that $\{\delta_{i,t}\}_{t\in T}$ is *finely increasing with* $Q_{i,t}$ if it is increasing and $\delta_{i,t'} = \delta_{i,t}$ whenever $Q_{i,t'} = Q_{i,t}$. We may also have reason to believe that some firm i in the industry has constant marginal costs and would like to confirm that the data supports that hypothesis. This can be checked by requiring $\delta_{i,t}$ to be independent of t. We state this formally in the next result, which we prove in the Appendix.

COROLLARY 2. *The following statements on* $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ *are equivalent:*

[A] *The set of observations is Cournot rationalizable with convex cost functions for all firms and with firms in* $J \subseteq I$ *having* C^2 *cost functions and firms in* $J' \subseteq J$ *having linear cost functions.*[8]

[8] It is clear from the proof that the cost functions could in fact be chosen to be differentiable to any degree, but there is no particular need to go beyond C^2, which is sufficient to ensure the differentiability of the marginal cost function.

[B] *There exists a set of positive numbers* $\{\delta_{i,t}\}_{(i,t)\in I\times T}$ *satisfying the common ratio property and the following:* (a) $\{\delta_{i,t}\}_{t\in T}$ *is increasing with* $Q_{i,t}$ *(for every firm* i*);* (b) *for a firm* $i\in J$, $\{\delta_{i,t}\}_{t\in T}$ *is finely increasing with* $Q_{i,t}$*; and* (c) *for a firm* $i\in J'$, $\delta_{i,t} = \delta_{i,t}$ *for all* $t\in T$.

3. Testing the degree of Collusion

A major concern in the empirical IO literature is estimating the degree of collusive behavior. This question is related to, but distinct from, the principal focus of our paper, which is to develop a revealed preference test for Cournot equilibrium. In this section, we shall explain this distinction and also consider what added information is needed in our framework to test specifically for collusion.

A subset of firms in an industry are said to be in *perfect collusion* if they are acting in concert to maximize their joint profit. In the proposition below we show that *any* data set is consistent with the hypothesis that all firms in the industry are in perfect collusion with each other. The simple proof provides rationalizing cost functions for each firm that are linear and identical across firms, and rationalizing demand functions at each observation t that are also linear.

Proposition 1. *For any set of observations* $\{[P_t,(Q_{i,t})_{i\in I}]\}_{t\in T}$ *with* $P_t>0$ *for all* t*, there is* $\epsilon>0$ *and downward-sloping inverse demand functions* $P_t:\mathbb{R}_+\to\mathbb{R}$ *for each* t*, such that, for every* t*,*

$$(Q_{i,t})_{i\in I}\in \operatorname{argmax}_{(q_i)_i\geq 0}\left[\left(\sum_{i\in I} q_i\right)P_t\left(\sum_{i\in I} q_i\right)-\epsilon\left(\sum_{i\in I} q_i\right)\right].$$

Proof: Suppose that every firm has cost function $C(q)=\epsilon q$. Then every output allocation is cost efficient and if firms are colluding they will act like a monopoly with the same cost function C. Choose ϵ sufficiently small so that $P_t>\epsilon$ for all t. It is straightforward to check that there is a linear and downward-sloping inverse demand function P_t such that $P_t(Q_t)=P_t$ and such that the marginal revenue at Q_t is ϵ. QED

Proposition 1 tells us that industry-wide perfect collusion is not testable, in the sense that any data set of the form $\{[P_t,(Q_{i,t})_{i\in I}]\}_{t\in T}$ is always consistent with such a hypothesis. However, it is possible for such a data set to be inconsistent with the hypothesis that a strict

subset of firms in the industry is perfectly colluding (i.e., maximizing their joint profit), while playing a Cournot game against other firms (or other groups of firms) in the industry. This is not surprising, since we have already shown through Examples 1 and 2 in the previous section that the Cournot hypothesis can be rejected by data. Indeed it is possible to develop a test (similar to that in Theorem 1) for this hybrid collusion-Cournot hypothesis and we do this in the Supplement.

Perfect collusion is a strong notion of collusion because it requires the colluding firms to be optimizing perfectly against firms outside the group: any output level chosen is divided in a cost-minimizing way across the group and the output level chosen is profit-maximizing given the group cost function. It is, of course, possible for firms to collude without collusion taking on this perfect form; for example, two firms may tacitly agree to restrict their output in a way that raises profit above the level given by a Cournot equilibrium, without necessarily dividing output between them in a way that precisely minimizes joint cost. Empirical IO studies of collusion have typically focussed on this broader notion of collusion and have used conduct parameters as a way of measuring collusive behavior. For the rest of this section we shall examine the extent to which conduct parameters can – or cannot – be recovered from data.

3.1 *Conduct parameters and conjectural variations*

Consider an industry with I firms, where P is the inverse demand function and where firm i has the cost function C_i. Given output $q_{-i} > 0$ and a real number $\theta_i \geq 0$, we denote by $\hat{Q}_i(\theta_i, q_{-i})$ the set of output levels such that, for $q_i \in \hat{Q}_i(\theta_i, q_{-i})$,

$$q_i \in \operatorname{argmax}_{q_i' \geq 0} \left\{ q_i' P(\theta_i(q_i' - q_i) + q_{-i}) - C_i(q_i') \right\}. \tag{4}$$

In other words, q_i is an output level at which firm i has no incentive to deviate, if other firms are producing q_{-i} and if it believes that any deviation will change total output by the deviation multiplied by the factor θ_i. An output vector $(q_i)_{i \in I}$ constitutes a $\theta = \{\theta_i\}_{i \in I}$ *conjectural variations equilibrium* (or θ-CV equilibrium, for short) if, for every i, $q_i \in \hat{Q}_i(\theta_i, \sum_{j \neq i} q_j)$. If $\theta_i = 1$ for all i, we have a Cournot equilibrium; if $\theta_i = 0$ then firm i is acting as though its output has no impact on total output, so it is a price-taker.

When P is log-concave and C_i has increasing marginal cost, the objective function on the right of (4) is quasiconcave in q_i (see Vives (1999, Section 6.2)) and $\hat{q}_i(\theta_i, q_{-i})$ is a singleton characterized by the first-order condition

$$P\left(\hat{q}_i(\theta_i, q_{-i}) + q_{-i}\right) - C_i'\left(\hat{q}_i(\theta_i, q_{-i})\right) = -\theta_i \hat{q}_i(\theta_i, q_{-i}) P'\left(\hat{q}_i(\theta_i, q_{-i}) + q_{-i}\right). \tag{5}$$

Furthermore, $\hat{q}_i$ *is decreasing in* θ_i, so that a large θ_i means that firm i is behaving less competitively. Note also that for any output vector $(q_i)_{i\in I}$ that satisfies $P(\sum_{j\in I} q_j) - C_i'(q_i) > 0$ for all i, there *must* be θ_i such that $q_i = \hat{q}_i(\theta_i, \sum_{j\neq i} q_j)$: just choose θ_i to solve

$$P\left(\sum_{j\in I} q_j\right) - C_i'(q_i) = -\theta_i q_i P'\left(\sum_{j\in I} q_j\right).$$

Given these observations, one could interpret θ_i simply as a *conduct parameter* measuring firm i's deviation from competitive behavior, rather than as a measure of what it literally believes will occur should it deviate from its equilibrium output.

Given $\theta_T = \{\theta_t\}_{t\in T}$, where $\theta_t = (\theta_{i,t})_{i\in I} \gg 0$, the data set $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ is said to be θ_T*-CV rationalizable* if there are cost functions C_i and inverse demand functions P_t such that $P_t(Q_t) = P_t$ and $(Q_{i,t})_{i\in I}$ is a θ_t-CV equilibrium. Notice that we allow the conduct parameters to vary across observations, which is important since models of dynamic collusion often exhibit this feature (where t is interpreted as the time period). The next result, which gives a linear program to test for θ_T-CV rationalizability (for a given θ_T), is a straightforward modification of Theorem 1 and is proved in the Appendix.

THEOREM 2. *The following statements on* $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ *are equivalent:*

[A] *The set of observations is* θ_T*-CV rationalizable with convex cost functions.*

[B] *There exists a set of positive real numbers,* $\{\delta_{i,t}\}_{(i,t)\in I\times T}$*, that satisfy the generalized common ratio property, i.e.,*

$$\frac{P_t - \delta_{1,t}}{\theta_{1,t}Q_{1,t}} = \frac{P_t - \delta_{2,t}}{\theta_{2,t}Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{\theta_{I,t}Q_{I,t}} > 0 \text{ for all } t \in T, \tag{6}$$

and for each i, $\{\delta_{i,t}\}_{t\in T}$ *is increasing with* $Q_{i,t}$.

Notice that if condition (6) is satisfied by $\theta_T = \{\theta_t\}_{t\in T}$, then it is satisfied by $\tilde{\theta}_T = \{\lambda_t\theta_t\}_{t\in T}$ for any $\lambda_t > 0$, which means that the conduct parameters can only be tested up to scalar multiples. In particular, we cannot distinguish between a Cournot equilibrium, in

which $\theta_t = (1, 1, ..., 1)$, and a CV equilibrium with $\theta_t = \lambda_t(1, 1, ..., 1)$. This observation is consistent with the results of Bresnahan (1982) and Lau (1982), who show that the identification of θ_t requires sufficiently rich variation in demand behavior, while our data set contains no information on the determinants of demand. Of course, none of this means that the Cournot hypothesis is not testable because the data still allow us to test the hypothesis that *conduct parameters are equal across firms at each observation*. The test of Cournot rationalizability in Theorem 1 could be construed as a test of this broader hypothesis.

Notice also in Theorem 2 that, even though θ_t is indeterminate up to scalar multiples, the $\{\delta_{i,t}\}_{i\in I}$ that solves (6) does *not* vary with the scale. If we are willing to believe (say) that the conduct parameters are equal across firms at each observation, and if a data set is consistent with this belief, then we are able to derive bounds on the cost functions using the array $\{\delta_{i,t}\}_{(i,t)\in I\times T}$ (see Corollary 1), without necessarily pinning down strategic behavior. In other words, cost information could be unambiguously recovered from the data set, even though the precise level of competitive/collusive behavior (λ_t) cannot be measured. This contrasts with the empirical IO literature, which typically estimates $\theta_{i,t}$ and the cost function of firm i simultaneously.

3.1. Bounding conduct parameters with demand information

It is not hard to see that more information on demand behavior will help to impose absolute bounds on the value of $\theta_{i,t}$. From the proof of Theorem 2, we know that the inverse demand function constructed to rationalize the data has slope $-(P_t - \delta_{i,t})/(\theta_{i,t}Q_{i,t})$. A proportionate reduction in $\theta_{i,t}$ across all firms does not upset condition (6), but the slope of the rationalizing inverse demand function provided in the proof will decrease. This suggests that if we have information on the demand curve that bounds the elasticity of demand within some range, then θ_t will no longer be indeterminate up to scalar multiples. We illustrate this next.

EXAMPLE 3. Consider a duopoly with firms i and j where
(i) at observation t, $P_t = 10$, $Q_{i,t} = 5/3$ and $Q_{j,t} = 5/3$; and
(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 2$ and $Q_{j,t'} = 5/3$.
In addition, suppose the modeler knows that $\mathrm{d}P_t/\mathrm{d}q \geq -3$. With this, we claim that the observations are compatible with $\theta_t = \theta_{t'} = (3, 3)$ but not with $\theta_t = \theta_{t'} = (1, 1)$.

Indeed, applying Theorem 2, compatibility with $\theta_t = \theta_{t'} = (3,3)$ is confirmed if we could find $\delta_{i,t}, \delta_{i,t'}$, $\delta_{j,t}$ and $\delta_{j,t'}$ that solve

$$\frac{10 - \delta_{i,t}}{5} = \frac{10 - \delta_{j,t}}{5} \text{ and } \frac{4 - \delta_{i,t'}}{6} = \frac{4 - \delta_{j,t'}}{5}.$$

In addition, because firm i's output is higher at t' than at t, we also require $\delta_{i,t} \le \delta_{i,t'}$. It is straightforward to check that these conditions are met if $\delta_{i,t} = 3$, $\delta_{i,t'} = 3$, $\delta_{j,t} = 3$ and $\delta_{j,t'} = 19/6$. In this case, the rationalizing inverse demand function P_t can be chosen to satisfy $\mathrm{d}P_t/\mathrm{d}q = -(10 - 3)/5 = -7/5 \ge -3$.

Suppose, contrary to our claim, that the data set is Cournot rationalizable, with P_t satisfying $\mathrm{d}P_t/\mathrm{d}q \ge -3$. In that case, the first-order condition of firm i gives

$$\frac{10 - m_{i,t}}{5/3} = -\frac{\mathrm{d}P_t}{\mathrm{d}q} \le 3,$$

where $m_{i,t}$ is a subgradient of firm i's cost function at output $Q_{i,t} = 5/3$. Therefore, $m_{i,t} \ge 5$, which means that the marginal cost at $Q_{i,t'} = 2$ must be at least 5 since firm i's cost function is convex. However, the price at t' is just 4, so there is a contradiction.

More generally, it is clear that with this lower bound on the slope of the inverse demand, there is a scalar $\bar\lambda$ such that the observations are CV rationalizable with $\theta_t = \theta_{t'} = (\lambda, \lambda)$ if and only if $\lambda \le \bar\lambda$. Lau (1982) showed that the identification of conduct parameters requires that demand be drawn from a family parameterized by at least two variables. In this example, we only have a bound on the slope of demand, which does not permit identification, but it is enough to recover a range of values of λ that is consistent with the data; in certain situations, this coarser information may be all that (say) an industry regulator is interested in.

We now outline a test for collusion that encompasses Example 3. We consider an observer who has a data set generated by demand varying across observations, while cost functions are fixed. Besides observing P_t and $(Q_{i,t})_{t \in T}$ at each t, the observer also observes n variables, represented by the vector $z \in Z \subseteq \mathbb{R}^n$, that have a known impact on the elasticity of demand. More precisely, we associate to each observation $(q, p, z) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times Z$ a set $S(q, p, z) \subseteq (-\infty, 0)$. Having observed (q, p, z), the observer assumes that the demand can be any downward sloping function P satisfying $P(q) = p$ and $P'(q) \in S(q, p, z)$.[9] This leads to the following test, which has the power to restrict the absolute value of conduct parameters.

