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Housing Prices and Robustly Optimal Monetary Policy

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Abstract

We analytically characterize robustly optimal monetary policy for an augmented New Keynesian model with a housing sector. In our setting, the housing stock delivers a service flow entering households' utility, houses are durable goods that depreciate over time, and new houses can be produced using a concave production technology.

We show that shocks to housing demand and to housing productivity have "cost-push" implications, which warrant temporary fluctuations in the inflation rate under optimal policy, even under an assumption of rational expectations, for reasons familiar from the literature on "flexible inflation targeting". However, under rational expectations optimal monetary policy can still be characterized by commitment to a "target criterion" that refers to inflation and the output gap only, just as in the standard model without a housing sector.

Instead, if policy is to be robust to potential departures of (house price and inflation) expectations from model-consistent ones, the target criterion must also depend on housing prices. In the empirically realistic case where the government subsidizes housing, the robustly optimal target criterion requires the central bank to "lean against" unexpected increases in housing prices, in the sense that it should adopt a policy stance that is projected to undershoot its normal targets for inflation and/or the output gap owing to the increase in housing prices, and similarly aim to overshoot those targets in the case of unexpected declines in housing prices.

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1 Introduction

The question of how (if at all) asset price movements should be taken into account in a central bank's interest-rate policy decisions has been much debated since at least the 1990s.[1] The importance of the issue has become even more evident after the fallout for the global economy of the recent real estate booms and busts in the US and several other countries, which at least some attribute to monetary policy decisions that failed to take account of the consequences for the housing market.[2]

Yet the issue is not easily addressed using standard frameworks for monetary policy analysis. One reason is that it is often supposed that large movements in asset prices are particularly problematic when they are not justified by economic "fundamentals," but instead represent mistaken valuations resulting from mistaken expectations. An analysis that evaluates alternative monetary policies under the assumption that the outcome resulting from each candidate policy will be a rational-expectations equilibrium assumes that there can never be any misallocation of resources due to speculative mispricing of assets, regardless of the monetary policy that is chosen. Such an analysis will accordingly conclude that there is no need for a central bank to monitor or respond to signs of such mispricing — but by assuming away the problem.

Some analyses of the question have accordingly allowed for potential departures of asset prices from "fundamental" values, introducing an expectational error term in the asset pricing equation that is specified as an exogenous stochastic process (*e.g.*, Bernanke and Gertler (1999, 2001)). But conclusions from such analyses depend on what is assumed about the nature of expectational errors, and not only on what is assumed about the probability distribution of errors under some given policy (perhaps the kind of policy that has historically been implemented), but also, crucially on what is assumed about how the probability distribution of errors would differ under each alternative policy that may be contemplated. Yet there is little basis for confidence about the correctness of a particular choice in this regard.

Here we propose a different approach to the problem. We do not assume that expectations must necessarily be model-consistent, but we do not assume that expectational errors must be of any specific type that can be predicted in advance, either; rather, we associate with any contemplated policy a *set* of possible probability beliefs, that includes all possible (internally coherent) probability beliefs that are not *too different* from those predicted by one's model, in the case of that policy and those beliefs. This is the hypothesis of "near-rational expectations" [NRE] introduced in Woodford (2010)

This makes the set of possible private-sector beliefs contemplated by the policy analysis dependent on the particular policy that is adopted, as in the case of the rational expectations hypothesis. In particular, beliefs are treated as possible if it

[1]See, for example, Bernanke and Gilchrist (1999, 2001), Gilchrist and Leahy (2002), Christiano et al. (2010)

[2]For example, Taylor (2007) or Adam, Marcet and Kuang (2011).

would not be too easy to disconfirm them using observed data from the equilibrium of the model, and whether this is so will depend on policy. But the set of beliefs that are considered will include ones that result in asset valuations different from the ones that will be judged correct according to the policy analyst's model; hence the policy analyst will consider the possibility of equilibria in which assets are mis-priced, and will therefore consider the consequences of responding to such asset price movements in different ways.

Because the set of possible "near-rational" beliefs associated with any given policy includes many elements, analysis of the kind proposed here will not associate a single predicted path for the economy (contingent on the realized values of exogenous shocks) with a given policy. It may therefore be wondered how welfare comparisons of alternative policies are possible. Our proposal, in the spirit of the robust policy analysis of Hansen and Sargent (2008), is to choose a policy that achieves the highest possible *lower bound* for expected utility of the representative household, across all of the equilibria with "near-rational" expectations consistent with that policy. We call a solution to this problem a "robustly optimal" policy rule.

We wish to consider the problem of robustly optimal policy within as broad a class of possible policy rules as possible; in particular, we do not wish to prejudge questions such as the way in which the policy rule may involve systematic response to housing-related variables or to indicators of market expectations. Our earlier paper (Adam and Woodford (2012)) shows how it is possible to characterize robustly optimal policy rules without restricting oneself *a priori* to some simple parametric family of policy rules. The basic idea (reviewed in more detail in section 2) is that we can derive an upper bound for the maximin level of welfare that is potentially achievable under any policy rule, without reference to any specific class of policy rules; if we can then display examples of policy rules that achieve this upper bound, we know that these are examples of robustly optimal policies. We show here that this method can be applied to a New Keynesian DSGE model with endogenous housing supply.

We are especially interested in a particular way of specifying the policy rule, in which the central bank commits itself to fulfill a quantitative *target criterion* at all times.[3] Under this commitment it uses its policy instrument at each point in time as necessary in order to ensure that the paths of various endogenous variables satisfy the relationship specified by the target criterion. In a basic New Keynesian model without a housing sector and under the assumption of rational expectations, it is well-known that an optimal policy commitment can be characterized in these terms; the required target criterion is a "flexible inflation targeting" rule in the sense of

[3]The robustly optimal policy rule is not unique, as is discussed in more detail in Adam and Woodford (2012). Different rules may be consistent with the same *worst-case* NRE equilibrium dynamics, and so achieve the same lower bound for expected utility, without being equivalent, either in terms of the out-of-equilibrium behavior that they would require from the central bank, or in terms of the boundaries of the complete set of NRE equilibria consistent with the policy in question.

Svensson (1999), in which short-run departures from the long-run inflation target are justified precisely to the extent that they are proportional to short-run variations in the rate of change of an “output gap” variable.[4] Here we show that a generalization of this criterion can be used to implement robustly optimal policy in our model with a housing sector and allowance for non-model-consistent beliefs. We show, however, that the robustly optimal target criterion must involve housing prices, as well as inflation (or the price level) in the non-housing sector and the output gap.

In the empirically realistic case in which housing is subsidized by the government, and is therefore supplies to excess in equilibrium, the robustly optimal target criterion requires the central bank to ”lean against” unexpected increases in housing prices. By this we mean that it should adopt a policy stance that is projected to result in smaller increases in inflation and/or the output gap than would be chosen in the absence of the surprise increase in housing prices; thus requires a ”tighter” monetary policy than would otherwise be chosen. Similarly, it should aim for larger increases in inflation and or the output gap in the case of an unexpected decline in housing prices.

The policy of ”leaning against” housing price increases is more robust than a correspondingly flexible inflation targeting rule that ignores housing price variations (and that would be optimal under rational expectations), in the sense that the distorted expectations that would lead to the worst possible outcome under this policy (among all possible beliefs that comply with a certain bound on the possible size of belief distortions) do not lower welfare as much as some possible beliefs distortions (consistent with the same bound on the size of distortions) would under the conventional policy. The degree to which the robustly optimal policy requires ”leaning against” housing prices increases depends, however, on model parameters. Notably, it depends both on the size of the housing subsidy and on the price elasticity of housing supply, as discussed further below.

Our linear approximation to the robustly optimal policy commitment can also be derived as the solution to a robust linear-quadratic policy problem. In this problem, the central bank’s quadratic loss function has three terms, representing three competing stabilization objectives: inflation stabilization, output-gap stabilization, and minimization of the variance of surprises in a composite variable that includes both inflation and housing prices. It is this additional stabilization objective, that appears only due to a concern for robustness and that requires housing prices to enter the robustly optimal target criterion.

Section 2 defines robustly optimal policy and presents the general approach that we use to characterize it. Section 3 then presents our New Keynesian monetary DSGE model with a housing sector, and defines an equilibrium with possibly distorted private sector expectations, generalizing the standard concept of rational-expectations equilibrium. Sections 4 and 5 characterize equilibrium dynamics in the case of a policy that achieves the highest possible lower bound for welfare of the representative

[4] See, for example, chapter 7 in Woodford (2003).

household, for a given bound on the possible size of belief distortions. Section 6 then derives a target cirterion that can implement this outcome, i.e., that achieves this highest lower bound for all belief distortions subject to the bound, and shows that it has the properties summarized above. Section 7 further discusses the reasons for our results, in terms of the implications of a linear-quadratic stabilization problem that approximates the exact problem solved in sections 4 and 5. Section 8 concludes.

2 The Policy Problem in General Terms

This section describes the general approach that we use to characterize robustly optimal policy. These general ideas are then applied to a New Keynesian model with a housing sector in section 3.

2.1 Robustly Optimal Policy

Consider a policymaker who cares about some vector x of endogenous economic outcomes in the sense of seeking to achieve as high a value as possible for some (welfare) objective $W(x)$. The value of x depends both on policy and on forward-looking private sector decisions, which in turn depend on the private-sector's belief distortions as parameterized by some vector m. Among the determinants of x is a set of structural economic equations, typically involving first-order conditions of private agents and market clearing conditions, that we write as

$$F(x; m) = 0. \tag{1}$$

We assume that the equations (1) are insufficient to completely determine the vector x, under given belief distortions m, so that the policymaker faces a non-trivial choice.