[9] For a recent study on bounding demand elasticities, see Nevo and Rosen (2010).

We omit the proof of this result, since it is a straightforward modification of the proofs of Theorems 1 and 2.

THEOREM 3. *The following statements on* $\{[P_t,(Q_{i,t})_{i\in\mathcal{I}},z_t]\}_{t\in\mathcal{T}}$ *are equivalent.*

[A] *The set of observations is* $\theta_{\mathcal{T}}$*-CV rationalizable (with* $\theta_t \gg 0$ *for all* t*) with convex cost functions and with the rationalizing inverse demand functions* P_t *satisfying* $P_t'(Q_t)\in S(Q_t,P_t,z_t)$*.*

[B] *There is a set of positive real numbers* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *that obey the generalized common ratio property* (6) *and are such that* $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ *is increasing with* $Q_{i,t}$ *for each* i*. Furthermore,* $-(P_t-\delta_{1,t})/(\theta_{1,t}Q_{1,t})\in S(Q_t,P_t,z_t)$ *for all* t*.*

In the model of Rotemberg and Saloner (1986), firms know the inverse demand function at each period t (so there is complete information for the stage game) while there is uncertainty about demand in future periods. The model predicts that firms will be more collusive in periods where the realized demand function is low, because the benefit of deviation is less compared to the threat of future punishment. This prediction could be translated to mean that the data set is CV rationalizable with $\theta_t=\lambda_t(1,1,...,1)$ and with λ_t being higher in periods with low realized demand; hypotheses like this could be tested with Theorem 3.

4. Cournot Rationalizability in a Multi-Product Oligopoly

In this section, we consider a market consisting of I firms, with each firm producing up to K goods. The production costs and demand for these goods are possibly inter-related,[10] which allows for synergies in production and complementarity/substitutability in demand. Each observation now consists of the prices of all the goods, and the output of each good by each firm. Formally, letting $\mathcal{K}=\{1,2,\ldots,K\}$ be the set of goods, each observation t consists of the price vector $P_t=(P_t^k)_{k\in\mathcal{K}}$, and the output vector of each firm, $Q_{i,t}=(Q_{i,t}^k)_{k\in\mathcal{K}}$. We require that this set of observations satisfy $Q_{i,t}>0$ for every firm i and at every observation t, and also that $\sum_{i\in\mathcal{I}}Q_{i,t}\gg 0$. In other words, every firm is always producing something (though a firm need not produce every one of the K goods) and strictly positive amounts of each good are produced at all observations.

[10] See, for example, Brander and Eaton (1984) and Bulow et al. (1985).

As before, the set of observations is Cournot rationalizable if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations. We impose some regularity conditions on the demand and cost functions. Generalizing our earlier definition, a *cost function of firm* i is a function $C_i : \mathbb{R}_+^K \to \mathbb{R}$ such that $C_i(0) = 0$, C_i is nondecreasing with respect to $q \in \mathbb{R}_+^K$, and C_i is strictly increasing along rays. We require the market inverse demand function $P_t : \mathbb{R}_+^K \to \mathbb{R}^K$ (for each t) to obey the *law of demand*; by this we mean that P_t is differentiable with a negative definite derivative matrix ∂P_t.[11]

The set of observations $\{[P_t, (Q_{i,t})_{i \in I}]\}_{t \in T}$ is said to be *Cournot rationalizable* if there exist cost functions C_i for each firm i, and demand functions P_t obeying the law of demand for each observation t, such that

(i) $P_t(Q_t) = P_t$; and

(ii) $Q_{i,t} \in \text{argmax}_{q_i \geq 0} \{ \sum_{k=1}^K q_i^k P_t^k(q_i + \sum_{j \neq i} Q_{j,t}) - C_i(q_i) \}$.

Theorem 4 below characterizes data sets that are Cournot rationalizable with convex cost functions. It is thus a generalization of Theorem 1.

THEOREM 4. *The following statements on the data set* $\{[P_t, (Q_{i,t})_{i \in I}]\}_{t \in T}$ *are equivalent.*

[A]. *The set of observations is Cournot rationalizable with with convex cost functions.*

[B]. *There exists real numbers* $\lambda_t^{\ell,k}$, *nonnegative numbers* $\delta_{i,t}^k$ *and positive numbers* $C_{i,t}$ *such that, for all* ℓ *and* $k \in K$, t *and* $t' \in T$, *and* $i \in I$, *the following holds:*

(i) *the* $K \times K$ *matrix* $\Lambda_t = [\lambda_t^{\ell,k}]$ *is positive definite;*

(ii) $\delta_{i,t}^k - P_t + \sum_{\ell=1}^K \lambda_t^{\ell,k} Q_{i,t}^\ell \geq 0$ *and* $(\delta_{i,t}^k - P_t + \sum_{\ell=1}^K \lambda_t^{\ell,k} Q_{i,t}^\ell) Q_{i,t}^k = 0$*;*

(iii) $C_{i,t'} \geq C_{i,t} + \sum_{k=1}^K \delta_{i,t}^k (Q_{i,t'}^k - Q_{i,t}^k)$*; and*

(iv) $0 \leq C_{i,t} \leq \sum_{k=1}^K \delta_{i,t}^k Q_{i,t}^k$.

In the case where there is just one good, statement [B] reduces to the following: there are numbers $\lambda_t > 0$, $\delta_{i,t} \geq 0$, $C_{i,t} > 0$ such that (a) $\delta_{i,t} = P_t - \lambda_t Q_{i,t}$, (b) $C_{i,t'} \geq C_{i,t} + \delta_{i,t}(Q_{i,t'} - Q_{i,t})$, and (c) $0 \leq C_{i,t} \leq \delta_{i,t} Q_{i,t}$. This gives an alternative characterization of

[11] For the use of this condition in the context of multi-product oligopolies, see Vives (1999). The microfoundations of this property has also been extensively studied; see Quah (2003) and also the survey of Jerison and Quah (2008). The literature tends to consider demand as a function of price, rather than the inverse demand function considered here. However, the two cases are equivalent: if $\partial^2 \bar{P}_t(\tilde{Q})$ is negative definite, then it is locally invertible, and its inverse, the demand function $\bar{D}_t$, has a negative definite matrix at $\bar{P}_t(\tilde{Q})$.

Cournot rationalizability to that given in Theorem 1. It is not hard to check directly that the two sets of conditions are equivalent.

Proof of Theorem 4: Suppose that [A] holds, so the data could be rationalized by inverse demand functions P_t^k, for $k \in \mathcal{K}$ and $t \in \mathcal{T}$ and cost functions C_i. We set

$$\lambda_t^{\ell,k} = -\frac{\partial P_t^\ell}{\partial q^k}(Q_t).$$

Since $(P_t^k)_{k\in\mathcal{K}}$ obeys the law of demand, Λ_t is positive definite as required by (i). At observation t, firm i's revenue function, given that firm $j \neq i$ is producing $Q_{j,t}$, is $R_{i,t}(q_i) = \sum_{\ell=1}^K q_i^\ell P_t^\ell(q_i^\ell + \sum_{j\neq i} Q_{j,t}^\ell)$; note that

$$\frac{\partial R_{i,t}}{\partial q_i^k}(Q_{i,t}) = P_t^k(Q_t) + \sum_{\ell=1}^K \frac{\partial P_t^\ell}{\partial q^k}(Q_t) Q_{i,t}^\ell = P_t^k - \sum_{\ell=1}^K \lambda_t^{\ell,k} Q_{i,t}^\ell. \tag{7}$$

By assumption, $Q_{i,t}$ maximizes $\Pi_{i,t}(q_i) = R_{i,t}(q_i) - C_i(q_i)$ (the profit of firm i). Therefore, there exists a vector $(\delta_{i,t}^k)_{k\in\mathcal{K}}$ in $\partial C_i(Q_{i,t})$ such that $\delta_{i,t}^k \geq P_t^k - \sum_{\ell=1}^K \lambda_t^{\ell,k} Q_{i,t}^\ell$ and with equality whenever $Q_{i,t}^k > 0$ for good k, so that (ii) holds. Since C_i is nondecreasing, we may choose the subgradient $(\delta_{i,t}^k)_{k\in\mathcal{K}}$ to be a nonnegative vector. Given that C_i is strictly increasing along rays and $Q_{i,t} > 0$, we have $C_i(Q_{i,t}) > 0$ for all (i,t). Set $C_{i,t} = C_i(Q_{i,t})$; since C_i is convex and $(\delta_{i,t}^k)_{k\in\mathcal{K}}$ is a subgradient, (iii) holds. Finally, (iv) holds since C_i is convex and $C_i(0) = 0$. This completes our proof that [A] implies [B].

Lemmas 3 and 4, which we state and prove below, show immediately that [B] implies [A]. These lemmas are analogous to Lemmas 1 and Lemma 2 respectively. *Q.E.D.*

LEMMA 3. *Suppose that, at some observation t, there are real numbers $\lambda_t^{\ell,k}$ and nonnegative numbers $\delta_{i,t}^k$ such that, for all $\ell, k \in \mathcal{K}$ and $i \in \mathcal{I}$, conditions* (i) *and* (ii) *in Theorem 4 hold. Suppose that there are convex cost functions C_i with $(\delta_{i,t}^k)_{k\in\mathcal{K}} \in \partial C_i(Q_{i,t})$. Then there exists an inverse demand function $\mathbf{P}_t$ obeying the law of demand such that $\mathbf{P}_t(Q_t) = P_t$ and, with each firm i having the cost function C_i, $\{Q_{i,t}\}_{i\in\mathcal{I}}$ constitutes a Cournot equilibrium.*

Proof: We define the inverse demand function for good k by $P_t^k(q) = a_t^k - \sum_{\ell=1}^K \lambda_t^{k,\ell} q^\ell$ with a_t^k chosen such that $P_t^k(Q_t) = P_t^k$. Firm i's profit at observation t, given that firm $j \neq i$ is producing $Q_{j,t}$ is $\Pi_{i,t}(q_i) = R_{i,t}(q_i) - C_i(q_i)$, where $R_{i,t}(q_i) = \sum_{\ell=1}^K q_i^\ell P_t^\ell(q_i^\ell + \sum_{j\neq i} Q_{j,t}^\ell)$ and marginal revenue is given by (7). Since $(\delta_{i,t}^k)_{k\in\mathcal{K}} \in \partial C_i(Q_{i,t})$, condition (ii) gives the Kuhn-Tucker conditions for profit maximization. These conditions are sufficient to guarantee that

firm i's choice is optimal if $\Pi_{i,t}$, is concave in q_i. Given that C_i is a convex function, it suffices to check that the $R_{i,t}$ is concave in q_i. It is straightforward to verify that, for all q_i, the Hessian $\partial^2 R_{i,t}(q_i) = -\,{}^{T}_{t}\ {}_{t}$. Condition (i) guarantees that this matrix is negative definite, so we conclude that $R_{i,t}$ is concave. *Q.E.D.*

LEMMA 4. *Suppose that for some firm i, there are numbers $\delta^k_{i,t} \geq 0$ and $C_{i,t} > 0$ such that, for all $k \in \mathcal{K}$ and $t, t' \in \mathcal{T}$, conditions* (iii) *and* (iv) *in Theorem 4 are satisfied. Then there is a convex cost function with $(\delta^k_{i,t})_{k\in\mathcal{K}} \in \partial C_i(Q_{i,t})$.*

Proof: Let $d = -\max_{t\in\mathcal{T}}\{C_{i,t} - \sum_{k=1}^{K}\delta^k_{i,t}Q^k_{i,t}\}$; by (iv), $d \geq 0$. Given this, $\hat{C}_{i,t} = C_{i,t} + d$ is a (strictly) positive number since $C_{i,t}$ is a positive number. Define the function $\tilde{C}_i$ by

$$\tilde{C}_i(q) = \max_{t\in\mathcal{T}}\left\{\hat{C}_{i,t} + \sum_{k=1}^{K}\delta^k_{i,t}(q^k - Q^k_{i,t})\right\}. \tag{8}$$

The function $\tilde{C}_i$ has all but one of the conditions we require on the cost function. First, notice that our choice of d guarantees that $\tilde{C}_i(0) = 0$. Since $\delta^k_{i,t} \geq 0$, the function $\tilde{C}_i$ is nondecreasing and since it is the upper envelope of linear functions, $\tilde{C}_i$ is a convex function. Condition (iii) implies that $\tilde{C}_i(Q_{i,t}) = \hat{C}_{i,t} > 0$, since

$$\hat{C}_{i,t} \geq \hat{C}_{i,s} + \sum_{k=1}^{K}\delta^k_{i,s}(Q^k_{i,t} - Q^k_{i,s}) \text{ for all } s \in \mathcal{T}.$$

Therefore, $(\delta^k_{i,t})_{k\in\mathcal{K}} \in \partial\tilde{C}_i(Q_{i,t})$.

However, $\tilde{C}_i$ may not be strictly increasing along rays. To guarantee this property we modify the function $\tilde{C}_i$ in the following way. Choose a vector $\varepsilon = (\varepsilon, \varepsilon, ..., \varepsilon)$ with $\varepsilon > 0$ and sufficiently small so that $C_{i,t} > \varepsilon \cdot Q_{i,t}$ for all t. Now, define the function C_i by $C_i(q) = \max\{\tilde{C}_i(q), \varepsilon \cdot q\}$; C_i is a convex and nondecreasing function, with $C_i(0) = 0$ and $C_i(Q_{i,t}) = \hat{C}_{i,t}$. Locally at $Q_{i,t}$, C_i and $\tilde{C}_i$ are identical, so $(\delta^k_{i,t})_{k\in\mathcal{K}} \in \partial C_i(Q_{i,t})$. In addition, C_i is strictly increasing along rays. Suppose, to the contrary, that C_i is locally constant along the ray through the point $q = \underline{q}$. In that case, there exists $s \in \mathcal{T}$ such that $\hat{C}_{i,\bar{s}} + \sum_{k=1}^{K}\delta^k_{i,\bar{s}}(\gamma\underline{q}^k - Q^k_{i,\bar{s}})$ is constant and positive for all values of $\gamma \geq 0$, which is not possible since $C_i(0) = 0$. *Q.E.D.*

Rationalizability in the sense of Theorem 4 imposes no observable restrictions on a data sets where each firm produces one and the same good at all observations and each of the K goods is produced by just one firm. Such a data set is always rationalizable because it can be

regarded as K single-product monopolies: for each firm, any sequence of prices and outputs can be justified by fluctuations to its inverse demand, which we can require to depend only on its own output. To have observable restrictions in such a setting, we require assumptions that exclude demand independence across goods; such a model of differentiated products is considered in the Supplement. On the other hand, when at least one of the K goods is supplied by two or more firms, observable restrictions exist, as demonstrated in the following example.

Example 4. Consider an industry with two goods, 1 and 2, where observations taken from the two firms in this industry are as follows:
(i) at observation t, P^1_t = 10, $Q^1_{i,t}$ = 13, $Q^2_{i,t}$ = 12, $Q^1_{j,t}$ = 4, $Q^2_{j,t}$ = 6.
(ii) at observation t', P^1_t = 1, P^2_t = 1, $Q^1_{j,t}$ = 8 and $Q^2_{j,t}$ = 8.
We claim that these observations are not Cournot rationalizable. Suppose, to the contrary, that observation t constitutes a Cournot equilibrium. In that case, the first-order condition for firm i says that there is $(\delta^1_{i,t}, \delta^2_{i,t}) \in \partial C_i(Q_{i,t})$ such that

$$P^1_t(Q_t) + Q^1_{i,t}\frac{\partial P^1_t}{\partial q_1} + Q^2_{i,t}\frac{\partial P^2_t}{\partial q_1} - \delta^1_{i,t} = 0.$$

Similarly, there is $(\delta^1_{j,t}, \delta^2_{j,t}) \in \partial C_j(Q_{j,t})$ such that

$$P^1_t(Q_t) + Q^1_{j,t}\frac{\partial P^1_t}{\partial q_1} + Q^2_{j,t}\frac{\partial P^2_t}{\partial q_1} - \delta^1_{j,t} = 0.$$

Multiplying the first equation by $Q^2_{j,t}$ and the second equation by $Q^2_{i,t}$ and taking the difference between them, we obtain

$$(Q^2_{j,t} - Q^2_{i,t})P^1_t(Q_t) + (Q^2_{j,t}Q^1_{i,t} - Q^2_{i,t}Q^1_{j,t})\frac{\partial P^1_t}{\partial q_1} - Q^2_{j,t}\delta^1_{i,t} + Q^2_{i,t}\delta^1_{j,t} = 0. \tag{9}$$

The significance of the numbers chosen for observation t is that they guarantee that $Q^2_{j,t} - Q^2_{i,t} < 0$ and $Q^2_{j,t}Q^1_{i,t} - Q^2_{i,t}Q^1_{j,t} > 0$. Note that $\delta^1_{i,t} \geq 0$ since firm i's cost function is nondecreasing and $\partial P^1_t/\partial q_1 < 0$ because of the law of demand. Therefore, the second and third terms on the left of equation (9) are both negative. Re-arranging that equation, we obtain

$$\delta^1_{j,t} \geq \left(\frac{Q^2_{i,t} - Q^2_{j,t}}{Q^2_{i,t}}\right) P^1_t(Q_t) = \left(\frac{6}{12}\right) 10 = 5. \tag{10}$$

In short, observation t provides us a with a lower bound on the marginal cost of firm j at its observed output of $(4, 6)$.

At observation t', firm j's output is $(8,8)$. The marginal cost of increasing output from $(4,6)$ to $(8,8)$ is no smaller than the marginal cost of increasing output from $(4,6)$ to $(8,6)$, which is in turn bounded below by $5 \times 4 = 20$ (because of (10) and the convexity of C_j). So, the total cost of producing $(8,8)$ is at least 20 but the total revenue of firm i at observation t' is just 16: firm i is better off choosing $(0,0)$ at observation t'.

Like Theorem 1, Theorem 4 establishes an equivalence between Cournot rationalizability and the solution to a programming problem. However, the program in statement [B] of Theorem 4 is not a linear program, because it requires checking that the matrix Λ is positive definite (condition (i) in statement [B]). It is possible to replace the law of demand with a stronger condition that is easier to check. For example, we could require the rationalizing inverse demand function P_t to obey *diagonal dominance with uniform weights*; by this, we mean that

$$2\frac{\partial P_t^k}{\partial q_k}(q) + \sum_{\ell \neq k} \left| \frac{\partial P_t^k}{\partial q_\ell}(q) + \frac{\partial P_t^\ell}{\partial q_k}(q) \right| < 0 \text{ for all } q \text{ and for all } k \in \mathcal{K}.$$

This intuitive condition says that own-price effects are larger than the sum of all cross-price effects. If we impose this condition on the rationalizing demand system, then the corresponding condition on $\delta_{i,t}^k$ (in place of condition (i) in [B]) is the following:[12]

$$2\lambda_t^{k,k} + \sum_{\ell \neq k} |\lambda_t^{\ell,k} + \lambda_t^{k,\ell}| < 0 \text{ for all } k \in \mathcal{K};$$

note that this can condition can be equivalently stated as a set of linear conditions.

In certain contexts, the modeler may have specific information on cross price effects which he would like to impose as conditions on the rationalizing demand system, on top of those required by the law of demand or diagonal dominance. For example, it is possible to interpret the different goods in this model as the same good sold in several distinct and isolated markets; in other words, this multi-product oligopoly is an instance of third degree price discrimination, with the same firms interacting in several markets. In that case, it may be reasonable to require all cross price effects to equal zero, i.e., $\partial P_t^k / \partial q^\ell = 0$ for all $k \neq \ell$. Correspondingly, one would have to impose the condition $\lambda_t^{\ell,k} = 0$ for all t and whenever $\ell \neq k$, in addition to the ones listed in statement [B] of Theorem 4.

[12] This property guarantees the positive definiteness of the symmetric matrix $\Lambda + \Lambda^T$, which is equivalent to the positive definiteness of Λ; see Mas-Colell et al. (1995, Appendix M.D) for more on diagonal dominance.

Similarly, the modeler may believe that the K goods are substitutes ($\partial P_t^k/\partial q^\ell \leq 0$ for all ℓ and k) or complements ($\partial P_t^k/\partial q^\ell \geq 0$ for all $\ell \neq k$). The corresponding conditions are $\lambda_t^{\ell,k} \geq 0$ for all ℓ and k, and $\lambda_t^{\ell,k} \leq 0$ for all $\ell \neq k$, respectively. If we impose the condition that all the goods are substitutes, then Cournot rationalizability requires that all observed prices be non-negative: if $P_t^{\bar{k}} < 0$ then any firm that is producing good k is strictly better off if it reduces its output of k. In the case when the goods are not necessarily substitutes, the model allows for the possibility that some observed prices are negative: firms can optimally pay for a good to be consumed in order that it may raise the demand for some other good.

5. Convincing Cournot Rationalizability

In all our results so far, we have studied Cournot rationalizability when cost functions display increasing marginal costs. This assumption on cost functions is of course ubiquitous in both theoretical and empirical work; nonetheless, in the context of oligopoly games, where increasing returns to scale may be present, it is useful to have a test for the Cournot hypothesis that is not necessarily linked to this condition. Returning to the one-good context, we could ask what conditions are needed for $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ to be Cournot rationalizable if we allow cost curves to be non-convex. The following result says that Cournot competition imposes no restrictions on any generic set of observations $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$, where by *generic* we mean that, for all i, $Q_{i,t} \neq Q_{i,t'}$ whenever $t \neq t'$.

Proposition 2. *Any generic set of observations $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ is Cournot rationalizable with firms having $\mathbf{C}^2$ cost functions.*

5.1. Cournot rationalizability with observed costs

To prove Proposition 2, and also to see how we could work around its seemingly negative implication, it is useful to first consider a scenario where the observer has more information at his disposal. Suppose the data set is $\{[P_t, (Q_{i,t})_{i\in I}, (C_{i,t})_{i\in I}]\}_{t\in T}$, where $C_{i,t} > 0$ is the total cost incurred by firm i at output $Q_{i,t} > 0$; we require $C_{i,t} = C_{i,t'}$ if $Q_{i,t} = Q_{i,t'}$. We say that this set is Cournot rationalizable if there are downward sloping (hence differentiable) inverse demand functions P_t (for all $t \in T$) and cost functions C_i (for each firm $i \in I$) such

that $P_t(Q_t) = P_t$, $C_i(Q_{i,t}) = C_{i,t}$, and $(Q_{i,t})_{i \in I}$ is a Cournot equilibrium when demand is P_t.

For each i and t, define the set $L_i(t) = \{t' \in T : Q_{i,t'} < Q_{i,t}\} \cup \{0\}$. Set $L_i(t)$ consists of those observations where firm i's output is strictly lower than Q_t, as well as a fictitious observation 0, for which $Q_{i,0} = 0$ and $C_{i,0} = 0$. We denote the set of observations where firm i's output is the lowest by t_i^*. It follows that $L_i(t_i^*) = \{0\}$ whilst, for any $t \notin t_i^*$, $L_i(t)$ will contain t_i^*, 0, and possibly other observations. We denote $l_i(t) = \text{argmax}_{t' \in L_i(t)} Q_{i,t'}$; that is, $l_i(t)$ is the set of observations corresponding to the highest output level strictly below $Q_{i,t}$.[13] In a similar fashion, the observation(s) with the highest output level for firm i is denoted by t_i^{**}. For $t \notin t_i^{**}$, the set of observations with outputs strictly higher than t is denoted by $U_i(t)$, with $u_i(t) = \text{argmin}_{t' \in U_i(t)} Q_{i,t'}$, so $u_i(t)$ is the set of observations with the lowest output level above $Q_{i,t}$.

For any t in T, define d$Q_{i,t} = Q_{i,t} - Q_{i,l_i(t)}$ and d$C_{i,t} = C_{i,t} - C_{i,l(t)}$. In words, d$C_{i,t}$ is the extra cost incurred by firm i when it increases its output from $Q_{i,l(t)}$ to $Q_{i,t}$. We denote the *average marginal cost* over that output range by $M_{i,t} = \text{d}C_{i,t}/\text{d}Q_{i,t}$.

We say that $\{C_{i,t}\}_{(i,t) \in I \times T}$ obeys the *discrete marginal property* if for every (i, t)

$$C_{i,t} - C_{i,t'} < P_t(Q_{i,t} - Q_{i,t'}) \text{ for } t' \in L_i(t). \tag{11}$$

For any $t' \in L_i(t)$, let $\mathbb{Q}_i(t', t)$ denote the set consisting of $Q_{i,t}$ and those observed output levels of firm i strictly between $Q_{i,t'}$ and $Q_{i,t}$.[14] Since $C_{i,t} - C_{i,t'} = \sum_{Q_{i,s} \in \mathbb{Q}_i(t',t)} M_{i,s}(Q_{i,s} - Q_{i,l(s)})$ the discrete marginal property may also be written as

$$\sum_{Q_{i,s} \in \mathbb{Q}_i(t',t)} M_{i,s}(Q_{i,s} - Q_{i,l(s)}) < P_t(Q_{i,t} - Q_{i,t'}) \text{ for } t' \in L_i(t). \tag{12}$$

We claim that this property is *necessary* for Cournot rationalizability. Indeed, notice that instead of producing at $Q_{i,t}$, firm i could have chosen to produce at $Q_{i,t'}$ for some $t' \in L_i(t)$. Given that $Q_{i,t}$ was chosen, the additional cost incurred, which is $C_{i,t} - C_{i,t'}$ must be less than the additional revenue gained, and the latter is bounded by $P_t(Q_{i,t} - Q_{i,t'})$, because the demand curve is downward sloping. The next result says that the discrete marginal property is both necessary and sufficient for Cournot rationalizability.

[13] In particular, $l_i(t_i^*) = \{0\}$.

[14] Formally, $\mathbb{Q}_i(t', t) = \{Q_{i,s} : s \in (L_i(t) \cup \{t\}) \setminus (L_i(t') \cup \{t'\})\}$.

Theorem 5. *A generic set of observations* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *is Cournot rationalizable with* $\mathbf{C}^2$ *cost functions if, and only if, it obeys the discrete marginal property.*

The proof of this result requires the following lemma.

Lemma 5. *Let* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *be a generic set of observations obeying the discrete marginal property and suppose that the positive numbers* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *satisfy* $0 < \delta_{i,t} < P_t$*, for all* (i,t)*, with* $\delta_{i,t} = \delta_{i,t'}$ *whenever* $Q_{i,t} = Q_{i,t'}$*. Then, there are* $\mathbf{C}^2$ *cost functions* $C_i : \mathbb{R}_+ \to \mathbb{R}$ *such that (i)* $C_i(Q_{i,t}) = C_{i,t}$*; (ii)* $C_i'(Q_{i,t}) = \delta_{i,t}$*; and (iii) for all* q_i *in* $[0, Q_{i,t})$*,*

$$P_t q_i - C_i(q_i) < P_t Q_{i,t} - C_i(Q_{i,t}). \tag{13}$$

Proof: Note that the inequality (13) may be re-written as

$$C_i(q_i) > P_t(q_i - Q_{i,t}) + C_i(Q_{i,t}). \tag{14}$$

The function $f_t(q_i) = P_t(q_i - Q_{i,t}) + C_{i,t}$, for q_i in $[0, Q_{i,t})$, is represented by a line with slope P_t passing through the point $(Q_{i,t}, C_{i,t})$ – see Figure 1. The discrete marginal property guarantees that for t' in $L_i(t)$, $(Q_{i,t'}, C_{i,t'})$ lies *above* the line f_t. We require a cost function that satisfies (14). One such function is the one given by the linear interpolation of all the points $(Q_{i,t}, C_{i,t})$, since its graph stays above every one of the lines representing the functions f_t. This cost function can in turn be replaced by a $\mathbf{C}^2$ function where the derivative at $Q_{i,t}$ is $\delta_{i,t}$, since $\delta_{i,t} < P_t$ and the latter is the slope of f_t. *Q.E.D.*

This lemma says that there is a cost function for firm i that (i) agrees with the observed values of firm costs, (ii) has marginal cost agreeing with specified values at the observed output levels, and (iii) guarantees that $q = Q_{i,t}$ is the optimal output level for firm i if the inverse demand function at t is $\bar{P}_t(q) = P_t$ for $q \leq Q_t$ and $\bar{P}_t(q) = 0$ for $q > Q_t$. So we have almost proved Theorem 5, and we fall short only because the rationalizing demand function $\bar{P}_t$ we just provided is not differentiable or downward sloping in our sense. However, as we show in the next result (proved in the Appendix), it is possible to replace $\bar{P}_t$ with a downward sloping inverse demand function P_t that preserves the optimality of firm i's output choice.