Let us suppose that the policymaker must choose a policy commitment c from some set C of feasible policy commitments. Our results about robustly optimal policy do not depend on the precise specification of the set C; for now, we simply assume that there exists such a set, but we make no specific assumption about what its boundaries may be. We only impose two general assumptions about the nature of the set C: first, we assume that each of the commitments in the set C can be defined independently of what the belief distortions may be[5]; and second, we shall require that for any $c \in C$, there exists an equilibrium outcome for any choice of $m \in M$. The latter assigns to the policymaker the responsibility for insuring existence of equilibrium for arbitrary belief distortions.

Given our general requirements, the set C may include many different types of policy commitments. For example, it may involve policy commitments that depend

[5] As is made more specific in the application below, we specify policy commitments by equations involving the endogenous and exogenous variables, but not explicitly the belief distortions. Of course, the endogenous variables referred to in the policy commitment will typically also be linked by structural equations that involve the belief distortions.

on the history of exogenous shocks; commitments that depend on the history of endogenous variables, as is the case with Taylor rules; and commitments regarding relationships between endogenous variables, as is the case with so-called targeting rules. Also, the endogenous variables in terms of which the policy commitment is expressed may include asset prices (futures prices, forward prices, etc.) that are often treated by central banks as indicators of private-sector expectations, as long as the requirement is satisfied that the policy commitment must be consistent with belief distortions of an arbitrary form.

In order to define the robustly optimal decision problem of the policymaker, we further specify an *outcome function* that identifies the equilibrium outcome x associated with a given policy commitment $c \in C$ and a given belief distortion m.

Definition 1 *T e economic outcomes associate wit e ief istortions* m *an com mitments* c *are given y an outcome function*

$$O : M \times C \to X$$

wit t e property t at for a $m \in M$ *an* $c \in C$*, t e outcome* $O(m;c)$ *an* m *joint y constitute an equi i rium of t e mo e In particu ar, t e outcome function must satisfy*

$$F(O(m;c);m) = 0$$

for a a $m \in M$ *an* $c \in C$

Here we have not been specific about what we mean by an "equilibrium," apart from the fact that (1) must be satisfied. In the context of the specific model presented in the next section, equilibrium has a precise meaning. For purposes of the present discussion, it does not actually matter how we define equilibrium; only the definition of the outcome function matters for our subsequent discussion.[6]

To complete the description of the robustly optimal policy problem, let M denote the set of all possible belief distortions and $V(m) \geq 0$ a measure of the size of the belief distortions. We assume that $V(m)$ is equal to zero only in the case of beliefs that agree precisely with those of the policymaker and that $V(m)$ is strictly increasing in the 'size of the distortions'. The functional form for $V(m)$ ultimately reflects our conception of 'near-rational expectations'. Section 2.2 introduces a specific functional form that is based on a relative entropy measure.

The robustly optimal policy problem can then be represented as a choice of a policy commitment that solves

$$\max_{c \in C} \left\{ \min_{m \in M} W(O(m;c)) \text{ s.t. } V(m) \leq \overline{V} \right\} \tag{2}$$

[6]If the set of equations (1) is not a complete set of requirements for x to be an equilibrium, this only has the consequence that the upper-bound outcome defined below might not be a tight enough upper bound; it does not affect the validity of the assertion that it provides an upper bound.

where $\overline{V} \geq 0$ measures the policymaker's degree of concern for robustness. For the special case with $\overline{V} = 0$ the robustly optimal policy problem reduces to a standard optimal policy problem with model-consistent private sector expectations. As $\overline{V}$ increases, the policymaker becomes concerned with increasingly larger deviations of private sector expectations from those that would be consistent with its own model used for policy analysis.

Let c^R denote the robustly optimal policy commitment and m^R the associated worst-case beliefs, i.e., the solution to the inner problem in (2). Suppose there exists a Lagrange multiplier $\theta \geq 0$ such that m^R also solves

$$\min_{m \in M} W(O(m; c^R)) + \theta V(m)$$

with $\theta(V(m^R) - \overline{V}) = 0$. Then c^R and m^R also jointly solve the alternative problem

$$\max_{c \in C} \min_{m \in M} W(O(m; c)) + \theta V(m) \tag{3}$$

where θ now parameterizes the concern for robustness. Let the resulting values for the endogenous variables be given by $x^R = O(m^R; c^R)$. Adam and Woodford (2012) show that

$$\begin{aligned} \max_{c \in C} \min_{m \in M} W(O(m; c)) + \theta V(m) \leq \min_{m \in M} \max_{x \in X} & \; W(x) + \theta V(m) \\ s.t. : & \; F(x; m) = 0; \end{aligned} \tag{4}$$

which allows us to determine the robustly optimal policy commitment as follows: first, we determine the optimal choices x^* and m^* solving the problem on the right-handside of (4). In a second step, we look for a policy-commimtment $\widetilde{c}$ and a belief disortion $\widetilde{m}$ such that $\widetilde{m}$ solves

$$\min_{m \in M} W(O(m; \widetilde{c})) + \theta V(m)$$

and for which $W(O(\widetilde{m}; \widetilde{c})) + \theta V(\widetilde{m}) = W(x^*) + \theta V(m^*)$. Since $W(x^*) + \theta V(m^*)$ represents an upper bound on what robustly optimal policy can achieve, see (3), no other policy commitment can achieve a better outcome, $\widetilde{c}$ indeed represents an optimal policy commitment, independently of the specific class C of policy commitments considered (as long as our two general requirements on the class C hold).

2.2 Distorted Private Sector Expectations

We next discuss our approach to the parameterization of belief distortions, and the distortion measure $V(m)$: At this point it becomes necessary to specify that our analysis concerns dynamic models in which information is progressively revealed over time, at a countably infinite sequence of successive decision points.

Let $(\Omega; \mathcal{B}; \mathcal{P})$ denote a standard probability space with Ω denoting the set of possible realizations of an exogenous stochastic disturbance process $\{\xi_0; \xi_1; \xi_2; :::\}$, $\mathcal{B}$ the $\sigma-$algebra of Borel subsets of $\Omega;$ and $\mathcal{P}$ a probability measure assigning probabilities to any set $B \in \mathcal{B}$. We consider a situation in which the policy analyst assigns probabilities to events using the probability measure $\mathcal{P}$ but fears that the private sector may make decisions on the basis of a potentially different probability measure denoted by $\widehat{\mathcal{P}}$.

We let E denote the policy analyst's expectations induced by $\mathcal{P}$ and $\widehat{E}$ the corresponding private sector expectations associated with $\widehat{\mathcal{P}}$. A first restriction on the class of possible distorted measures that the policy analyst is assumed to consider — part of what we mean by the restriction to "near-rational expectations" — is the assumption that the distorted measure $\widehat{\mathcal{P}}$, when restricted to events over any finite horizon, is absolutely continuous with respect to the correspondingly restricted version of the policy analyst's measure $\mathcal{P}$.

The Radon-Nikodym theorem then allows us to express the distorted private sector expectations of some $t+j$ measurable random variable X_{t+j} as

$$\widehat{E}[X_{t+j}|\xi^t] = E[\frac{\mathcal{M}_{t+j}}{\mathcal{M}_t} X_{t+j}|\xi^t]$$

for all $j \geq 0$ where ξ^t denotes the partial history of exogenous disturbances up to period t. The random variable $\mathcal{M}_{t+j}$ is the Radon-Nikodym derivative, and completely summarizes belief distortions.[7] The variable $\mathcal{M}_{t+j}$ is measurable with respect to the history of shocks ξ^{t+j}, non-negative and is a martingale, i.e., satisfies

$$E[\mathcal{M}_{t+j}| !^t] = \mathcal{M}_t$$

for all $j \geq 0$. Defining

$$m_{t+1} = \frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}$$

one step ahead expectations based on the measure $\widehat{\mathcal{P}}$ can be expressed as

$$\widehat{E}[X_{t+1}|\xi^t] = E[m_{t+1} X_{t+1}|\xi^t];$$

where m_{t+1} satisfies

$$E[m_{t+1}|\xi^t] = 1 \text{ and } m_{t+1} \geq 0: \tag{5}$$

This representation of the distorted beliefs of the private sector is useful in defining a measure of the distance of the private-sector beliefs from those of the policy analyst. As discussed in Hansen and Sargent (2005), the relative entropy

$$R_t = E_t[m_{t+1} \log m_{t+1}]$$

[7]See Hansen and Sargent (2005) for further discussion.

is a measure of the distance of (one-period-ahead) private-sector beliefs from the policymaker's beliefs with a number of appealing properties.

We wish to extend this measure of the size of belief distortions to an infinite-horizon economy with a stationary structure. In the kind of model with which we are concerned, the policy objective in the absence of a concern for robustness is of the form

$$W(x) \equiv E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t) \right]; \tag{6}$$

for some discount factor $0 < \beta < 1;$ where $U(\cdot)$ is a time-invariant function, and x_t is a vector describing the real allocation of resources in period t. Correspondingly, we propose to measure the overall degree of distortion of private-sector beliefs by a discounted criterion of the form

$$V(m) \equiv E_0 \left[\sum_{t=0}^{\infty} \beta^{t+1} m_{t+1} \log m_{t+1} \right]; \tag{7}$$

as in Woodford (2010). This is a discounted sum of the one-period-ahead distortion measures $\{R_t\}:$ We assign relative weights to the one-period-ahead measures R_t for different dates and different states of the world in this criterion that match those of the other part of the policy objective (6). Use of this cost function implies that the policymaker's degree of concern for robustness (relative to other stabilization objectives) remains constant over time, regardless of past history.