Lemma 6. *Let* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *be a set of positive numbers, with* $\delta_{i,t} = \delta_{i,t'}$ *whenever* $Q_{i,t} = Q_{i,t'}$*, satisfying the common ratio property* (1) *and suppose that the* $\mathbf{C}^2$ *cost functions* $C_i : \mathbb{R}_+ \to \mathbb{R}$

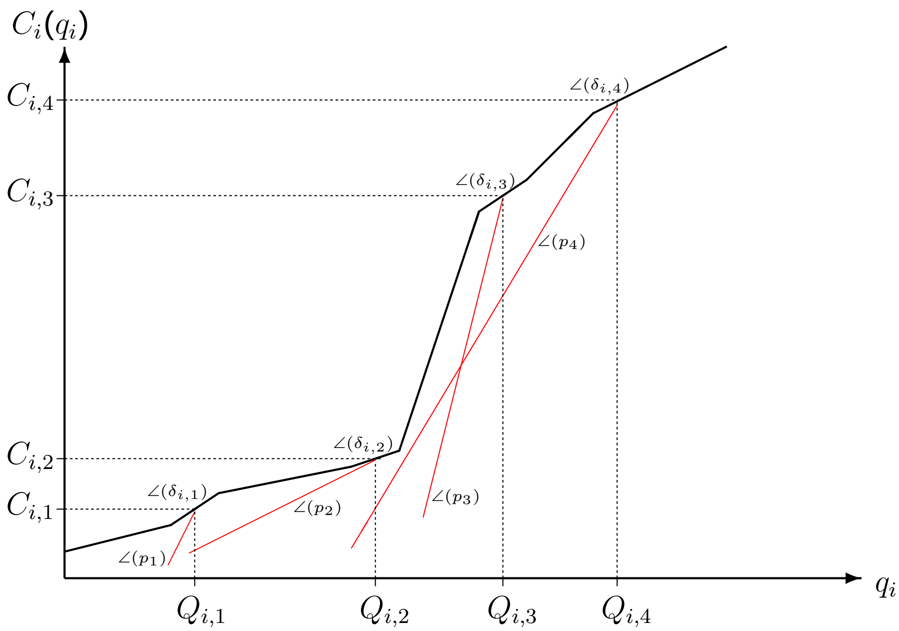

Figure 1: **Construction of a cost function:** The notation $\angle(\delta)$ is used to denote the slope (δ) at a point on the curve or of a line. The straight, thin lines represent the functions $f_t(q_i) = C_{i,t} + P_t(q_i - Q_{i,t})$. The discrete marginal property guarantees that if $Q_{i,t'} < Q_{i,t}$, then $(Q_{i,t'}, C_{i,t'})$ lies above the graph of f_t.

satisfy properties (i)-(iii) *in Lemma 5. Then there are downward sloping inverse demand functions* $P_t : \mathbb{R}_+ \to \mathbb{R}$ *such that,* $P_t(\sum_{i \in I} Q_{i,t}) = P_t$ *and, for every* i,

$$\operatorname{argmax}_{q_i \geq 0} \left\{ q_i P_t(q_i + \sum_{j \neq i} Q_{j,t}) - C_i(q_i) \right\} = Q_{i,t}.$$

Proof of Theorem 5: We have already explained why the discrete marginal property is necessary for rationalizability. For sufficiency, the crucial observation to make is that the common ratio property by itself imposes no restrictions on the data set, i.e., given P_t and $\{Q_{i,t}\}_{i \in I}$ there *always* exist positive numbers $\{\delta_{i,t}\}_{i \in I}$ such that (1) holds. Indeed, suppose firm k produces more than any other firm at observation t, i.e., $Q_{k,t} \geq Q_{i,t}$ for all i in I. Let $\delta_{k,t}$ be any positive number smaller than P_t, and define $\beta = (P_t - \delta_{k,t})/Q_{k,t}$. Then,

$$\delta_{i,t} := P_t - \beta Q_{i,t} \geq P_t - \beta Q_{k,t} = \delta_{k,t} > 0.$$

Note also that the genericity of the data set means that the condition that $\delta_{i,t} = \delta_{i,t'}$ when $Q_{i,t} = Q_{i,t'}$ is vacuously satisfied. These observations, together with Lemmas 5 and 6, establish the sufficiency of the discrete marginal property. *Q.E.D.*

Proof of Proposition 2: This is straightforward, given Theorem 5. By that theorem, it suffices that we find an array of individual costs, $\{C_{i,t}\}_{(i,t)\in I\times T}$, that satisfies the discrete marginal property. Equivalently, we need to find $\{M_{i,t}\}_{(i,t)\in I\times T}$ that obeys (12). But since the right side of that inequality is always positive and bounded away from zero for any t and t', it is clear that (12) holds if $M_{i,t}$ is sufficiently small. *Q.E.D.*

5.2. Convincing cost functions

What makes Proposition 2 possible — and also what makes its seemingly negative conclusion less than persuasive — is that there need be no link between the infinitesimal and average marginal costs, i.e, between $\delta_{i,t}$ and $M_{i,t}$. In fact the proof relies on the freedom to choose $M_{i,t}$ to be arbitrarily small, so that it could be much smaller than both $C_i'(Q_{i,l(t)}) = \delta_{i,l(t)}$ and $C_i'(Q_{i,t}) = \delta_{i,t}$. Since, apart from continuity, we impose no restrictions on C_i', this is permissible, but a rationalizing cost function for firm i that requires the modeler to believe in such a disconnection between infinitesimal and average marginal costs is not persuasive.[15]

One way of avoiding such ill-behaved marginal cost functions is to take as the marginal cost function between $Q_{i,t}$ and $Q_{i,l_i(t)}$ the linear interpolation of the hypothesized marginal costs at those two outputs, i.e, the marginal cost curve is the straight line joining $(Q_{i,l(t)}, \delta_{i,l(t)})$ and $(Q_{i,t}, \delta_{i,t})$. In that case, the average marginal cost between those two outputs is exactly $[\delta_{i,l_i(t)} + \delta_{i,t}]/2$. We wish to have more flexibility in our choice of the marginal cost function than simply taking a linear interpolation, but we *could* require the rationalizing cost function's average marginal cost between those two outputs is at least $[\delta_{i,l_i(t)} + \delta_{i,t}]/2$.

Formally, a $\mathbf{C}^2$ cost function C_i for firm i is said to be *convincing* or to satisfy *the convincing criterion* (given observed output $\{Q_{i,t}\}_{t\in T}$) if

$$\frac{C_i(Q_{i,t}) - C_i(Q_{i,l_i(t)})}{Q_{i,t} - Q_{i,l(t)}} \geq \frac{1}{2}\left[C_i'(Q_{i,l(t)}) + C_i'(Q_{i,t})\right] \quad \text{for all } t \notin t_i . \tag{15}$$

A set of observations $\{[P_t, (Q_{i,t})_{i\in I}, (C_{i,t})_{i\in I}]\}_{t\in T}$ is said to be *convincingly Cournot rationalizable* if it is Cournot rationalizable with convincing cost functions. The convincing criterion is by no means the only defensible restriction that one could impose on the rationalizing

[15] Put another way, the observer is asked to believe that the marginal cost information he could surmise by observing market shares convey no information at all about costs *between* observed output levels.

cost functions; for example, 1/2 in (15) could be replaced by a lower or higher fraction.[16] However, the possibility of reasonable alternatives is not an argument in favor of imposing no restriction at all and whenever a restriction is imposed that gives a lower bound to a firm's average marginal cost, one avoids the indeterminacy conclusion of Proposition 2.

EXAMPLE 5. Consider the following observations of firms i and j:

(i) at observation t, $P_t = 14$, $Q_{i,t} = 50$ and $Q_{j,t} = 100$;

(ii) at observation t', $P_{t'} = 4$, $Q_{i,t'} = 60$ and $Q_{j,t'} = 120$.

We claim that these observations are not convincingly Cournot rationalizable. Suppose instead that it is. By (2), we have $C_1'(Q_{i,t}) \geq P_t(1 - Q_{i,t}/Q_{j,t})$. This gives $C_i'(50) \geq 7$. The analogous inequality at t' gives $C_i'(60) \geq 2$. By the convincing criterion, the added cost incurred by firm i as it increases output from 50 to 60 is at least $4.5 \times 10 = 45$. On the other hand, at observation t', the added revenue made by firm i as it increases output from 50 to 60 is no more than 40, which means that the firm is better off producing at 50 rather than 60 at t' and is *not* optimizing.

Our final result, which we prove in the Appendix, characterizes data sets that are convincingly Cournot rationalizable via an easy-to-implement linear program.

THEOREM 6. *A set of observations* $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ *is convincingly Cournot rationalizable if, and only if, the following conditions are satisfied:*[17]

(a) *there are positive scalars* $\{\delta_{i,t}\}_{(i,t)\in I\times T}$, *with* $\delta_{i,t} = \delta_{i,t'}$ *if* $Q_{i,t} = Q_{i,t'}$, *that has the common ratio property;*

(b) *there are positive scalars* $\{\Delta_{i,t}\}_{(i,t)\in I\times T}$, *with* $\Delta_{i,t} = \Delta_{i,t'}$ *if* $Q_{i,t} = Q_{i,t'}$, *that has the discrete marginal property, i.e.,*

$$\sum_{Q_{i,s}\in\mathbb{Q}(t',t)} \Delta_{i,s}\left(Q_{i,s} - Q_{i,l(s)}\right) < P_t(Q_t - Q_{t'}) \text{ for all } t' \in L_i(t);\text{ and} \tag{16}$$

[16] There is, of course, a strong case that 1/2 is the most natural choice since the cost function constructed using that coefficient approximates the Riemann integral of the marginal cost function and will in the limit tend towards the true cost function. A coefficient smaller or larger than 1/2 may lead (respectively) to an under- or over-estimate of the true cost function.

[17] Corollary 2 says that there is a Cournot rationalization with C^2 convex cost curves if the common ratio property holds and $\delta_{i,t}$ is finely increasing with $Q_{i,t}$. Note that the latter condition is stronger than conditions (b) and (c) in this theorem, as we would expect. To see this, let $\Delta_{i,t} = \delta_{i,t}$ and it is clear that (16) and (17) are satisfied.

(c) *for all i and $t \notin t_i$,*

$$_{i,t} \quad \frac{1}{2}\left(\delta_{i,l_i(t)} + \delta_{i,t}\right). \tag{17}$$

It should be quite clear that there are potentially many variations on Theorem 6 corresponding to different restrictions on each firm's cost function. For example, we may know, a priori, that a particular firm j has increasing marginal cost. In that case, for that particular firm, we could dispense with (b) and (c) in that theorem and simply require $\{\delta_{j,t}\}_{t \in T}$ to be finely increasing with $Q_{j,t}$. The method we have outlined in this section also allows us to deal with other shape restrictions on the cost curves (besides convincing and increasing marginal costs). In particular, we could restrict a firm's marginal cost to be decreasing, or -shaped or -shaped; all that needs to be done is to impose the right conditions on $\delta_{j,t}$ and $_{j,t}$, which will eventually serve as the infinitesimal and average marginal cost of that firm's cost function. To illustrate this, suppose we wish to restrict firm j to a cost function with decreasing marginal cost. In that case, it is not hard to see that the following conditions on firm j are both necessary and sufficient: condition (b) as in Theorem 6, (c) $\delta_{j,t}$ (=) $\delta_{j,t}$ if $Q_{j,t} <$ (=) $Q_{j,t}$, and (d) $_{j,t}$ $\delta_{j,t}$ for all $t \in T$; we leave the details to the reader.

6. Allowing for changes to cost functions

Up to this point, we have assumed that firms' cost functions remain the same across observations in a data set. This assumption is not crucial and there are a number of ways in which it could be relaxed. We illustrate how this can be done with one method of allowing for cost changes that we think is intuitive and instructive.

Assume that, in addition to prices and firm-level outputs, the observer also observes some parameter α_i that has an impact on firm i's cost function, which we denote as $C_i(\cdot\,; \alpha_i)$. We assume that $\alpha_i \in A_i \subseteq \mathbb{R}^n$. The firm i has a differentiable cost function and its marginal cost is affected by α_i; specifically, higher values of α_i (in the product order) lead to higher marginal costs while costs are unchanged if the parameter is unchanged. In other words,

$$C_i'(q_i; \alpha_i) \quad (=)\, C_i'(q_i; \hat{\alpha}_i) \text{ for all } q_i > 0 \text{ if } \alpha_i > (=)\, \hat{\alpha}_i. \tag{18}$$

In the case where α_i is the observable price of some major input in the production process, a well-known result says that marginal cost will increase with α_i if the demand for this input

(as a function of the output level) is normal. Note that we allow α_i to be a vector (for example, the prices of different inputs), so observed parameters may not be comparable; for two non-comparable parameters, we allow their marginal cost functions to differ without being ordered.

In this context, a set of observations takes the form $\{[P_t, (Q_{i,t})_{i\in I}, (a_{i,t})_{i\in I}]\}_{t\in T}$, where $a_{i,t}$ is the observed value of α_i at observation t. This set is said to be *Cournot rationalizable with* $\mathbf{C}^2$ *and convex (convincing) cost functions that agree with* $\{a_{i,t}\}_{(i,t)\in I\times T}$ if there are downward sloping demand functions P_t with $P_t(Q_t) = P_t$ and $\mathbf{C}^2$ and convex (convincing) cost functions $C_i(\cdot\,; a_{i,t})$ obeying (18) such that $(Q_{i,t})_{i\in I}$ is a Cournot equilibrium at observation t.

Corollary 3. *The set of observations* $\{[P_t, (Q_{i,t})_{i\in I}, (a_{i,t})_{i\in I}]\}_{t\in T}$ *is Cournot rationalizable with* $\mathbf{C}^2$ *and convex cost functions that agree with* $\{a_{i,t}\}_{(i,t)\in I\times T}$ *if, and only if, there are positive scalars* $\{\delta_{i,t}\}_{(i,t)\in I\times T}$ *satisfying the common ratio property, with*

$$\delta_{i,t} \geq (=)\, \delta_{i,t'} \text{ whenever } Q_{i,t} \geq (=)\, Q_{i,t'} \text{ and } a_{i,t} \geq (=)\, a_{i,t'}. \tag{19}$$

Corollary 4. *The set of observations* $\{[P_t, (Q_{i,t})_{i\in I}, (a_{i,t})_{i\in I}]\}_{t\in T}$ *is Cournot rationalizable with convincing cost functions that agree with* $\{a_{i,t}\}_{(i,t)\in I\times T}$ *if and only if the following conditions are satisfied:*

(a) There are positive scalars $\{\delta_{i,t,\tilde{t}}\}_{(i,t,\tilde{t})\in I\times T\times T}$ *such that (i) if* $a_{i,t} = a_{i,t'}$ *and* $Q_{i,t} = Q_{i,t'}$ *then* $\delta_{i,t,t} = \delta_{i,t',t'}$*; (ii) if* $a_{i,\tilde{t}} = a_{i,\tilde{t}'}$ *then* $\delta_{i,t,\tilde{t}} = \delta_{i,t,\tilde{t}'}$ *for all* $t \in T$*; (iii) if* $a_{i,\tilde{t}} > a_{i,\tilde{t}'}$ *then* $\delta_{i,t,\tilde{t}} \geq \delta_{i,t,\tilde{t}'}$ *for all* $t \in T$*; and (iv)* $\{\delta_{i,t,t}\}_{(i,t)\in I\times T}$ *satisfy the common ratio property, in the sense that, for all* t,

$$\frac{P_t - \delta_{1,t,t}}{Q_{1,t}} = \frac{P_t - \delta_{2,t,t}}{Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t,t}}{Q_{I,t}} > 0.$$

(b) There are positive scalars $\{\gamma_{i,t,\tilde{t}}\}_{(i,t,\tilde{t})\in I\times T\times T}$ *such that (i) if* $a_{i,t} = a_{i,t'}$ *and* $Q_{i,t} = Q_{i,t'}$ *then* $\gamma_{i,t,t} = \gamma_{i,t',t'}$*; (ii) if* $a_{i,\tilde{t}} = a_{i,\tilde{t}'}$ *then* $\gamma_{i,t,\tilde{t}} = \gamma_{i,t,\tilde{t}'}$ *for all* $t \in T$*; (iii) if* $a_{i,\tilde{t}} > a_{i,\tilde{t}'}$ *then* $\gamma_{i,t,\tilde{t}} \geq \gamma_{i,t,\tilde{t}'}$ *for all* $t \in T$*; and (iv)* $\gamma_{i,t,\tilde{t}}$ *has the discrete marginal property, i.e.,*

$$\sum_{Q_{i,s}\in\mathbb{Q}(t',t)} \gamma_{i,s,t}\left(Q_{i,s} - Q_{i,l(s)}\right) < P_t(Q_t - Q_{t'}) \text{ for all } t' \in L_i(t). \tag{20}$$

(c) For all i *and* $t \neq t_i^*$,

$$\gamma_{i,t,\tilde{t}} \leq \frac{1}{2}\left(\delta_{i,l_i(t),\tilde{t}} + \delta_{i,t,\tilde{t}}\right). \tag{21}$$

7. Measurement error and unobserved cost shifters

The revealed preference tests we have constructed so far are binary – either a data set passes a test or it fails. In this section, we show that it is possible to modify these tests in a way that allows the observer to take formal account of the *extent* of any failure and use this information in deciding whether or not to reject the model being tested. One reason why a data set may fail a test even if the model is true is that the data contain errors. Varian (1985) suggested a way of dealing with this problem, primarily in the context of demand and production theory; it can also be applied to the Cournot model and we do so in Section 8.1. In Section 8.2, we construct a statistical test of the Cournot hypothesis by positing unobserved fluctuations in cost functions. To keep the exposition simple, we confine our discussion to the case of an industry producing a homogeneous good, with firms having convex cost functions.

7.1. Measurement Error

We denote the collection of data sets which are Cournot rationalizable with convex cost functions by $\mathfrak{D}$. Suppose that an observed data set $\{[P_t, (Q_{i,t})_{i\in I}]\}_{t\in T}$ is *not* Cournot rationalizable with convex cost functions. We postulate that this is because it has been contaminated with measurement error. In other words, the 'true' data set is $\{[P_t + \varepsilon_t^P, (Q_{i,t} + \varepsilon_{i,t}^Q)_{i\in I}]\}_{t\in T} \in \mathfrak{D}$, where the error terms $\Xi = \{\varepsilon_t^P, (\varepsilon_{i,t}^Q)_{i\in I}\}_{t\in T}$ are assumed to be classical with variance σ^2. Consider now the null hypothesis that the true data set is Cournot rationalizable with convex cost functions. We construct a test statistic for this null hypothesis based on the loss function $\mathbb{L} = [\xi'\xi]/\sigma^2$, where ξ is the vector formed by concatenating the errors in the set Ξ. Since the errors are normally distributed, $\mathbb{L}$ has a chi-squared distribution with $T(I+1)$ degrees of freedom.

Following Varian (1985), we can estimate a lower bound on $\mathbb{L}$ by solving the following optimization problem:

$$L = \min \left\{ \frac{\hat{\xi}'\hat{\xi}}{\sigma^2} : \{(P_t + \hat{\varepsilon}_t^P, (Q_{i,t} + \hat{\varepsilon}_{i,t}^Q)_{i\in I})\}_{t\in T} \in \mathfrak{D} \right\}, \tag{22}$$

where $\hat{\xi}$ is the concatenation of $\{\hat{\varepsilon}_t^P, (\hat{\varepsilon}_{i,t}^Q)_{i\in I}\}_{t\in T}$. This problem involves finding the minimum perturbation to the observed data so that the perturbed data is Cournot rationalizable. If

we denote by χ_α the critical value of the chi-squared distribution, it follows that if we reject the null hypothesis whenever $L > \chi_\alpha$, we know that $\mathbb{L} > \chi_\alpha$ and thus the probability of Type I error is no greater than α.

Besides the fact that the variance of the measurement error is typically unknown,[18] a difficulty with implementing this method is that it is not easy to compute L, since the set of $\hat{\xi}$ obeying the constraint of Problem (22) is not characterized by a linear program: by Theorem 1, the constraint holds if and only if there are $\hat{\varepsilon}^P_{i,t}$, $\hat{\varepsilon}^Q_{i,t}$, and $\delta_{i,t}$ such that the common ratio property holds and $\delta_{i,t}$ is increasing with $Q_{i,t} + \hat{\varepsilon}^Q_{i,t}$; the former is a linear condition but the latter is not. This computational issue also arises when measurement errors are included in the Afriat inequalities, and is well-known.[19]

7.2. Unobserved cost fluctuations

Instead of assuming measurement error, let us now postulate that there are unobserved fluctuations to the cost functions, so that firm i's cost function at observation t is

$$C_i(q_i, t) = C_i(q_i) + \varepsilon_{i,t} q_i,$$

where $\varepsilon_{i,t}$ is drawn from a normal distribution with zero mean and variance σ^2. The null hypothesis is that, for every $t \in T$, $(P_t, (Q_{i,t})_{i \in I})$ is a Cournot equilibrium for some perturbation of the convex cost functions. Assuming the null hypothesis, the test statistic $\mathbb{L} = [\xi' \xi]/\sigma^2$, where ξ is the concatenation of perturbations in the set $= \{(\varepsilon_{i,t})_{i \in I}\}_{t \in T}$, has a chi-squared distribution with IT degrees of freedom.

While ξ is not directly observable, it is possible to form a lower bound on $\mathbb{L}$ by solving the following optimization problem. Let $\mathfrak{D}$ be the set of all $(\delta_{i,t}, \delta_{i,t}, \hat{\varepsilon}_{i,t})_{(i,t) \in I \times T}$ such that

(i) $\delta_{i,t} > 0$ and $\{\delta_{i,t}\}_{(i,t) \in I \times T}$ obeys the common ratio property,

(ii) $\delta_{i,t} > 0$ is increasing with $Q_{i,t}$, and

(iii) $\hat{\varepsilon}_{i,t} = \delta_{i,t} - \delta_{i,t}$.

[18] Varian (1985) suggests that estimates of the variance can be obtained from parametric or nonparametric fits of the data, from knowledge of how the variables were actually measured, or from other data sources. Alternatively, he suggests computing how large the variance would need to be in order for the null hypothesis to be accepted and comparing this value to our prior opinion regarding the precision of the data.

[19] See Varian (1985); in particular, the optimization problem on page 449 and comment (1) on page 456. For a recent discussion of this issue, see Kuosmanen et al. (2007).

Letting $\hat{\xi}$ denote the concatenation of $\{(\hat{\varepsilon}_{i,t})_{i\in I}\}_{t\in T}$, we can define

$$L = \min \left\{ \frac{\hat{\xi}'\hat{\xi}}{\sigma^2} : (\delta_{i,t}, \delta_{i,t}, \hat{\varepsilon}_{i,t})_{(i,t)\in I\times T} \in \mathfrak{D} \right\}. \tag{23}$$

Since it is always possible to find $\delta_{i,t}$ obeying the common ratio property (see the proof of Theorem 5), there must be $(\delta_{i,t}, \delta_{i,t}, \varepsilon_{i,t})_{(i,t)\in I\times T}$ obeying (i)–(iii) and, hence, problem (23) is well-defined. When $(\delta_{i,t}, \delta_{i,t}, \varepsilon_{i,t})_{(i,t)\in I\times T}$ obeys (i)–(iii), the observation $(P_t, (Q_{i,t})_{i\in I})$ can be rationalized as a Cournot equilibrium in which firm i has the cost function $\hat{C}_i(q_i; t) = \hat{C}_i(q_i) + \hat{\varepsilon}_{i,t} q_i$, where $\hat{C}_i$ is any convex cost function with $\delta_{i,t} \in \hat{C}_i'(q_i)$. Finally, note that the constraints (i)–(iii) are linear, so the minimization problem here is computationally straightforward, unlike problem (22). As before, if we denote by χ_α the critical value of the chi-squared distribution, we reject the null hypothesis whenever $L > \chi_\alpha$, and since $\mathbb{L} > \chi_\alpha$ we know that the probability of Type I error is at most α.

It is easy to adapt the above procedure to develop a test which can simultaneously account for observed and unobserved cost shifters. This would involve augmenting conditions (i)–(iii) to account for observable cost shifters as shown in Corollaries 3.

8. Application: The world market for crude oil

Accounting for roughly one third of global oil production, OPEC is a dominant player in the international oil market. OPEC was founded in 1960 and exists, in its own words, "to co-ordinate and unify petroleum policies among Member Countries, in order to secure fair and stable prices for petroleum producers; an efficient, economic and regular supply of petroleum to consuming nations; and a fair return on capital to those investing in the industry." OPEC's stated aims are effectively those of a cartel, but its ability to set world oil prices is questionable. Hence, a large literature has emerged that attempts to model its actions and to test whether these models fit its observed behavior. For the most part, the literature suggests that OPEC is a "weakly functioning cartel" of some sort, and is not "competitive" in either the price-taking or non-cooperative Cournot senses – see, for example, Alhajji and Huettner (2000), Dahl and Yücel (1991), Griffin and Neilson (1994), or Smith (2005). Many of these tests rely on parametric assumptions about the functional forms taken by market demand, countries' objective functions and production costs. Typically, they also

require that factors shifting the cost and inverse demand functions be observed, and rely on constructed proxies such as estimates of countries' extraction costs, the presence of US price controls, and involvement of an OPEC member in a war. Given the ambitious questions they are trying to answer, this seems unavoidable.

Our objective is more specific. All we wish to do is to use the results developed in the previous sections to test whether the behavior of the oil-producing countries is consistent with the Cournot model or, more generally (given the discussion in Section 3), any *symmetric* CV model. Our tests make use of very few ancillary assumptions, giving, so to speak, the greatest benefit of the doubt to the hypothesis, by allowing for a very large class of cost functions and by not making any assumptions at all about the evolution of demand (apart from the assumption that it is downward sloping with respect to output). In spite of this apparent permissiveness, our tests can reject the restrictions of the Cournot model in real world data.

Two sources of data are used for this study. The first is the *Monthly Energy Review (MER)*, published by the US Energy Information Administration. This provides full-precision series of monthly crude oil production in thousands of barrels per day by the twelve current OPEC members (Algeria, Angola, Ecuador, Iran, Iraq, Kuwait, Libya, Nigeria, Qatar, Saudi Arabia, the United Arab Emirates, and Venezuela) and seven non-members (Canada, China, Egypt, Mexico, Norway, the United States, and the United Kingdom).[20] This series also contains total world output. The data are available from January 1973 until April 2009, giving a total length of T = 436 months and $M \times T$ = 8248 country-month observations. The second source of data is a series of oil prices published by the St. Louis Federal Reserve, in dollars per barrel. This series is deflated by the monthly consumer price index reported by the Bureau of Labor Statistics, so that prices are in 2009 US dollars. Since the time windows over which Cournot behavior is tested are short (twelve months or less), the adjustment for inflation should not matter to the results.[21]

Each test consists of using a linear programming algorithm to find whether there exists

[20] Russia and the former Soviet Union are not used here, because the two are not comparable units. Although the composition of OPEC has changed over the course of the data (Ecuador left in 1994 and returned in 2007, Gabon left in 1995, Angola joined in 2007, and Indonesia left in 2007), the overall pattern of rejecting Cournot behavior below does not depend on what countries are considered to be part of OPEC.