3 A Sticky Price Model with a Housing Sector

We shall begin by deriving the exact structural relations describing a New Keynesian model featuring a long-lived asset and potentially distorted private sector expectations. The existing stock of assets is assumed to generate a service flow that directly enters agents' utility. Assets depreciate over time but can be produced using a technology with decreasing returns to scale. For convenience we interpret the long-lived asset as housing, though other interpretations are possible.

The model is completely standard, except for the presence of the long-lived asset and the fact that the private sector holds potentially distorted expectations. The exposition here extends the framework of Adam and Woodford (2011), who write the exact structural relations for a simpler model without a housing sector.

3.1 Model Structure

The economy is made up of identical infinite-lived households, each of which seeks to maximize

$$U \equiv \widehat{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\tilde{u}(C_t; \xi_t) - \int_0^1 \tilde{v}(H_t(j); \xi_t) dj + ! (D_t; \xi_t) \right]; \tag{8}$$

subject to a sequence of flow budget constraints[8]

$$
\begin{aligned}
&P_t C_t + B_t + (D_t + (1-\delta)D_{t-1})\, q_t P_t + k_t P_t \\
&\leq (1+s^d)\tilde{d}(k_t;\xi_t) q_t P_t + \int_0^1 w_t(j) P_t H_t(j) dj + B_{t-1}(1+i_{t-1}) + \Sigma_t + T_t;
\end{aligned}
$$

where $\widehat{E}_0$ is the common distorted expectations held by consumers conditional on the state of the world in period t_0, C_t an aggregate consumption good which can be bought at nominal price P_t; $H_t(j)$ is the quantity supplied of labor of type j and $w_t(j)$ the associated real wage, D_t the stock of durable assets or houses, $\delta \in [0;1]$ the housing depreciation rate, q_t the real price of houses, k_t investment in new houses and $\tilde{d}(k_t;\xi_t)$ the resulting production of new houses, s^d a government subsidy (which may be positive or negative) applied to the value of newly produced houses, B_t nominal bond holdings, i_t the nominal interest rate, and ξ_t is a vector of exogenous disturbances, which may induce random shifts in the functions $\tilde{u}$, $\tilde{v}$, $?$ and $\tilde{d}$. The variable T_t denotes lump sum taxes levied by the government and Σ_t profits accruing to households from the ownership of firms.

The aggregate consumption good is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods,

$$
C_t \equiv \left[\int_0^1 c_t(i)^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}; \tag{9}
$$

with an elasticity of substitution equal to $\eta > 1$. We further assume isoelastic functional forms

$$
\tilde{u}(C_t;\xi_t) \equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}}; \tag{10}
$$

$$
\tilde{v}(H_t;\xi_t) \equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{-\nu}; \tag{11}
$$

$$
?(D_t;\xi_t) = \xi_t^d D_t; \tag{12}
$$

$$
\tilde{d}(k_t;\xi_t) = \frac{A_t^d}{\tilde{\alpha}} k_t^{\tilde{\alpha}}; \tag{13}
$$

where $\tilde{\sigma};\nu > 0$; $\tilde{\alpha} \in (0;1)$ and $\{\bar{C}_t; \bar{H}_t; \xi_t^d; A_t^d\}$ are bounded exogenous and positive disturbance processes which are among the exogenous disturbances included in the vector ξ_t. Our specification includes two housing related disturbances that will be of particular interest for our analysis, namely ξ_t^d which captures shocks to housing preferences and A_t^d shocks to the productivity in the construction of new houses. We

[8] We abstract from state-contingent assets in the household budget constraint because the representative agent assumption implies that in equilibrium there will be no trade in these assets.

impose linearity in the utility function (12) as this greatly facilitates the analytical characterization of optimal policy.

Each differentiated good is supplied by a single monopolistically competitive producer; there is a common technology for the production of all goods, in which (industry-specific) labor is the only variable input,

$$y_t(i) = A_t f(h_t(i)) = A_t h_t(i)^{1/\phi}; \tag{14}$$

where A_t is an exogenously varying technology factor, and $\phi > 1$. The Dixit-Stiglitz preferences (9) imply that the quantity demanded of each individual good i will equal[9]

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\eta}; \tag{15}$$

where Y_t is the total demand for the composite good defined in (9), $p_t(i)$ is the (money) price of the individual good, and P_t is the price index,

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\eta} di\right]^{\frac{1}{1-\eta}}; \tag{16}$$

corresponding to the minimum cost for which a unit of the composite good can be purchased in period t. Total demand is given by

$$Y_t = C_t + k_t + g_t Y_t; \tag{17}$$

where g_t is the share of the total amount of composite good purchased by the government, treated here as an exogenous disturbance process.

3.2 Household Optimality Conditions

Each household maximizes utility by choosing state contingent sequences $\{C_t; H_t(j); D_t; k_t; B_t\}$ taking as given the process for $\{P_t; w_t(j); q_t; i_t; \Sigma_t; T_t\}$. The first order conditions give rise to an optimal labor supply relation

$$w_t(j) = \frac{\tilde{v}_H(H_t(j); \xi_t)}{\tilde{u}_C(C_t; \xi_t)}; \tag{18}$$

a consumption Euler equation

$$\tilde{u}_C(C_t; \xi_t) = \beta \widehat{E}_t \left[\tilde{u}_C(C_{t+1}; \xi_{t+1}) \frac{1 + i_t}{\Pi_{t+1}}\right]; \tag{19}$$

[9] In addition to assuming that household utility depends only on the quantity obtained of C_t; we assume that the government also cares only about the quantity obtained of the composite good defined by (9), and that it seeks to obtain this good through a minimum-cost combination of purchases of individual goods.

an equation characterizing optimal investment in new houses

$$k_t = \left(\left(1 + s^d\right) A_t^d q_t\right)^{\frac{1}{1-\tilde{\alpha}}}; \tag{20}$$

and an asset pricing equation

$$q_t^u = \xi_t^d + \beta(1-\delta)\widehat{E}_t q_{t+1}^u; \tag{21}$$

where

$$q_t^u \equiv q_t C_t^{-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}} \tag{22}$$

is the market valuation of housing in period t, expressed in *marginal-utility* units. The variable q_t^u provides a measure of whether housing is currently expensive or inexpensive, in units that are particularly relevant for determining housing demand. More importantly, because of (21), it is expectations about *the future value of* q_T^u, rather than the future value of q_T as such, that influence the current market value of housing, so that the degree of distortion that may be present in expectations regarding the former variable is of particular importance for equilibrium determination. The housing-price variable q_t^u is accordingly of particular interest.

Equations (18)-(21) jointly characterize optimal household behavior under distorted beliefs. Using (17) and (20), one an express aggregate demand as

$$Y_t = \frac{C_t + \Omega_t C_t^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}}}{1 - g_t} \tag{23}$$

where

$$\Omega_t \equiv \left(\left(1 + s^d\right) A_t^d \bar{C}_t^{-\tilde{\sigma}^{-1}} q_t^u\right)^{\frac{1}{1-\tilde{\alpha}}} > 0 \tag{24}$$

is a term that depends on exogenous shocks and belief distortions only.

3.3 Optimal Price Setting by Firms

The producers in each industry fix the prices of their goods in monetary units for a random interval of time, as in the model of staggered pricing introduced by Calvo (1983) and Yun (1996). Let $0 \leq \alpha < 1$ be the fraction of prices that remain unchanged in any period. A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$\widehat{E}_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i); p_T^j; P_T; Y_T; q_T^u; \xi_T); \tag{25}$$

where $\widehat{E}_t$ is the distorted expectations of price setters conditional on time t information, which are assumed identical to the expectations held by consumers, $Q_{t,T}$ is the stochastic discount factor by which financial markets discount random nominal income in period T to determine the nominal value of a claim to such income in period

t, α^{T-t} is the probability that a price chosen in period t will not have been revised by period T, and the function $\Pi(p_t(i); \ldots)$ indicates the nominal profits of the firm in period t (discussed further below). In equilibrium, the discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_C(C_T; \xi_T)}{\tilde{u}_C(C_t; \xi_t)} \frac{P_t}{P_T}. \tag{26}$$

Profits are equal to after-tax sales revenues net of the wage bill. Sales revenues are determined by the demand function (15), so that (nominal) after-tax revenue equals

$$(1 - \tau_t) p_t(i) Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\eta}.$$

Here τ_t is a proportional tax on sales revenues in period t; $\{\tau_t\}$ is treated as an exogenous disturbance process, taken as given by the monetary policymaker. We assume that τ_t fluctuates over a small interval around a non-zero steady-state level $\underline{\tau}$. We allow for exogenous variations in the tax rate in order to include the possibility of "pure cost-push shocks" that affect equilibrium pricing behavior while implying no change in the efficient allocation of resources.

The real wage demanded for labor of type j is given by equation (18) and firms are assumed to be wage-takers. Because the right-hand side of (23) is a monotonically increasing function of C_t, (23) implies the existence of a differentiable function

$$C_t = C(Y_t; q_t^u; \xi_t) \tag{27}$$

solving (23) with the derivative C_Y satisfying $0 < C_Y(Y_t; q_t^u; \xi_t) < 1 - g$. Using this function and the assumed functional forms for preferences and technology, the nominal wage bill will equal

$$\begin{aligned} P_t w_t(j) h_t(i) &= P_t \frac{\lambda H_t(i)^\nu \overline{H}_t^{-\nu}}{C_t^{-\tilde{\sigma}^{-1}} \overline{C}_t^{\tilde{\sigma}^{-1}}} h_t(i) \\ &= \lambda P_t \left(\frac{p_t(i)}{P_t} \right)^{-\eta\phi} \left(\frac{p_t^j}{P_t} \right)^{-\eta\phi\nu} \overline{H}_t^{-\nu} \left(\frac{Y_t}{A_t} \right)^{1+\omega} \left(\frac{C(Y_t; q_t^u; \xi_t)}{\overline{C}_t} \right)^{\tilde{\sigma}^{-1}} \end{aligned}$$

where

$$\omega \equiv \phi(1 + \nu) - 1 > 0$$

is the elasticity of real marginal cost in an industry with respect to industry output. Subtracting the nominal wage bill from the above expression for nominal after tax revenue, we obtain the function $\Pi(p_t(i); p_T^j; P_T; Y_T; q_T^u; \xi_T)$ used in (25). The vector of exogenous disturbances ξ_t now includes $A_t; g_t$ and τ_t, in addition to the shocks $(\bar{C}_t; \bar{H}_t; \xi_t^d; A_t^d)$.

Each of the suppliers that revise their prices in period t chooses the same new price p_t^*, that maximizes (25). Note that supplier i's profits in (25) are a concave function of

the quantity sold $y_t(i)$; since revenues are proportional to $y_t(i)^{\frac{\eta-1}{\eta}}$ and hence concave in $y_t(i)$, while costs are convex in $y_t(i)$. Moreover, since $y_t(i)$ is proportional to $p_t(i)^{-\eta}$; the profit function is also concave in $p_t(i)^{-\eta}$. The first-order condition for the optimal choice of the price $p_t(i)$ is the same as the one with respect to $p_t(i)^{-\eta}$; hence the first-order condition with respect to $p_t(i)$;

$$\hat{E}_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi_1(p_t(i); p_T^j; P_T; Y_T; q_T^u; \xi_T) = 0;$$

is both necessary and sufficient for an optimum. The equilibrium choice p_t^* (which is the same for each firm in industry j) is the solution to the equation obtained by substituting $p_t(i) = p_t^j = p_t^*$ into the above first-order condition.

Under the assumed isoelastic functional forms, the optimal choice has a closed-form solution

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\eta}}; \tag{28}$$

where F_t and K_t capture the effects of discounted marginal costs and revenues, respectively, and are defined by

$$F_t \equiv \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} f(Y_T; q_T^u; \xi_T) \left(\frac{P_T}{P_t}\right)^{\eta-1}; \tag{29}$$

$$K_t \equiv \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T; \xi_T) \left(\frac{P_T}{P_t}\right)^{\eta(1+\omega)}; \tag{30}$$

where

$$f(Y; q^u; \xi) \equiv (1-\tau)\bar{C}^{\tilde{\sigma}^{-1}} Y \ C(Y; q^u; \xi)^{-\tilde{\sigma}^{-1}}; \tag{31}$$

$$k(Y; \xi) \equiv \frac{\eta}{\eta-1} \lambda\phi \frac{\bar{H}^{-\nu}}{A^{1+\omega}} Y^{1+\omega} \tag{32}$$

Relations (29)–(30) can also be written in the recursive form

$$F_t = f(Y_t; q_t^u; \xi_t) + \alpha\beta \hat{E}_t[\Pi_{t+1}^{\eta-1} F_{t+1}] \tag{33}$$

$$K_t = k(Y_t; \xi_t) + \alpha\beta \hat{E}_t[\Pi_{t+1}^{\eta(1+\omega)} K_{t+1}]; \tag{34}$$

where $\Pi_t \equiv P_t / P_{t-1}$.[10] The price index then evolves according to a law of motion

$$P_t = \left[(1-\alpha) p_t^{*1-\eta} + \alpha P_{t-1}^{1-\eta}\right]^{\frac{1}{1-\eta}}; \tag{35}$$

[10]It is evident that (29) implies (33); but one can also show that processes that satisfy (33) each period, together with certain bounds, must satisfy (29). Since we are interested below only in the characterization of bounded equilibria, we can omit the statement of the bounds that are implied by the existence of well-behaved expressions on the right-hand sides of (29) and (30), and treat (33)–(34) as necessary and sufficient for processes $\{F_t; K_t\}$ to measure the relevant marginal conditions for optimal price-setting.

as a consequence of (16). Substitution of (28) into (35) implies that equilibrium inflation in any period is given by

$$\frac{1 - \alpha \Pi_t^{\eta-1}}{1 - \alpha} = \left(\frac{F_t}{K_t} \right)^{\frac{\eta-1}{1+\omega\eta}} : \tag{36}$$

Equations (33), (34) and (36) jointly define a short-run aggregate supply relation between inflation, output and house prices, given the current disturbances ξ_t; and (potentially distorted) expectations regarding future inflation, output, house prices and disturbances.

3.4 Summary and Equilibrium Definition

For the subsequent analysis it will be helpful to express the model in terms of the endogenous variables $(Y_t; K_t; F_t; \Delta_t; q_t^u; m_t; i_t)$ only, where m_t is the belief distortions of the private sector and

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t} \right)^{-\eta(1+\omega)} di \geq 1$$

a measure of price dispersion at time t. The vector of exogenous disturbances is given by $\xi_t = \left(A_t; g_t; \tau_t; \bar{C}_t; \bar{H}_t; \xi_t^d; A_t^d \right)'$.

We begin by expressing expected household utility (evaluated under the objective measure $\mathcal{P}$) in terms of these variables. Inverting the production function (14) to write the demand for each type of labor as a function of the quantities produced of the various differentiated goods, it is possible to write the utility of the representative household as a function of the expected production plan $\{y_t(i)\}$. One thereby obtains

$$U \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left[u(Y_t; q_t^u; \xi_t) - \int_0^1 v(y_t^j; \xi_t) dj + ! (D_t; \xi_t) \right] ; \tag{37}$$

with

$$\begin{aligned} u(Y_t; q_t^u; \xi_t) &\equiv \tilde{u}(C(Y_t; q_t^u; \xi_t); \xi_t) \\ v(y_t^j; \xi_t) &\equiv \tilde{v}(f^{-1}(y_t^j = A_t); \xi_t) \end{aligned}$$

where in this last expression we make use of the fact that the quantity produced of each good in industry j will be the same, and hence can be denoted y_t^j; and that the quantity of labor hired by each of these firms will also be the same, so that the total demand for labor of type j is proportional to the demand of any one of these firms.

One can furthermore express the relative quantities demanded of the differentiated goods each period as a function of their relative prices, using (15). This and the linear

dependence of utility on the stock of assets allows us to write the utility flow to the representative household in the form

$$u(Y_t; q_t^u; \xi_t) - v(Y_t; \xi_t)\Delta_t + \bar{\xi}_t^d \frac{A_t^d}{\tilde{\alpha}} k_t^{\tilde{\alpha}};$$

where

$$\bar{\xi}_t^d \equiv \sum_{T=t}^{\infty} E_t[(1-\delta)^{T-t}\,\beta^{T-t}\xi_T^d]: \tag{38}$$

We can use (20), (22) and (27) to express k_t in terms of Y_t, q_t^u and exogenous shocks. Hence we can express the household objective (37) as

$$U = E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t; \Delta_t; q_t^u; \xi_t): \tag{39}$$

where the explicit expression for the flow utility is given by

$$\begin{aligned} U(Y_t; \Delta_t; q_t^u; \xi_t) = {} & \frac{\overline{C}_t^{\tilde{\sigma}^{-1}}\, C(Y_t; q_t^u; \xi_t)^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} \\ & - \frac{\lambda}{1+\nu}\bar{H}_t^{-\nu}\left(\frac{Y_t}{A_t}\right)^{1+\omega}\Delta_t \\ & + \frac{A_t^d \bar{\xi}_t^d}{\tilde{\alpha}}\Omega(q_t^u; \xi_t)^{\tilde{\alpha}}\; C(Y_t; q_t^u; \xi_t)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}}; \end{aligned} \tag{40}$$

which is a a monotonically decreasing function of Δ given Y, q^u and ξ and where $\Omega(q_t^u; \xi_t)$ is the function defined in (24).

The consumption Euler equation (19) can be expressed as

$$\tilde{u}_C(C(Y_t; q_t^u; \xi_t); \xi_t) = \beta E_t\left[m_{t+1}\tilde{u}_C(C(Y_t; q_t^u; \xi_t); \xi_{t+1})\frac{1+i_t}{\Pi_{t+1}}\right]; \tag{41}$$

Using (36) to substitute for the variable Π_t equations (33) and (34) can be expressed as

$$F_t = f(Y_t; q_t^u; \xi_t) + \alpha\beta E_t\left[m_{t+1}\phi_F(K_{t+1}; F_{t+1})\right] \tag{42}$$

$$K_t = k(Y_t; q_t^u; \xi_t) + \alpha\beta E_t\left[m_{t+1}\phi_K(K_{t+1}; F_{t+1})\right]; \tag{43}$$

where the functions $\phi_F; \phi_K$ are both homogeneous degree 1 functions of K and F.