[21] The rejection rates reported below are similar with nominal price series.

a solution to the specified linear program. If a solution exists, this subset of the data can be rationalized within the model, i.e. Cournot behavior by these M countries is not rejected by the data during the period tested. Clearly, as M and W increase, it is more likely that at least one country is not behaving optimally in at least one period, and so it is more likely that it will not be possible to satisfy the set of inequalities. Rather than performing a single test for whether the entire data series can be rationalized, we select a number of countries M, and then test whether the data for each of the $\binom{\bar{M}}{M}$ possible combinations of countries in each of the $T + 1 - W$ periods of length W can be rationalized. We then report the percentage of these $\binom{\bar{M}}{M}(T + 1 - W)$ cases in which optimal behavior is rejected. The time windows selected are short; W is either 3 months, 6 months, or 12 months. This is in keeping with the assumption that cost functions do not change over the period of the test. If a test is able to reject for a small amount of data (for example, three countries over three months), it demonstrates that, despite the generality of the non-parametric framework, the test has sufficient power to detect non-equilibrium behavior in real data.

Table 1: Rejection rates with convex cost functions

		OPEC sample			
		Number of Countries			
		2	3	6	12
	3 Months	0.28	0.54	0.89	1.00
Window	6 Months	0.65	0.89	1.00	1.00
	12 Months	0.90	0.99	1.00	1.00
		Non-OPEC sample			
		Number of Countries			
		2	3	6	7
	3 Months	0.44	0.75	0.99	1.00
Window	6 Months	0.83	0.98	1.00	1.00
	12 Months	0.96	1.00	1.00	1.00

Notes: The rejection rate reported is the proportion of cases that were rejected. For example, there are $436 + 1 - 3 = 434$ three month periods in the data. There are 66 possible combinations of two out of twelve OPEC members. The entry for two countries and three months, then, reports that out of the $434 \times 66 = 28,644$ possible tests of two OPEC members over three months, 8138, or 28% could not be rationalized.

We use the linear program specified in Theorem 1 to test whether the data sets are Cournot rationalizable with convex cost curves. Table 1 presents the percentage of cases (in the sense explained in the Notes below the table) for which the data is *not* Cournot

rationalizable with convex costs over groups of 2, 3, 6 and 12 OPEC countries within windows of 3, 6, and 12 months. The results are unambiguous { once there are more than a handful of observations used for the test, the behavior of OPEC members cannot be explained by the Cournot model with convex costs. For nearly 90% of six-month periods with three countries, the test rejects optimal behavior. Once six countries are included, fewer than one six-month case in ten thousand can be rationalized. The same test was performed for the non-OPEC countries (see Table 1). Once again the results are strongly against the Cournot model. For almost all six month periods, when at least three countries are considered, the data cannot be rationalized by the Cournot model with convex costs.

Table 2: Rejection rates with convincing cost functions

		OPEC sample			
		Number of Countries			
		2	3	6	12
	3 Months	0.21	0.41	0.76	0.98
Window	6 Months	0.40	0.66	0.92	1.00
	12 Months	0.60	0.84	0.98	1.00
		Non-OPEC sample			
		Number of Countries			
		2	3	6	7
	3 Months	0.06	0.13	0.36	0.43
Window	6 Months	0.12	0.25	0.63	0.73
	12 Months	0.20	0.45	0.84	0.90

Notes: See Table 1.

It is, in principle, possible that the tests reported in Table 1 reject the Cournot hypothesis because convexity of the cost functions is too strong an assumption. To address this potential problem, tests for convincing Cournot rationalizability, using the linear program speci ed in Theorem 6, were also carried out. These are reported in Table 2. Given the very permissive setup, one may expect the test to have little power, but that is not the case. Rejection rates for the countries in OPEC exceed 50% with 3 countries and 6 observations. In the case of the non-OPEC countries the drop in the rejection rate is sharper and the picture becomes mixed, but rejection rates still exceed 50% with 6 countries and 6 observations.

9. Conclusion

The purpose of this paper is to develop revealed preference tests for equilibrium in a multi-firm oligopoly setting akin to that of the classical revealed preference techniques for consumption and production. We show that these tests can take into account various features of oligopoly models – the possibility of changes to both cost and demand functions, non-convex cost functions, multi-product industries, etc. When a data set passes the test, the test also yields cost information on the firms in the industry. In spite of its minimal data requirements (in particular, effectively no assumptions are placed on the demand curves), we show that these tests are powerful enough to reject the Cournot hypothesis in the case of the world market for crude oil.

The techniques developed in this paper can potentially be adapted to other models of oligopoly or games. One potential application is to models of differentiated product Bertrand oligopoly. Revealed preference tests of this model may have some of the features of the tests of differentiated product Cournot oligopoly we discuss in the Supplement. We leave this for future work.

Appendix

Proof of Corollary 2: To see that [A] implies [B], suppose that the data is rationalized with demand functions $\{P_t\}_{t\in T}$ and cost functions $\{C_i\}_{i\in I}$. We have already shown in Theorem 1 that the first-order condition guarantees the existence of $\delta_{i,t} \in C_i'(Q_{i,t})$ obeying the common ratio property and condition (a). Condition (b) holds since for a firm in J, $C_i'(Q_{i,t})$ is unique, so clearly $\delta_{i,t} = \delta_{i,t'}$ whenever $Q_{i,t} = Q_{i,t'}$. Lastly, a firm in J' has constant marginal cost, so $\delta_{i,t}$ does not vary with t, which is condition (c).

To see that [B] implies [A], it suffices to notice that the proof of Lemma 2 can be strengthened to say that (I) if the positive scalars $\{\delta_{i,t}\}_{t\in T}$ are finely increasing with $Q_{i,t}$, C_i can be chosen to be a $\mathbf{C}^2$ function and (II) if the positive scalars $\{\delta_{i,t}\}_{t\in T}$ are independent of $Q_{i,t}$, then C_i can be chosen to be linear. Consequently, C_i (as defined by equation (3)) is, respectively, $\mathbf{C}^2$ and linear. *Q.E.D.*

Proof of Theorem 2: We first show that [A] implies [B]. Suppose that the data is ratio-

nalized with demand functions $\{P_t\}_{t\in T}$ and cost functions $\{C_i\}_{i\in I}$. At observation t, firm i's choice of $Q_{i,t}$ is optimal given the output of other firms and given its conjecture θ_i. By the first-order condition, there is $\delta_{i,t} \in C_i'(Q_{i,t})$ (the set of subgradients of C_i at $Q_{i,t}$) such that $\theta_i Q_{i,t} P_t'(Q_t) + P_t(Q_t) - \delta_{i,t} = \theta_i Q_{i,t} P_t'(Q_t) + P_t - \delta_{i,t} = 0$. Re-arranging this equation, we obtain $-P_t'(Q_t) = (P_t - \delta_{i,t})/(\theta_i Q_{i,t})$ for all i. This gives us equation (6). Since C_i is convex, $\delta_{i,t}$ must increase with $Q_{i,t}$.

To proof that [B] implies [A] we need only mimic the two-step procedure used in the proof of Theorem 1. Lemma 2 guarantees that firm i has a convex and well-behaved cost function C_i such that $\delta_{i,t} \in C_i(Q_{i,t})$. A modified version of Lemma 1 is then needed to show that $Q_{i,t}$ is firm i's optimal choice at observation t, given it's conjecture θ_i. As in the proof of Lemma 1, we use the linear function $P_t(Q) = a_t - b_t Q$, where $b_t = (P_t - \delta_{i,t})/(\theta_i Q_{i,t})$. This is well-defined because of (6). *Q.E.D.*

Proof of Lemma 6: For each firm i, define $g_i(q_i) = k_i(q_i - Q_{i,t}) + \delta_{i,t}$. The graph of g_i is a line, with slope k_i that passes through the point $(Q_{i,t}, \delta_{i,t})$. Since $\delta_{i,t} < P_t$, and C_i is $\mathbf{C}^2$, there is $\varepsilon > 0$ and k_i (for $i \in I$) such that, $P_t > g_i(Q_{i,t} - \varepsilon)$ and for q_i in the interval $[Q_{i,t} - \varepsilon, Q_{i,t})$, we have

$$g_i(q_i) > C_i'(q_i). \tag{24}$$

(Note that k_i must be a negative number if $C_i'''(Q_{i,t}) < 0$.) For q_i in $[0, Q_{i,t} - \varepsilon]$, there exists $\zeta > 0$ such that

$$Pq_i - C_i(q_i) < PQ_{i,t} - C_i(Q_{i,t}) \text{ for } P_t < P < P_t + \zeta; \tag{25}$$

this follows from property (iii) in Lemma 5. Note that ζ is common across all firms.

We shall specify the function P_t', so P_t can be obtained by integration. Holding the output of firm j (for $j \neq i$) at $Q_{j,t}$, we denote the marginal revenue function for firm i by $m_{i,t}$; i.e., $m_{i,t}(q_i) = P_t'(\sum_{j\neq i} Q_{j,t} + q_i)q_i + P_t(\sum_{j\neq i} Q_{j,t} + q_i)$. We first consider the construction of P_t' in the interval $[0, Q_t]$, where $Q_t = \sum_{i\in I} Q_{i,t}$. Choose P_t' with the following properties: (a) $P_t'(Q_t) = (\delta_{i,t} - P_t)/Q_{i,t}$ (which is equivalent to the first-order condition $m_{i,t}(Q_{i,t}) = C_i'(Q_{i,t}) = \delta_{i,t}$; note that there is no ambiguity here because of (1)), (b) P_t' is negative, decreasing and concave in $[0, Q_t]$, (c) $\int_0^{Q_t} P_t'(q)dq = P_t - P_t(0) > -\zeta$ and (d) $P_t'(Q_t - \varepsilon)$ is sufficiently close to zero so that $m_{i,t}(Q_{i,t} - \varepsilon) > g_i(Q_{i,t} - \varepsilon)$. Property (b) guarantees that

$m_{i,t}$ is decreasing and concave (as a function of q_i). This fact, together with (a) and (d), ensures that $m_{i,t}(q_i) > g_i(q_i)$ for all i and q_i in $[Q_{i,t} - \varepsilon, Q_{i,t})$; combining with (24), we obtain $m_{i,t}(q_i) > C_i'(q)$. Therefore, in the interval $[Q_{i,t} - \varepsilon, Q_{i,t}]$, firm i's profit is maximized at $q_i = Q_{i,t}$. Because of (c), $P_t < P_t(q) < P_t + \zeta$, so by (25), $P_t(\sum_{j \neq i} Q_{j,t} + q_i)q_i - C_i(q_i) < P_t Q_{i,t} - C_i(Q_{i,t})$ for q_i in $[0, Q_{i,t} - \varepsilon]$.

To recap, we have constructed P_t' (and hence P_t) such that, with this inverse demand function, firm i's profit at $Q_{i,t}$ is higher than at any output below $Q_{i,t}$, so long as other firms are producing $\sum_{j \neq i} Q_{j,t}$. Our next step is to show how to specify P_t' for $q > Q_t$ in such a way that firm i's profit at $q_i = Q_{i,t}$ is higher than at any output level above $Q_{i,t}$, for every firm i. It suffices to have P_t such that, for $q_i > Q_{i,t}$,

$$m_{i,t}(q_i) = P_t'\Big(\sum_{j \neq i} Q_{j,t} + q_i\Big)q_i + P_t\Big(\sum_{j \neq i} Q_{j,t} + q_i\Big) < C_i'(q_i),$$

so firm i's marginal cost always exceeds its marginal revenue for $q_i > Q_{i,t}$. Provided P_t is decreasing, it suffices to have $P_t'(\sum_{j \neq i} Q_{j,t} + q_i)q_i + P_t < C_i'(q_i)$, which is equivalent to

$$P_t'(Q_t + x) > \frac{P_t - C_i'(Q_{i,t} + x)}{Q_{i,t} + x} \quad \text{for } x > 0 \text{ and all firms } i. \tag{26}$$

The right side of this inequality is a finite collection of continuous functions of x and at $x = 0$, the two sides are equal to each other (because of (1)). Clearly, we can choose $P_t' < 0$ such that (26) holds for $x > 0$. *Q.E.D.*

Proof of Theorem 6: To see that these conditions are necessary, set $\delta_{i,t} = C_i'(Q_{i,t})$ and $\gamma_{i,t} = [C_i(Q_{i,t}) - C_i(Q_{i,l_i(t)})]/[Q_{i,t} - Q_{i,l_i(t)}]$. Then (c) is just the convincing criterion on the cost functions, (a) follows from the first-order condition at the equilibrium and we have already explained why (b) (see the discussion following (12)).

For sufficiency, set $C_{i,t} = \sum_{Q_{i,s} \in \mathbb{Q}_i(t_i, t)} \gamma_{i,s}(Q_{i,s} - Q_{i,l(s)})$; then (16) guarantees that the data set $\{[P_t, (Q_{i,t})_{i \in I}, (C_{i,t})_{i \in I}]\}_{t \in T}$ obeys the discrete marginal property (see (11) and (12)). Lemmas 5 and 6 then guarantee the existence of P_t and C_i with the desired properties. In particular, $C_i(Q_{i,t}) = C_{i,t}$, $C_i'(Q_{i,t}) = \delta_{i,t}$ and the average marginal cost between $Q_{i,l(t)}$ and $Q_{i,t}$, is $\gamma_{i,t}$, so that C_i is convincing because of (17). *Q.E.D.*

Proof of Corollary 3: Let $\delta_{i,t} = C_i'(Q_{i,t}; a_{i,t})$. The common ratio property follows from the first-order condition. If $Q_{i,t} = Q_{i,t'}$ and $a_{i,t} = a_{i,t'}$, we have $C_i'(Q_{i,t}; a_{i,t}) = C_i'(Q_{i,t'}; a_{i,t'})$,

so $\delta_{i,t} = \delta_{i,t'}$. If $Q_{i,t} \geq Q_{i,t'}$ and $a_{i,t} \geq a_{i,t'}$, then $C_i'(Q_{i,t'}; a_{i,t'}) \leq C_i'(Q_{i,t}; a_{i,t'}) \leq C_i'(Q_{i,t}; a_{i,t})$, given the convexity of $C_i(\cdot\,; a_{i,t'})$ and (18); so $\delta_{i,t} \geq \delta_{i,t}$.