Because the relative prices of the industries that do not change their prices in period t remain the same, one can use (35) to derive a law of motion for the price dispersion term Δ_t of the form

$$\Delta_t = h(\Delta_{t-1}; \Pi_t); \tag{44}$$

where

$$h(\Delta; \Pi) \;\equiv\; \alpha\Delta\Pi^{\eta(1+\omega)} + (1-\alpha)\left(\frac{1-\alpha\Pi^{\eta-1}}{1-\alpha}\right)^{\frac{\eta(1+\omega)}{\eta-1}} :$$

This is the source of welfare losses from inflation or deflation. Using once more (36) to substitute for the variable Π_t one obtains

$$\Delta_t = \tilde{h}(\Delta_{t-1}; K_t{=}F_t): \tag{45}$$

The asset pricing equation (21) and equations (41)-(45) represent five constraints on the equilibrium paths of the seven endogenous variables $(Y_t; F_t; K_t; \Delta_t; q_t^u; m_{t+1}; i_t)$. For a given sequence of belief distortions m_t satisfying restriction (5) there is thus one degree of freedom left, which can be determined by monetary policy. We are now in a position to define the equilibrium with distorted private sector expectations:

Definition 2 (DEE) *A istorte expectations equi i rium (DEE) is a oun e stoc as tic process for* $\{Y_t; F_t; K_t; \Delta_t; q_t^u; m_{t+1}; i_t\}_{t=0}^{\infty}$ *satisfying equations (5), (21) an (41) (45)*

4 Upper Bound in the Model with Housing

We shall now formulate the upper bound problem on the right-hand side of (4) for the nonlinear New Keynesian model with a housing market and distorted private sector expectations, and characterize its solution. The upper bound problem can be written in the form[11]

$$\min_{\{m_{t+1}, q_t^u\}_{t=0}^{\infty}} \max_{\{Y_t, F_t, K_t, \Delta_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{l} U(Y_t; \Delta_t; q_t^u; \xi_t) + \theta\beta m_{t+1} \log m_{t+1} \\ +\gamma_t \left(\tilde{h}(\Delta_{t-1}; K_t{=}F_t) - \Delta_t \right) \\ +\Gamma_t'[z(Y_t; q_t^u; \xi_t) + \alpha\beta m_{t+1}\Phi(Z_{t+1}) - Z_t] \\ +\Psi_t[\xi_t^d + \beta(1-\delta) m_{t+1} q_{t+1}^u - q_t^u] \\ +\beta \quad_t (m_{t+1} - 1) \end{array} \right] + \alpha\Gamma_{-1}'\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u; \tag{46}$$

where $\gamma_t; \Gamma_t; \Psi_t$ and $\;_t$ denote Lagrange multipliers and we use the shorthand notation

$$Z_t \equiv \begin{bmatrix} F_t \\ K_t \end{bmatrix}; \quad z(Y; q^u; \xi) \equiv \begin{bmatrix} f(Y; q^u; \xi) \\ k(Y; \xi) \end{bmatrix}; \quad \Phi(Z) \equiv \begin{bmatrix} \phi_F(K; F) \\ \phi_K(K; F) \end{bmatrix}; \tag{47}$$

[11] From the constraint (21) follows that the choice of $\{m_{t+1}\}_{t=0}^{\infty}$ and the exogenous shocks jointly determine $\{q_t^u\}_{t=0}^{\infty}$. Therefore, worst case beliefs effectively determine the minimizing sequence for $\{m_{t+1}; q_t^u\}_{t=0}^{\infty}$.

and added the initial pre-commitments $\alpha\Gamma'_{-1}\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u$ to obtain a time-invariant solution. The Lagrange multiplier vector Γ_t is associated with constraints (42) and (43) and given by $\Gamma'_t = (\Gamma_{1t}; \Gamma_{2,t})$. The multiplier γ_t relates to equation (45), the multiplier Ψ_t to equation (21) and the multiplier $\;_t$ to constraint (5). We also eliminated the interest rate and the constraint (41) from the problem. Under the assumption that the zero lower bound on nominal interest rates is not binding, constraint (41) imposes no restrictions on the path of the other variables.[12] The path for the nominal interest rates can thus be computed ex-post using the solution for the remaining variables and equation (41).

The nonlinear FOCs for the policymaker are then given by

$$U_Y(Y_t; \Delta_t; q_t^u; \xi_t) + \Gamma'_t z_Y(Y_t; q_t^u; \xi_t) = 0 \tag{48}$$

$$-\gamma_t \tilde{h}_2(\Delta_{t-1}; K_t{=}F_t)\frac{K_t}{F_t^2} - \Gamma_{1t} + \alpha m_t \Gamma'_{t-1} D_1(K_t{=}F_t) = 0 \tag{49}$$

$$\gamma_t \tilde{h}_2(\Delta_{t-1}; K_t{=}F_t)\frac{1}{F_t} - \Gamma_{2t} + \alpha m_t \Gamma'_{t-1} D_2(K_t{=}F_t) = 0 \tag{50}$$

$$U_\Delta(Y_t; \Delta_t; q_t^u; \xi_t) - \gamma_t + \beta E_t[\gamma_{t+1}\tilde{h}_1(\Delta_t; K_{t+1}{=}F_{t+1})] = 0 \tag{51}$$

for all $t \geq 0$. The nonlinear FOC for the worst-case belief distortions m_{t+1} and the FOC for q_t^u take the form

$$\theta(\log m_{t+1} + 1) + \alpha\Gamma'_t\Phi(Z_{t+1}) + (1-\delta)\Psi_t q_{t+1}^u + \;_t = 0 \tag{52}$$

$$U_q(Y_t; \Delta_t; q_t^u; \xi_t) + \Gamma'_t z_q(Y_t; q_t^u; \xi_t) + \Psi_{t-1}(1-\delta)m_t - \Psi_t = 0 \tag{53}$$

for all $t \geq 0$. Above, $\tilde{h}_i(\Delta; K{=}F)$ denotes the partial derivative of $\tilde{h}(\Delta; K{=}F)$ with respect to its i-th argument, and $D_i(K{=}F)$ is the i-th column of the matrix

$$D(Z) \equiv \begin{bmatrix} \partial_F \phi_F(Z) & \partial_K \phi_F(Z) \\ \partial_F \phi_K(Z) & \partial_K \phi_K(Z) \end{bmatrix}: \tag{54}$$

Since the elements of $\Phi(Z)$ are homogeneous degree 1 functions of Z, the elements of $D(Z)$ are all homogenous degree 0 functions of Z, and hence functions of $K{=}F$ only. Thus we can alternatively write $D(K{=}F)$. The optimal upper-bound dynamics are then bounded stochastic processes $\{Y_t; F_t; K_t; \Delta_t; q_t^u; m_{t+1}\}$ that satisfy the structural equation (5), (21), (42)-(45) and the first order conditions (48)-(53).

5 Optimal Upper Bound Dynamics

We shall be concerned solely with optimal outcomes that involve small fluctuations around a deterministic optimal steady state. An *optimal steady state* is a

[12] This assertion also depends on our assumption here that the central bank chooses its interest-rate operating target i_t with full information about the state of the economy at date t.

set of constant values $(\underline{Y}; \underline{Z}; \underline{\Delta}; \underline{q}^u; \underline{m}; \gamma; \underline{\Gamma}; \underline{\Psi}; \underline{\ })$ that solve the structural equations (5),(21),(42)-(45) and the FOCs (48)-(53) in the case that $\xi_t = \underline{\xi}$ at all times and initial conditions consistent with the steady state are assumed. We now compute the steady-state, then derive the local dynamics implied by these equations.

5.1 The Optimal Steady State and Its Properties

In a deterministic steady state, restriction (5) implies $\underline{m} = 1$. Equation (21) then implies $\underline{q}^u = \underline{\bar{\xi}}^d$. Moreover, as in the model without housing, considered in Adam and Woodford (2012), the optimal steady state satisfies $\underline{F} = \underline{K} = (1 - \alpha\beta)^{-1} k(\underline{Y}; \underline{\xi})$, which implies $\underline{\Pi} = 1$ (no inflation) and $\underline{\Delta} = 1$ (zero price dispersion), where the value of $\underline{Y}$ is implicitly defined by

$$f(\underline{Y}; \underline{q}^u; \underline{\xi}) = k(\underline{Y}; \underline{\xi}). \tag{55}$$

As shown in appendix A.1, there exists a unique steady state consumption level $\underline{Y}$ solving (55).

Furthermore, with $\tilde{h}_2(1; 1) = 0$ (the effects of a small non-zero inflation rate on the measure of price dispersion are of second order), conditions (49)–(50) reduce in the steady state to the eigenvector condition

$$\underline{\Gamma}' = \alpha \underline{\Gamma}' D(1). \tag{56}$$

Moreover, since when evaluated at a point where $F = K$,

$$\frac{\partial \log(\phi_K / \phi_F)}{\partial \log K} = -\frac{\partial \log(\phi_K / \phi_F)}{\partial \log F} = \frac{1}{\alpha},$$

we observe that $D(1)$ has a left eigenvector $[1 \ -1]$, with eigenvalue $1/\alpha$; hence (56) is satisfied if and only if $\underline{\Gamma}_2 = -\underline{\Gamma}_1$. Condition (48) provides then one additional condition to determine $\underline{\Gamma}_1$. It implies

$$U_Y(\underline{Y}; 1; \underline{q}^u; \underline{\xi}) + \underline{\Gamma}_1 (f_Y(\underline{Y}; \underline{q}^u; \underline{\xi}) - k_Y(\underline{Y}; \underline{q}^u; \underline{\xi})) = 0. \tag{57}$$

Appendix A.1 shows that

$$k_Y - f_Y > 0,$$

so that $\underline{\Gamma}_1$ has the same sign as U_Y. Appendix A.1 also proves that

$$U_Y(\underline{Y}; 1; \underline{q}^u; \underline{\xi}) = \bar{C}^{\tilde{\sigma}^{-1}} C\left(\underline{Y}; \underline{q}^u; \underline{\xi}\right)^{-\tilde{\sigma}^{-1}} \left(\frac{1-g}{1+s^d} + \frac{s^d}{1+s^d} C_Y\left(\underline{Y}; \underline{q}^u; \underline{\xi}\right) - \frac{\eta - 1}{\eta}(1 - \underline{\tau}) \right) \tag{58}$$

which shows that in the absence of a housing subsidy ($s^d = 0$) we have $U_Y = 0$ and thus $\underline{\Gamma}_1 = 0$, whenever the output subsidy eliminates the steady state monopoly

distortion, i.e., when $1-\underline{\tau} = \left(1-\underline{g}\right)\frac{\eta}{\eta-1}$. The resulting steady state consumption level is then determined by (98) in appendix A.1.