Now suppose that there are $\delta_{i,t}$ obeying the common ratio property and (19). Choose positive scalars $d_{i,t,\tilde{t}}$ for every $(i, t, \tilde{t}) \in I \times T \times T$ with the following properties: (i) $d_{i,t',t'} = \delta_{i,t'}$, (ii) if $a_{i,\hat{t}} = a_{i,\tilde{t}}$, then $d_{i,t,\hat{t}} = d_{i,t,\tilde{t}}$ for all t and $d_{i,t',\hat{t}} = d_{i,t'',\tilde{t}}$ whenever $Q_{i,t'} = Q_{i,t''}$ and, (iii) $d_{i,t',\hat{t}} \geq d_{i,t'',\tilde{t}}$ whenever $Q_{i,t'} > Q_{i,t''}$ and $a_{i,\hat{t}} = a_{i,\tilde{t}}$, and (iv) if $a_{i,\hat{t}} > a_{i,\tilde{t}}$ then $d_{i,t,\hat{t}} \geq d_{i,t,\tilde{t}}$ for all t. This is possible because of (19). Due to (ii), we may associate, for each distinct value of $a_{i,\tilde{t}}$ a single $\mathbf{C}^2$ cost function $C_i(\cdot\,; a_{i,\tilde{t}})$ such that

$$C_i'(Q_{i,t}; a_{i,\tilde{t}}) = d_{i,t,\tilde{t}}. \tag{27}$$

Condition (iii) guarantees that this function can be chosen to be convex. Furthermore, because of (iv), we could choose C_i in such a way that $C_i'(q_i; a_{i,\hat{t}}) \geq C_i'(q_i; a_{i,\tilde{t}})$ for all $q_i > 0$ if $a_{i,\hat{t}} > a_{i,\tilde{t}}$. (These claims follow from Lemma 2 and straightforward modifications of its proof.) Notice that equation (27) and property (i) imply that $C_i'(Q_{i,\tilde{t}}; a_{i,\tilde{t}}) = \delta_{i,\tilde{t}}$, so the common ratio property on $\{\delta_{i,t}\}_{(i,t) \in I \times T}$ tells us that

$$\frac{P_{\tilde{t}} - C_1'(Q_{1,\tilde{t}}; a_{1,\tilde{t}})}{Q_{1,\tilde{t}}} = \frac{P_{\tilde{t}} - C_2'(Q_{2,\tilde{t}}; a_{2,\tilde{t}})}{Q_{2,\tilde{t}}} = \ldots = \frac{P_{\tilde{t}} - C_I'(Q_{I,\tilde{t}}; a_{I,\tilde{t}})}{Q_{I,\tilde{t}}} > 0. \tag{28}$$

By Lemma 1, there is $P_{\tilde{t}}$ such that $P_{\tilde{t}}(Q_{\tilde{t}}) = P_{\tilde{t}}$ and, with each firm i having the cost function $C_i(\cdot\,; a_{i,\tilde{t}})$, $\{Q_{i,\tilde{t}}\}_{i \in I}$ constitutes a Cournot equilibrium. *Q.E.D.*

Proof of Corollary 4: Setting $\delta_{i,t,\tilde{t}} = C_i'(Q_{i,t}; a_{i,\tilde{t}})$, it is clear that (a)-(i) holds and also (a)-(ii) holds because the cost function of firm i is C^2 and unchanged if $a_{i,\tilde{t}} = a_{i,\tilde{t}'}$. The condition (a)-(iii) holds because of the inequality part of (18) and (a)-(iv) holds because of the first order condition at equilibrium. If we let $\Delta_{i,t,\tilde{t}} = [C_i(Q_{i,t}; a_{i,\tilde{t}}) - C_i(Q_{i,l_i(t)}; a_{i,\tilde{t}})]/[Q_{i,t} - Q_{i,l_i(t)}]$, then (c) is just the requirement that $C_i(\cdot\,; a_{i,\tilde{t}})$ obeys the convincing criterion. The necessity of (b)-(i), (b)-(ii), and (b)-(iii) follows from (18); for the necessity of (b)-(iii) see the discussion following (12).

To establish the sufficiency of (a), (b), and (c) for Cournot rationalizability, first note that because the inequality (20) is strict, there is no loss of generality in assuming that

$$\Delta_{i,t,\tilde{t}} > \Delta_{i,t,\tilde{t}'} \quad \text{for all } t \in T \text{ if } a_{i,\tilde{t}} > a_{i,\tilde{t}'}. \tag{29}$$

Set $C_{i,t,\tilde{t}} = \sum_{Q_{i,s} \in \mathbb{Q}_i(t_i,t)} \delta_{i,s,\tilde{t}}(Q_{i,s} - Q_{i,l(s)})$; then (20) guarantees that

$$C_{i,t,t} - C_{i,t',t} < P_t(Q_{i,t} - Q_{i,t'}) \text{ for } t' \in L_i(t).$$

Given this, Lemma 5 guarantees that there are cost functions $C_i(\cdot\,; a_{i,t})$ (one for each distinct value of $a_{i,t}$) such that $C_i(Q_{i,t,\tilde{t}}; a_{i,\tilde{t}}) = C_{i,t,\tilde{t}}$, $C_i'(Q_{i,t,\tilde{t}}; a_{i,\tilde{t}}) = \delta_{i,t,\tilde{t}}$, and for all q in $[0, Q_{i,t})$,

$$P_t q_i - C_i(q_i; a_{i,t}) < P_t Q_{i,t} - C_i(Q_{i,t}; a_{i,t}).$$

Furthermore, it is clear from a closer inspection of the proof of Lemma 5 that given conditions (a)(i)-(iv), (b)(i)-(iv), and (29), we may in fact choose C_i such that $C_i'(q_i; a_{i,\tilde{t}}) \geq C_i'(q_i; a_{i,\tilde{t}'})$ for all $q_i > 0$. Finally, Lemma 6 guarantees the existence of P_t such that the optimal choice of firm i at t is $Q_{i,t}$ given the output of the other firms and given its cost function $C_i(\cdot\,; a_{i,t})$. *Q.E.D.*

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SUPPLEMENT TO "REVEALED PREFERENCE TESTS OF THE COURNOT MODEL"

By Andrés Carvajal, Rahul Deb, James Fenske and John K.-H. Quah [1]

In this Supplement, we show how perfect collusion is testable on subsets of firms and develop tests which can account for product differentiation.

1. Perfect Collusion Amongst Subsets of Firms

In Section 3 of the main paper we showed that the hypothesis that all firms in the industry are colluding to maximize joint profit cannot be contradicted by a data set consisting only of prices and firm-level outputs. However, we also claimed that it *is* possible for a data set to be inconsistent with the hypothesis that a strict subset of firms in the industry is perfectly colluding (i.e., maximizing their joint profit), while playing a Cournot game against other firms (or other groups of firms) in the industry. In this section, we explain this claim in greater detail.

We consider an industry consisting of I firms producing a homogeneous good; we denote the set of firms by $\mathcal{I} = \{1, 2, \ldots, I\}$. Given a subset O of these $\mathcal{I}$ firms, we denote the total quantity produced at time t by this subset of firms by $Q_{O,t} = \sum_{i \in O} Q_{i,t}$,; similarly, we use $Q_{-O,t} = Q_{\mathcal{I} \setminus O,t}$ to denote the total quantity of the good produced by all the firms in $\mathcal{I} \setminus O$. Perfect collusion by the set of firms in O would involve each firm in O choosing a firm-specific output levels to maximize their *joint profit* in response to the remaining firms in the economy.

Consider a partition $\mathcal{O}$ of the firms in this industry, i.e., $\mathcal{O} = \{O_1, O_2, \ldots, O_J\}$ such that $O_j \cap O_{j'} = \emptyset$ for all $j \neq j'$, and where $\cup_{j=1}^{J} O_j = \mathcal{I}$. We refer to each element of $\mathcal{O}$ as a *coalition*, though we allow for the possibility that an element of $\mathcal{O}$ may contain just one firm. We denote the element of the partition that contains a firm i as O^i. For example, if we have 3 firms and we want to test if the data is consistent with the first two firms colluding against the third, then the appropriate partition $\mathcal{O}$ consists of $O_1 = \{1, 2\}$ and $O_2 = \{3\}$.

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Here $O^1 = O^2 = O_1$ and $O^3 = O_2$.

A given data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is *Cournot rationalizable with perfectly collusive firms having convex costs* corresponding to a partition $\mathcal{O}$ of the set of firms if there exist convex cost functions $\bar{C}_i$ for each firm i and downward sloping demand functions $\bar{P}_t$ for each observation t such that

(i) $\bar{P}_t(Q_t) = P_t$ for all $t \in \mathcal{T}$; and

(ii) $\{Q_{i,t}\}_{i\in O} \in \operatorname{argmax}_{\{q_i\}_{i\in O}} \left\{ \sum_{i\in O} q_i \; \bar{P}\left(\sum_{i\in O} q_i + Q_{-O,t}\right) - \sum_{i\in O} \bar{C}_i(q_i) \right\}$ for all $O \in \mathcal{O}$.

In other words, the firms in a coalition O are behaving like a single firm, choosing their joint output to maximize their joint profit. Note that profit maximization implies that any level of output is divided amongst the firms in O in a way that minimizes total cost. Therefore, we may understand the coalition O's total output as chosen to maximize its profit given the output of other firms in the industry and subject to the convex cost function

$$\bar{C}_O(q) = \min \left\{ \sum_{i\in O} \bar{C}_i(q_i) : \sum_{i\in O} q_i = q \right\}. \tag{1}$$

By definition, firm i is in the coalition O^i; if $Q_{i,t}$ is the optimal output of firm i at observation t, then the first order condition guarantees that there is $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$ such that

$$\frac{\bar{P}\left(Q_t\right) - \delta_{i,t}}{Q_{O^i,t}} = -\bar{P}'\left(Q_t\right). \tag{2}$$

Equation (2) leads to a modified common ratio property (see (3) below). Furthermore, since each firm has a convex cost function, $\delta_{i,t}$ must be increasing with the output of firm i. Given the intuition developed in the main paper, it should not be surprising that these two conditions are also sufficient for rationalizability.

Theorem 1. *The following statements on $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ are equivalent.*

[A] *The set of observations is Cournot rationalizable with perfectly collusive firms having convex costs corresponding to a partition $\mathcal{O} = \{O_j\}_{1\le j\le J}$ of the set of firms.*

[B] *There exists a set of positive numbers $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying*

$$\frac{P_t - \delta_{1,t}}{Q_{O^1,t}} = \frac{P_t - \delta_{2,t}}{Q_{O^2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{Q_{O^I,t}} > 0. \tag{3}$$

and such that, for each i, $\{\delta_{i,t}\}_{t\in\mathcal{T}}$ is increasing with $Q_{i,t}$ in the sense that $\delta_{i,t'} \ge \delta_{i,t}$ whenever $Q_{i,t'} > Q_{i,t}$.

Proof: We have already shown that [A] implies [B]. For the other direction, we first construct a convex cost function $\bar{C}_i$ for each firm i such that $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$. This can be done – see Lemma 2 in the main paper. Using an argument analogous to that in Lemma 1 in the main paper, it is clear that $Q_{i,t}$ maximizes the profit of coalition O^i when $\bar{P}$ is chosen to be linear with slope $-(P_t - \delta_{i,t})/Q_{O^i,t}$. *Q.E.D.*

If [B] in Theorem 1 holds, then a rationalization can be chosen in which firm i has a cost function $\bar{C}_i$ with $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$. It follows from (3) (or (2)) that $\delta_{i,t} = \delta_{k,t}$ if firms i and k are in the same coalition, i.e., if $O^i = O^k$. The resulting cost function of the coalition O^i, $\bar{C}_{O^i}$ (see (1)), will satisfy $\delta_{i,t} \in \bar{C}_{O^i}'(Q_{O^i,t})$.

Recall that in Section 3 of the main paper we pointed out that, without demand information, it is not possible to pin down the conduct parameter of each firm in the industry, even though their *relative* values are testable. The same feature is present in Theorem 1. To be precise, if a data set is consistent with the hypothesis of Cournot oligopoly across some partition $\mathcal{O}$ of firms, then it will be consistent with the hypothesis that the coalitions represented by the partition $\mathcal{O}$ are playing a CV-equilibrium, with every coalition having the same conduct parameter λ, for any $\lambda > 0$. This essentially follows from the fact that the condition

$$\frac{P_t - \delta_{1,t}}{\lambda Q_{O^1,t}} = \frac{P_t - \delta_{2,t}}{\lambda Q_{O^2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{\lambda Q_{O^I,t}} > 0$$

is identical to (3). (The reader should consult Section 3, and in particular Theorem 2, to fill in the details.)

We now provide two examples. The first one gives a data set that is Cournot rationalizable if the three firms are acting independently but not if two firms are colluding; the second example gives a data set that is Cournot rationalizable with two firms colluding but not with all three firms acting independently.

Example 1. Consider an industry with three firms, i, j and k where
(i) at observation t, $P_t = 16$, $Q_{i,t} = Q_{j,t} = 4$ and $Q_{k,t} = 16$; and
(ii) at observation t', $P_{t'} = 38$, $Q_{i,t} = Q_{j,t} = 8$ and $Q_{k,t} = 10$.
These observations are rationalizable as the outcome of a Cournot game with each firm acting independently. Indeed it is trivial to check that the common ratio property is satisfied with

$\delta_{i,t} = \delta_{j,t} = \delta_{i,t} = \delta_{j,t'} = 14$ and $\delta_{k,t} = \delta_{k,t'} = 8$. Therefore, by Corollary 2 of the main paper these observations are Cournot rationalizable with firms i and j having constant marginal cost of 14 and firm k having a constant marginal cost of 8.

On the other hand, these observations are *not* consistent with the hypothesis that firms i and j form a coalition, because in that case, the output of the coalition will be 8 in observation t and 16 in observation t'. On the other hand, the output of firm k falls from 16 to 10 between observations t and t'. Furthermore, it produces more than the coalition at observation t and less than the coalition at observation t': this sort of output variation cannot occur in a Cournot duopoly (see Example 1 of the main paper).

Example 2. Consider an industry with three firms, i, j and k where
(i) at observation t, $P_t = 16$, $Q_{i,t} = 3.5$, $Q_{j,t} = 0.5$ and $Q_{k,t} = 16$; and
(ii) at observation t', $P_{t'} = 38$, $Q_{i,t} = 3$, $Q_{j,t} = 5$ and $Q_{k,t} = 10$.
These observations are not compatible with the three firms acting independently. Notice that, between observations t and t', the output of firm i has fallen while that of firm j has gone up. Furthermore, the two firms have switched ranks. This is not possible in a Cournot oligopoly with three independent firms (see Example 1 of the main paper).

On the other hand, these observations are rationalizable as an oligopoly where firms i and j are colluding and playing a Cournot game with k. Indeed, in this case the coalition produces 4 at observation t and 8 at observation t'. As we pointed out in Example 1 above, this is compatible with a Cournot oligopoly in which firm k has a constant marginal cost of 8 and the coalition has a constant marginal cost of 14. The latter will hold if firms i and j both have the same constant marginal cost of 14; furthermore, in that case, any output division between firms i nd j, including the divisions observed, is cost efficient.

2. Cournot oligopoly with differentiated goods

As we have argued in the main text, the distinctive feature of our tests of the Cournot model is that we require no knowledge at all on the factors that influence demand. This is possible because of a crucial feature of the model which is captured by the common ratio property and which holds no matter how demand changes: *at equilibrium, when firm i chooses to*

same at every firm because the good is homogeneous.

Now consider the case when each firm produces a good that is differentiated from that of another firm, so that different firms can charge different prices and may face different own-price elasticities. (For a discussion of the Cournot model with differentiated goods, see Vives (1999).) In this situation, it is still possible to construct a test for the Cournot model that generalizes the one we have for a homogeneous good, provided the model has the following feature: the price impact on good i as firm i increases its output is related in some definite way with the price impact on good j as firm j increases its output and this relationship does not vary with the shocks that generate the data. Provided this holds, we can tease out cost information from the data (and, therefore, potentially disprove the model), even with minimal knowledge of the factors influencing demand. We illustrate our claim with a model of an industry containing these features.

Assume that each firm produces one good and the inverse demand for firm i's output is $\bar{P}_i(q;\tilde{t})$, where $q = (q_i)_{i\in\mathcal{I}}$ is the vector of outputs of the firms in the industry and $\tilde{t} \in \tilde{\mathcal{T}}$ is some unobservable shock that affects the demand for good i.[2] As usual, we require $\bar{P}_i$ to be decreasing in q_i. We also assume that product differentiation in this industry has an affine structure or, more formally, that the collection $\{\bar{P}_i(\cdot;\tilde{t})\}_{(i,\tilde{t})\in\mathcal{I}\times\tilde{\mathcal{T}}}$ form an *affine family*. By this we mean that there are scalars c_{ij} (with $c_{jj} = 0$), $m_j > 0$ and n_j such that, for all $\tilde{t} \in \tilde{\mathcal{T}}$,

1. $\frac{\partial \bar{P}_j}{\partial q_i}(q;\tilde{t}) = \frac{\partial \bar{P}_j}{\partial q_j}(q;\tilde{t}) + c_{ij}$;

2. $\frac{\partial \bar{P}_j}{\partial q_1}(q;\tilde{t}) = m_j \frac{\partial \bar{P}_1}{\partial q_1}(q;\tilde{t}) + n_j$.

Property 1 says that a change in $\tilde{t}$ that leads to (say) an increase in $d\bar{P}_j/dq_j$ leads to an equal increase in $d\bar{P}_j/dq_i$ (though $d\bar{P}_j/dq_i$ is distinct from $d\bar{P}_j/dq_j$ and will always be smaller if $c_{ij} < 0$). Property 2 relates the price impact on good 1 of an increase in the output of 1 with its impact on the price of good j; it specifies an affine and co-monotone relationship between the two. We should think of $\tilde{t}$ as reflecting general conditions that affect every good in the industry and the source of the observed variation in a data set. The coefficients c_{ij}, m_j and n_j (loosely speaking) measure the imperfect substitution possibilities across firms;

[2] There is no loss of generality in assuming that $\tilde{t}$ does not depend on i since $\tilde{t}$ can be multi-dimensional.

these do not vary with $\tilde{t}$ and remain the same at every observation.[3] The following result gives a characterization of an affine family of inverse demand functions.

Proposition 1. *Suppose that the collection $\{\bar{P}_i(\cdot;\tilde{t})\}_{(i,\tilde{t})\in\mathcal{I}\times\tilde{\mathcal{T}}}$ form an affine family. Then the inverse demand functions have the following semi-parametric form: there exist $\Phi : \mathbb{R}_+ \times \tilde{\mathcal{T}} \to \mathbb{R}$ with $\Phi_x(x;\tilde{t}) < 0$, functions $\gamma_j : \tilde{\mathcal{T}} \to \mathbb{R}$, and vectors $d_j = (d_{j1}, d_{j2}, ..., d_{jI}) \in \mathbb{R}^I$ such that*

$$\bar{P}_1(q;\tilde{t}) = \Phi\left(\sum_{i=1} q_i\, ;\tilde{t}\right) + d_1 \cdot q \ \text{ with } d_{11} = 0 \text{ and} \tag{4}$$

$$\bar{P}_j(q;\tilde{t}) = m_j\Phi\left(\sum_{i=1} q_i\, ;\tilde{t}\right) + \gamma_j(\tilde{t}) + d_j \cdot q \ \text{ for } j = 2, 3, ..., I. \tag{5}$$

Proof: Choose any nonzero vector v that is orthogonal to $(1, 1, ..., 1)$. Property 1 guarantees that

$$\sum_{i=1}\left(\frac{\partial \bar{P}_1}{\partial q_i}(q;\tilde{t}) - c_{i1}\right) v_i \equiv 0$$

so there is $\Phi(\cdot;\tilde{t})$ such that $\bar{P}_1(q;\tilde{t}) - c \cdot q = \Phi\left(\sum_{i=1}^{I} q_i; t\right)$, where the vector $c = (c_{i1})_{i\in\mathcal{I}}$. This gives us (4), with $d_1 = c$. Note that $\Phi'(\cdot;\tilde{t}) < 0$ because we require $\bar{P}_1(\cdot;\tilde{t})$ to be decreasing in its own price. By the same argument applied to good j, there is $\Psi(\cdot;\tilde{t})$ such that $\bar{P}_j(q;\tilde{t}) = \Psi\left(\sum_{i=1}^{I} q_i; t\right) + b_j \cdot q$ for $b_j = (c_{ij})_{i\in\mathcal{I}}$. By Property 2,

$$\Psi'\left(\sum_{i=1} q_i;\tilde{t}\right) + b_{j1} = m_j\Phi'\left(\sum_{i=1} q_i;\tilde{t}\right) + n_j.$$

Therefore, there is a scalar $\gamma_j(\tilde{t})$ such that

$$\Psi_j\left(\sum_{i=1} q_i;\tilde{t}\right) = m_j\Phi\left(\sum_{i=1} q_i;\tilde{t}\right) + (n_j - b_{j1})\left(\sum_{i=1} q_i\right) + \gamma_j(\tilde{t}).$$

Substituting this into the formula for $\bar{P}_j(\cdot;\tilde{t})$ gives us (5). *Q.E.D.*

A data set $\{(P_{i,t}, Q_{i,t})_{i\in\mathcal{I}}\}_{t\in\mathcal{T}}$ consisting of firm-level prices and outputs is *rationalizable as Cournot equilibria in a model of affine differentation* if there are cost functions for each firm and an affine family of demand functions $\{\bar{P}_i(\cdot;t)\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ such that each observation

[3] The formulation does not give any special status to good 1; the lack of symmetry is apparent and not real.

is a Cournot equilibrium outcome. The assumption of an affine structure *alone* imposes no restrictions on the data set, essentially because one is free to choose the scalars $\gamma^j(t)$ to match the data; however, it does imply that

$$\frac{\partial \bar{P}_1}{\partial q_1}(q;t) = \Phi'\left(\sum_{i=1} q_i\,;t\right) \quad \text{and} \quad \frac{\partial \bar{P}_j}{\partial q_j}(q;t) = m_j \Phi'\left(\sum_{i=1} q_i\,;t\right) + d_{jj}, \tag{6}$$

so that these two derivatives must have an affine relationship. If the observation at t constitutes a Cournot equilibrium, then each firm obeys its first order condition and there is $\delta_{j,t} \in \bar{C}'_j(Q_{j,t})$ (the set of subgradients of $\bar{C}_j$ at $Q_{j,t}$) such that

$$\frac{P_{j,t} - \delta_{j,t}}{Q_{j,t}} = -\frac{\partial \bar{P}_j}{\partial q_j}\left((Q_{1,t}, Q_{2,t}, ..., Q_{I,t});t\right) = -m_j \Phi'(Q_t, t) - d_{jj} \tag{7}$$

(recall that $Q_t = \sum_{i=1}^{I} Q_{i,t}$). It follows from (6) and (7) that we have the following generalized common ratio property: for every $t \in \mathcal{T}$,

$$\frac{P_{1,t} - \delta_{1,t}}{Q_{1,t}} = \frac{P_{2,t} - \delta_{2,t}}{m_2 Q_{2,t}} + k_2 = \ldots = \frac{P_t - \delta_{I,t}}{m_I Q_{I,t}} + k_I > 0, \tag{8}$$

where $k_i = d_{ii}/m_i$. The next result, which we prove in the Appendix, provides the revealed preference test for our model of differentiated products.

THEOREM 2. *The following statements on $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ are equivalent.*

[A] *The set of observations is rationalizable as Cournot equilibria in a model of affine differentation, with firms having convex cost functions.*

[B] *There are positive numbers $\{\delta_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$, positive numbers $\{m_i\}_{i \geq 2}$, and numbers $\{k_i\}_{i \geq 2}$ such that the following holds:* (i) *$P_{i,t} - \delta_{i,t} > 0$ for all (i,t);* (ii) *for each t, the generalized common ratio property (8) holds; and* (iii) *for each i, $\{\delta_{i,t}\}_{t \in \mathcal{T}}$ is increasing with $Q_{i,t}$.*

Proof: By Lemma 2 in the main paper, firm i has a convex cost function $\bar{C}_i$ with $\delta_{i,t} \in \bar{C}'_i(Q_{i,t})$. Let $\bar{P}_1(\cdot;t)$ have the form (9), with $b_t = (P_{1,t} - \delta_{1,t})/Q_{1,t}$ and a_t chosen such that $\bar{P}_1(Q_{1,t};t) = P_{1,t}$. It is straightforward to see that Firm 1's profit function is concave in q_1 and (with this choice of b_t) the first order condition is satisfied at $q_1 = Q_{1,t}$; therefore, Firm 1 is profit-maximizing at $q_1 = Q_{1,t}$. For firm j, let $\bar{P}_j(\cdot;t)$ have the form (10) and choose $\gamma_j(t)$ so that $\bar{P}_j(Q_{j,t};t) = P_{j,t}$. Because of (8), the first order condition of firm j's profit-maximization problem is satisfied at $q_j = Q_{j,t}$ and since firm j's profit function is also concave in q_j, the first order condition suffices for optimality. *Q.E.D.*

As in Theorem 1 of the main paper, the proof of Theorem 2 in fact shows that if a data set is rationalizable then it is rationalizable with *linear* inverse demand functions. To be specific, the rationalization provides Firm 1 with the inverse demand function

$$\bar{P}_1(q;t) = a_t - b_t \left(\sum_{i=1} q_i \right) \tag{9}$$

where $a_t > 0$, $b_t > 0$, and firm j (for $j \geq 2$) has the inverse demand function

$$\bar{P}_j(q;t) = m_j \left[a_t - b_t \left(\sum_{i=1} q_i \right) \right] + k_j m_j q_j + \gamma_j(t). \tag{10}$$

Notice that, unlike a_t, b_t, and $\gamma_j(t)$, k_j and m_j do not vary with t. We end this section with a simple example to show that there are observable restrictions in this model.

Example 3. Consider the following observations of two firms 1 and 2:
(i) at observation t, $P_{1,t} = 2$, $Q_{1,t} = 50$, $P_{2,t} = 4$ and $Q_{2,t} = 60$;
(ii) at observation t', $P_{1,t'} = 4$, $Q_{1,t'} = 40$, $P_{2,t'} = 3$ and $Q_{2,t'} = 70$.
We claim that these observations are not Cournot rationalizable with convex cost functions. Indeed, if they are, then the generalized common ratio property says that

$$\frac{P_{1,t} - \delta_{1,t}}{Q_{1,t}} = \frac{P_{2,t} - \delta_{2,t}}{m_2 Q_{2,t}} + k_2, \tag{11}$$

where $\delta_{1,t} \in \bar{C}_1'(Q_{1,t})$ and $\delta_{2,t} \in \bar{C}_2'(Q_{2,t})$. Since $Q_{1,t} > Q_{1,t'}$, the convexity of firm 1's cost function guarantees that, for any $\delta_{1,t'} \in \bar{C}_1'(Q_{1,t'})$, we have $\delta_{1,t'} \leq \delta_{1,t}$; given the observed prices and output of firm 1, we conclude that

$$\frac{P_{1,t} - \delta_{1,t}}{Q_{1,t}} < \frac{P_{1,t'} - \delta_{1,t'}}{Q_{1,t'}}.$$

By a similar argument, for any $\delta_{2,t'} \in \bar{C}_2'(Q_{2,t'})$, we have $\delta_{2,t'} \geq \delta_{2,t}$; given the observed prices and output of firm 2, we obtain

$$\frac{P_{2,t} - \delta_{2,t}}{Q_{2,t}} > \frac{P_{2,t'} - \delta_{2,t'}}{Q_{2,t'}}.$$

It follows from (11) that

$$\frac{P_{1,t'} - \delta_{1,t'}}{Q_{1,t'}} > \frac{P_{2,t'} - \delta_{2,t'}}{m_2 Q_{2,t'}} + k_2.$$

Hence there does *not* exist $\delta_{1,t'} \in \bar{C}_1'(Q_{1,t'})$ and $\delta_{2,t'} \in \bar{C}_2'(Q_{2,t'})$ satisfying the generalized common ratio property at t'.

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