More generally, we shall consider the case with a non-zero housing subsidy/tax. Conditional on the value of housing subsidy s^d one can then define an efficient steady state output subsidy $\underline{\tau}^{eff}(s^d)$, which is the value of $\underline{\tau}$ such that $U_Y = 0$ in (58). Appendix A.2 shows the following result:

Lemma 1 *If $s^d \geq 0$, or if $s^d < 0$ ut sufficient y c ose to zero, t en $U_{YY} < 0$*

The previous lemma shows that $\underline{\tau}^{eff}(s^d)$ indeed maximizes steady state utility in the presence of a housing subsidy or a housing tax that is not too large. Appendix A.1 then establishes the following result:

Lemma 2 *Given a ousing su si y s^d an t e efficient output su si y $1-\underline{\tau} = 1-\underline{\tau}^{eff}(s^d)$, we ave*

$$U_Y = \underline{\Gamma}_1 = 0:$$

If t e output su si y fa s s ort of its efficient va ue, $1-\underline{\tau} < 1-\underline{\tau}^{eff}(s^d)$, t en

$$U_Y > 0,\ \underline{\Gamma}_1 > 0;$$

w i e if instea $1-\underline{\tau} > 1-\underline{\tau}^{eff}(s^d)$ one o tains

$$U_Y < 0,\ \underline{\Gamma}_1 < 0:$$

The previous lemma shows that the marginal utility of output is positive (negative) in the steady state, whenever the output subsidy falls short of (exceeds) the output subsidy that would be efficient given the assumed level of housing subsidies.

Condition (51) provides a restriction that determines the steady state value of $\bar{\gamma}$:

$$U_\Delta(\underline{Y};1;\underline{q}^u;\underline{\xi}) - \underline{\gamma} + \beta\underline{\gamma}\tilde{h}_1(1;1) = 0:$$

Since $U_\Delta < 0$ and $\tilde{h}_1(1;1) = \alpha$, we have

$$\underline{\gamma} = \frac{U_\Delta(\underline{Y};1;\underline{q}^u;\underline{\xi})}{(1-\beta\alpha)} < 0:$$

Condition (53) implies

$$\underline{\Psi} = \frac{1}{\delta}\left(U_q(Y;1;\underline{q}^u;\underline{\xi}) + \underline{\Gamma}_1 f_q(\underline{Y};\underline{q}^u;\underline{\xi})\right) \tag{59}$$

and appendix A.3 proves the following result:

Lemma 3 *If* $s^d = 0$, $U_q(Y; 1; \underline{q}^u; \underline{\xi}) = 0$ *If instea* $s^d > 0$, $U_q < 0$, *an if* $s^d < 0$, $U_q > 0$ *T en since* $\underline{\Gamma}_1 = 0$ *if* $\underline{\tau} = \underline{\tau}^{eff}(s^d)$, *it fo ows t at*

$$\begin{aligned}
&\underline{\Psi} = 0 \quad \text{if } s^d = 0 \text{ an } \underline{\tau} = \underline{\tau}^{eff}(0) \\
&\underline{\Psi} < 0 \quad \text{if } s^d > 0; \underline{\tau} \text{ sufficient y c ose to } \underline{\tau}^{eff}(s^d) \\
&\underline{\Psi} > 0 \quad \text{if } s^d < 0, \underline{\tau} \text{ sufficient y c ose to } \underline{\tau}^{eff}(s^d)
\end{aligned}$$

The previous lemma shows that for a positive housing subsidy, the representative household's utility is decreasing with further house price increases, whenever the output subsidy is sufficiently close to its efficient level. Correspondingly, in the presence of a housing tax, houshold utility decreases with a fall in housing prices. Intuitively, holding the level of total output Y fixed, an increase in the housing price leads to a further increase in housing investment, which is already inefficiently high (low) when there is a housing subsidy (tax).

5.2 Characterizing the Upper Bound Dynamics

It is useful to implicitly define a variable Y_t^* (a function of the exogenous disturbances), as the solution to the equation

$$U_Y(Y_t^*; 1; \bar{\xi}_t^d; \xi_t) + \underline{\Gamma}' z_Y(Y_t^*; \bar{\xi}_t^d; \xi_t) = 0 \tag{60}$$

or alternatively as the output level that maximizes $U(Y_t; 1; \bar{\xi}_t^d; \xi_t) + \underline{\Gamma}' z(Y_t; \bar{\xi}_t^d; \xi_t)$. One can then derive a first order approximation of the upper bound dynamics for the variables

$$\begin{aligned}
\pi_t &\equiv \log \Pi_t \\
\widehat{m}_t &\equiv \log m_t \\
x_t &= \log Y_t - \log \widehat{Y}_t^*
\end{aligned}$$

where x_t denotes the output gap. We also make the following assumption:

Assumption 1: The output subsidy falls short of its efficient level ($1 - \underline{\tau} < 1 - \underline{\tau}^{eff}(s^d)$), but $1 - \underline{\tau}$ is sufficiently close to $1 - \underline{\tau}^{eff}(s^d)$ for the conclusions of lemma 3 to be valid. Either $s^d \geq 0$ or if $s^d < 0$ then s^d is sufficiently small for the conclusions of lemma 1 to hold. In addition, initial price dispersion Δ_{-1} is small ($\Delta_{-1} = 0 + O(2)$) and the initial commitments are such that $\Gamma_{1,-1} = -\Gamma_{2,-1}$.

We then have the following result:

Proposition 1 *Suppose assumption 1 o s T e first or er con itions (48) (53) an t e constraints in (46) imp y (to first or er)*

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t \tag{61}$$

$$0 = \xi_\pi \pi_t + \lambda \left(x_t - x_{t-1}\right) + \xi_m \widehat{m}_t \tag{62}$$

$$\widehat{m}_t = \lambda_m \left(\pi_t - E_{t-1}\pi_t\right) + \lambda_q (\widehat{q}_t^u - E_{t-1}\widehat{q}_t^u): \tag{63}$$

w ere t e constants $(\kappa; \xi_\pi; \xi_m; \lambda; \lambda_m; \lambda_q)$ *are functions of t e eep mo e parameters (exp icit expression are provi e in appen ix A 4) wit* $\kappa > 0; \xi_\pi > 0; \lambda > 0; \lambda_m > 0; \xi_m > 0$ *Furt ermore,*

$$\lambda_q > 0 \text{ if } s^d > 0$$

$$\lambda_q < 0 \text{ if } s^d < 0$$

In t e imiting case wit out ro ustness concerns $(\theta \to \infty)$*, we ave* $\lambda_m \to 0$ *an* $\lambda_q \to 0$

Equation (63) shows that the worst-case belief distortions are of the kind that they increase the likelihood of positive inflation surprises. Intuitively, overweighing positive inflation surprises (and underweighing negative ones) increases average expected inflation and thereby via (61) current inflation rates, which contributes (ceteris paribus) to reducing output, which due to $1 - \underline{\tau} < 1 - \underline{\tau}^{eff}(s^d)$ is already below its optimal level (given the assumed housing subsidy). In the presence of a housing subsidy ($s^d > 0$) worst case beliefs also overweigh positive housing price suprises (and underweigh negative ones). Doing so increases increases the expected value of future housing prices, thereby the average house price today, see the asset pricing equation (21), which in turn increases housing supply. The latter is harmful in welfare terms because the presence of an housing subsidy implies that the housing stock is already suboptimally high.

The proof of proposition 1 also shows that to first order

$$\widehat{q}_t^u = \widehat{\bar{\xi}}_t^d; \tag{64}$$

so that $\widehat{q}_t^u = \log q_t^u = \log \underline{q}$ is determined (to this order of approximation) purely by exogenous disturbances. Importantly, this does not mean that endogenous belief distortions have no consequences for the marginal-utility value of housing, only that these effects are of *second order* in the amplitude of the exogenous disturbances, in our local approximation. Such second-order effects remain welfare-relevant, since our log-linear approximation to the optimal policy rule depends on *second-order* terms in a local approximation to the expected utility of the representative household, see Benigno and Woodford (2005) for discussion of this general issue.[13]

[13] In Adam and Woodford (2012), belief distortions similarly have no effect on a log-linear approximation to the aggregate supply tradeoff (given as usual by the "New Keynesian Phillips curve"); yet the *second-order* effects of belief distortions on this relationship explain why in that model, a robustly optimal policy commitment requires different inflation dynamics than an optimal policy commitment under rational expectations, *even to a log-linear approximation.*

We are particularly interested in the analyzing effects of shocks origniating in the housing sector, i.e., the disturbances A^d and $\bar{\bar{\xi}}^d$. The following result determines their effects on the 'cost-push' disturbance u_t:

Proposition 2 *Suppose s^d is not too negative, so t at emma 1 o s T en (to first or er)*

$$u_t = \kappa_u \left(\widehat{\bar{\bar{\xi}}}_t^d + \widehat{A}_t^d \right) + n{:}h{:}s{:}$$

w ere $n{:}h{:}s{:}$ enotes t e effects of non ousing s oc s an t e constant κ_u is a func tion of eep mo e parameters (an exp icit expression is erive in appen ix A 5) At t e efficient stea y state w ere $s^d = 0$ an $\tau = \tau^{eff}(0)$ we ave $\kappa_u = 0$ In t e presence of a ousing su si y ($s^d > 0$) an for τ sufficient y c ose to $\tau^{eff}(s^d)$, we ave $\kappa_u < 0$ w i e wit a ousing tax ($s^d < 0$) an for τ sufficient y c ose to $\tau^{eff}(s^d)$;we ave $\kappa_u > 0$

The proof of the proposition can be found in appendix A.5. The proposition shows that cost-push effects are absent (to first order) whenever the steady state is first best. In the presence of a housing subsidy or tax, however, housing demand shocks and shocks to the productivity of housing production give both rise to cost-push effects, with the sign of the effect depending on the sign of the housing subsidy s^d.

5.3 Impulse Responses to Housing Sector Shocks

We now derive a closed form solution for the impulse response to housing sector disturbances implied by the upper bound dynamics. For simplicity we assume that the evolution of the disturbances is described by

$$\widehat{\bar{\bar{\xi}}}_t^d = \rho_\xi \widehat{\bar{\bar{\xi}}}_{t-1}^d + !_{\xi t}$$
$$\widehat{A}_t^d = \rho_A \widehat{A}_t^d + !_{At}$$

where $\rho_i \in [0;1)$ captures the persistence of the disturbance and $!_{it}$ is an iid innovation ($i = \xi; A$).

Substituting (63) and (64) into (62) to eliminate $\widehat{m}_t$ and $\widehat{q}_t^u$, and using (61) to substitute for x_t, one obtains

$$0 = \left(\xi_\pi + \frac{\lambda}{\kappa} \right) \pi_t - \frac{\lambda}{\kappa} \left(\beta \left(E_t \pi_{t+1} - E_{t-1} \pi_t \right) + \pi_{t-1} + u_t - u_{t-1} \right)$$
$$+ \xi_m \lambda_m \left(\pi_t - E_{t-1} \pi_t \right) + \xi_m \lambda_q \left(\widehat{\bar{\bar{\xi}}}_t^d - E_{t-1} \widetilde{\bar{\bar{\xi}}}_t^d \right); \qquad (65)$$

which characterizes the inflation response to exogenous shocks with cost-push effects, under the upper-bound dynamics. For a housing sector shock hitting the economy in period t_0 , i.e., for

$$u_{t_0} = \kappa_u !_{\xi t_0} \text{ or } u_{t_0} = \kappa_u !_{A t_0},$$

and an economy starting out in $t_0 - 1$ at its deterministic steady state and with no further shocks occurring after t_0, the inflation response is characterized by

$$0 = \left(\xi_\pi + \frac{\lambda(1+\beta)}{\kappa}\right)\pi_t - \frac{\lambda}{\kappa}\left(\beta\pi_{t+1} + \pi_{t-1} + u_t - u_{t-1}\right) \quad \text{for } t > t_0 \tag{66}$$

and for t_0 and $u_{t_0} = \kappa_u \,!_{\xi t_0}$ by

$$0 = \left(\xi_\pi + \frac{\lambda}{\kappa} + \xi_m \lambda_m\right)\pi_{t_0} - \frac{\lambda\beta}{\kappa}\pi_{t_0+1} - \left(\frac{\lambda}{\kappa}\kappa_u - \xi_m\lambda_q\right) !_{\xi t_0}, \tag{67}$$

and for t_0 and $u_{t_0} = \kappa_u \,!_{A t_0}$ by

$$0 = \left(\xi_\pi + \frac{\lambda}{\kappa} + \xi_m \lambda_m\right)\pi_{t_0} - \frac{\lambda\beta}{\kappa}\pi_{t_0+1} - \frac{\lambda\kappa_u}{\kappa} !_{A t_0}: \tag{68}$$

As shown in Adam and Woodford (2012), equation (66) has for $t > t_0$ a unique stable solution given by

$$\pi_t = a\pi_{t-1} + b_i u_{t-1} \tag{69}$$

where $a \in (0; 1)$ and $b_i = -(1-\rho_i)a < 0$ with $a; b_i \;\; (i = \xi; A)$ being independent of the policymakers' concern for robustness θ. Combining (69) for $t = t_0 + 1$ with (67) and (68), respectively, delivers the optimal initial inflation response at time t_0 :

$$\pi_{t_0} = \frac{\left(b_\xi + \beta^{-1}\right)\kappa_u - \frac{\kappa}{\lambda\beta}\xi_m\lambda_q}{\frac{\kappa}{\lambda_c\beta}\left(\xi_\pi + \frac{\lambda}{\kappa} + \xi_m\lambda_m\right) - a} !_{\xi t_0} \qquad \text{for } u_{t_0} = \kappa_u \,!_{\xi t_0} \tag{70}$$

$$\pi_{t_0} = \frac{\left(b_A + \beta^{-1}\right)\kappa_u}{\frac{\kappa}{\lambda_c\beta}\left(\xi_\pi + \frac{\lambda}{\kappa} + \xi_m\lambda_m\right) - a} !_{A t_0} \qquad \text{for } u_{t_0} = \kappa_u \,!_{A t_0} \tag{71}$$

where $\frac{\kappa}{\lambda\beta}\left(\xi_\pi + \frac{\lambda}{\kappa}\right) - a > 0$. From proposition 1 follows that in the limiting case without robustness concerns ($\theta \to \infty$) we have $\lambda_m \to 0$ and $\lambda_q \to 0$, so that the impulses reponses to housing demand and productivity shocks are identical, provided both shocks have the same persistence. Moreover, under the optimal response the price level returns to its initial starting value.

In the presence of a housing subsidy ($s^d > 0$) and with robustness concerns we have $\lambda_q > 0$, so that the initial inflation response is dampened compared to the case with rational expectations optimal policy, so that in the long-run the price level undershoots its initial level following positive disturbances. The dampening and undershooting effects are thereby more pronounced for the housing demand shock.

In the presence of a housing tax $\left(s^d < 0\right)$ we have $\lambda_q < 0$. Following a positive disturbance, inflation thus optimally increases more in the initial period than under rational expectations optimal policy. As a result, the price level will not fully return and stay at an elevated level. The initial and terminal increase is thereby more pronounced following a demand disturbance.

For $s^d > 0$ ($s^d < 0$) the response to housing demand shocks is dampenend (amplified) more by robustness concerns relative to the response to housing technology shocks. This occurs because housing demand shocks show up (to first order) in the asset pricing equation, while housing productivity shock do not.

5.4 The Case of Inelastic Housing Supply

Equation (63) shows that in the presence of a housing subsidy ($s^d > 0$), worst case beliefs increase the likelihood of future positive surprises to housing prices while decreasing the probability of negative surprises. Worst case beliefs thus increase, via the asset price equation (21), current house prices and thereby lead to a further upward distortion of housing supply (an effect that is of second order).

These effects depend on the presence of an elastic supply of housing in response to relative price changes. We can consider the limiting case of an inelastic housing supply ($\widetilde{\alpha} \to 0$) by fixing an exogenous stochastic process for $\widetilde{A}_t^d = A_t^d$=$\widetilde{\alpha}$. All of our previous equations then remain well-behaved in this limit, if written in terms of $\widetilde{\alpha}\widetilde{A}_t^d$ rather than A_t^d.

In the limit with $\widetilde{\alpha} = 0$, zero resources are used to produce houses ($k_t = 0$), housing supply is equal to $\widetilde{A}_t^d$, and the housing stock therefore independent of agents' beliefs. Equation (23) then holds for $\Omega = 0$, and the utility function (39) becomes independent of q^u. Similarly, the function $f(Y_t; q_t^u; \xi_T)$ becomes independent of q_t^u, so that equation (59) implies that $\underline{\Psi} = 0$. From equation (117) in the appendix it then follows that $\lambda_q = 0$, so that worst case belief distortions become independent of house price surprises. Since $\underline{\Omega} = 0$, it follows from equation (131) in the appendix that $\kappa_u = 0$, so that housing sector shocks no longer have cost-push effects. One can then show that the upper bound dynamics for inflation and the output gap are independent of both types of housing shocks; in fact they depend on other fundamental disturbances in exactly the same way as in the model without a housing sector analyzed in Adam and Woodford (2012). In this case, the robustly optimal conduct of policy requires no reference to housing prices or other housing variables.

6 A Robustly Optimal Target Criterion for Monetary Policy

It remains to be shown that the lower bound for welfare associated with the upper-bound dynamics characterized in the previous section can in fact be achieved by some policy rule. This requires not only that we find a policy rule consistent with the upper-bound dynamics in the case of the worst-case belief distortions also characterized above, but also that we can show that these distorted beliefs are indeed worst-case belief distortions in the case of that policy rule. In fact, not only is it possible to find such a rule, but there are many of them, for the same reasons as are explored

in Adam and Woodford (2012) in the case of the model without a housing sector. Indeed, our analysis here is a direct extension of our results in the earlier paper. The only important new work required here is the modification of our derivations of the policy rules consistent with the upper-bound dynamics, showing the way in which the policy rules must be modified because of the additional complications in the model.

As in Adam and Woodford (2012), it is of particular interest to note that robustly optimal monetary policy can be specified in terms of a *target criterion* — a log-linear relationship among endogenous variables that the central bank commits itself to maintain at all times (regardless of the evolution of exogenous disturbances), through appropriate adjustment of its policy instrument (here, a short-term nominal interest rate). As in the case of optimal policy commitments under rational expectations, analyzed under general conditions in Giannoni and Woodford (2010), the robustly optimal target criterion can be derived from the first-order conditions that characterize the upper-bound dynamics, in a way that is independent of the specification of the exogenous disturbance processes.

Substituting (63) into (62) to eliminate $\widehat{m}_t;$ we obtain

$$\pi_t + \phi_x (x_t - x_{t-1}) + \phi_s (\pi_t - E_{t-1}\pi_t) + \phi_q(\widehat{q}^u_t - E_{t-1}\widehat{q}^u_t) = 0; \tag{72}$$

where $\phi_x \equiv \lambda=\xi_\pi > 0;$ $\phi_s \equiv \xi_m\lambda_m=\xi_\pi > 0$ and $\phi_q \equiv \xi_m\lambda_q=\xi_\pi$, so that $\phi_q > 0$ if $s^d > 0$ and $\phi_q < 0$ if $s^d < 0$. Condition (72) makes no explicit reference to belief distortions or to exogenous disturbances, except insofar as the latter are involved in the definitions of x_t and $\widehat{q}^u_t$.

This condition provides a target criterion that represents a possible form of monetary policy commitment, as established by the following result.

Proposition 3 *Suppose assumption 1 o s an suppose t at t e centra an commits to use interest rate po icy to ensure t at (72) o s eac perio T en for any e ief istortions* $\{m_{t+1}\}_{t=0}^{\infty}$ *c ose enoug to t e istortions* $\{m^{ub}_{t+1}\}_{t=0}^{\infty}$ *associate wit t e upper oun ynamics (an c aracterize to first or er in proposition 1), t ere exist equi i rium pat s for a en ogenous varia es for w ic (72) o s eac perio (so t at t e po icy commitment is feasi e) Moreover, t e equi i rium pat s are o ca y unique, so t at t e target criterion (72) fu y etermines t e necessary po icy actions*

The proof of the previous result is in appendix A.6. The same appendix also establishes that condition (72) is indeed the robustly optimal target criterion that we seek:

Proposition 4 *T e e ief istortion* $\{m^{ub}_{t+1}\}_{t=0}^{\infty}$ *an t e associate pat s for t e en ogenous varia es (equa to first or er to t e ynamics c aracterize in proposition 1) represent a oca so ution to t e inner pro em on t e eft an si e of (4); t at is, t e istortions* $\{m^{ub}_{t+1}\}_{t=0}^{\infty}$ *represent (oca y) worst case e iefs un er t e po icy (72), an t e associate worst case outcomes are t ose c aracterize (to first or er) in proposition 1*

Note that (72) is a form of "flexible inflation-targeting rule": it implies a definite long-run average inflation rate (namely, zero) regardless of the history of disturbances, and allows transitory departures of the actual inflation rate from that long-run target only to the extent that they are justified by the sign and magnitude of transitory variations in the output gap x_t and/or the housing price $\widehat{q}_t^u$. Moreover, in the limiting case of no concern for robustness to possible departures from model-consistent beliefs ($\theta \to \infty$), we have $\phi_s \to 0$ and $\phi_q \to 0$, in which case the criterion (72) reduces to the optimal target criterion for the basic New Keynesian model under rational expectations and without a housing sector, as discussed in Benigno and Woodford (2005) and Giannoni and Woodford (2010). In particular, under rational expectations, expected utility can be maximized by committing to a target criterion

$$\pi_t + \phi_x (x_t - x_{t-1}) = 0; \tag{73}$$

that can be stated purely in terms of the paths of inflation and the output gap, without any reference to the behavior of housing prices, even in our model with a housing sector subject to tax distortions. Robustness to belief distortions instead requires a more complex rule with two additional terms, one of which involves surprise changes in inflation an the other suprise changes in the index of housing prices (measured in marginal-utility units). The terms involving surprise changes in inflation also appeared in the the robustly optimal target criterion derived Adam and Woodford (2012) in the case of a New Keynesian model without a housing sector. The new term arising from robustness concerns in our model with housing is thus the term involving housing price surprises.

The case of empirical relevance for economies like the US is the one in which there is overproduction of housing owing to tax subsidies ($s^d > 0$), so that the robustly optimal target criterion involves $\phi_q > 0$: This means that the central bank should "lean against" unexpected increases in housing prices, in the sense that it adopts a policy stance under which it deliberately undershoots its normal targets for inflation and/or the output gap because of an unexpected rise in housing prices.

7 Belief Distortions and Stabilization Goals

Further insight into the way in which robustly optimal policy should respond to housing variables can be obtained through an explicit consideration of the way in which welfare is affected by variations in the inflation rate, real activity, and housing prices.

In the basic New Keynesian model, without a housing sector, it is well known that the expected utility of the representative household can be approximated by a quadratic objective function, that varies negatively with a discounted sum of squared inflation rates and squared output gaps.[14] This result makes it natural to express the

[14]The approximation consists of a second order Taylor series approximation, valid in the case of

target criterion for optimal policy in terms of those same two variables, the inflation rate and the output gap.[15] Here we show how such a quadratic approximation is modified by the introduction of a housing sector, in order to obtain insight into the way in which the robustly optimal target criterion must be modified.

We begin by observing that in the case of small enough exogenous disturbances, small enough belief distortions, and small enough departures of endogenous variables from their steady state values, the period contribution to the welfare objective (39) can be locally approximated by

$$U(Y_t; \Delta_t; q_t^u; \xi_t) = U_Y \widetilde{Y}_t + U_\Delta \widetilde{\Delta}_t + U_q \left(q_t^u - \bar{\xi}_t^d \right) + \frac{1}{2} U_{YY} \widetilde{Y}_t^2 + \widetilde{Y}_t \left(U_{Y\xi} \widetilde{\xi}_t + U_{Yq} \widetilde{\bar{\xi}}_t^d \right) + t{:}i{:}p{:} + O(3); \tag{74}$$

where $\widetilde{Y}_t = Y_t - \underline{Y}$, $\widetilde{\Delta}_t = \Delta_t - \underline{\Delta}$, $\widetilde{\xi}_t = \xi_t - \underline{\xi}$, $\widetilde{\bar{\xi}}_t^d = \bar{\xi}_t^d - \underline{\bar{\xi}}^d$ denote the deviation from the steady state values of each variable, $t{:}i{:}p{:}$ denotes terms which are independent of policy (constants and exogenous disturbances), and $O(3)$ indicates a residual that is at most of third order in the amplitude of the disturbances.[16]

The terms labeled $t{:}i{:}p{:}$ need not be written out explicitly, as their value will be independent of monetary policy *an* of the nature of belief distortions, so that these terms affect none of the comparison made below. In writing (74), we restrict attention to possible paths for the economy in which fluctuations in the endogenous variable $\widetilde{Y}_t$ are of a magnitude proportional to that of the exogenous disturbances $\widetilde{\xi}_t^d$ ("of first order", in our terminology), and the fluctuations in the endogenous variables $\widetilde{\Delta}_t$, $(q_t^u - \bar{\xi}_t^d)$ are of a magnitude proportional to that of squared disturbance terms ("of second order"). These bounds on the magnitude of fluctuations in the endogenous variables hold for the solution of the upper bound dynamics characterized in sections 4 and 5, given initial conditions satisfying assumption 1.

One can similarly show that the worst-case belief distortions associated with these dynamics will be such that $\widetilde{m}_{t+1} = m_{t+1} - 1$ is of first order as well; in fact, only fluctuations $\{\widetilde{m}_{t+1}\}_{t=0}^{\infty}$ of first order will be consistent with a finite relative entropy bound, and so in our local approximations we assume that $\widetilde{m}_{t+1}$ is of first order. The fact that belief distortions can only be of first order in the amplitude of the disturbances then implies that the discrepancy $q_t^u - \bar{\xi}_t^d$ can be at most of second order in the amplitude of the disturbances (as a consequence of (21)). This is why it is

small enough exogenous disturbances, see chapter 6 in Woodford (2003) and Benigno and Woodford (2003).

[15] In fact, the appropriate definition of the output gap is precisely the one required in order to express the quadratic approximation to expected utility in terms of squared inflation terms and squared output gaps.

[16] See chapter 6 in Woodford (2003) and Benigno and Woodford (2003) for further discusssion of the method of linear-quadratic approximation of optimal policy problems used here.

convenient to expand (74) in powers of $\left(q_t^u - \bar{\xi}_t^d\right)$ and $\tilde{\bar{\xi}}_t^d$, rather than powers of $q_t^u - \underline{q}$ alone.[17]

Again under the assumption of initial conditions satisfying assumption 1, a second order Taylor approximation to the dynamic equation (44) implies that[18]

$$\sum_{t=0}^{\infty} \beta^t \widetilde{\Delta}_t = \sigma_\Delta \sum_{t=0}^{\infty} \beta^t \pi_t^2 + t{:}i{:}p{:} + O(3); \tag{75}$$

where

$$\sigma_\Delta = \frac{1}{2} \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \eta(1+\,!\,)(1+\,!\,\eta) > 0{:}$$

Using this to approximate the discounted sums of $\widetilde{\Delta}_t$ terms, one can then approximate the welfare objective (39) by a discounted sum of squared inflation rates (as in (75)), a discounted sum of terms that are quadratic functions of $\widetilde{Y}_t$ and the exogenous disturbances, and discounted sums of the form

$$\sum_{t=0}^{\infty} \beta^t \left[U_Y \widetilde{Y}_t + U_q \left(q_t^u - \bar{\xi}_t^d \right) \right]{:} \tag{76}$$

From lemma 3, if there is no housing subsidy ($s^d = 0$), then $U_q = 0$. If in addition, there is an efficient output subsidy ($\underline{\tau} = \underline{\tau}^{eff}(0)$), by lemma 2, $U_Y = 0$. In this case, all terms in (76) would vanish, and the welfare objective can be approximated by

$$\sum_{t=0}^{\infty} \beta^t U(Y_t; \Delta_t; q_t^u; \xi_t) = -\sum_{t=0}^{\infty} \beta^t \left(\Lambda_\pi \pi_t^2 + \Lambda_x x_t^2 \right) + t{:}i{:}p{:} + O(3); \tag{77}$$

where[19]

$$\Lambda_\pi = -U_\Delta \sigma_\Delta > 0$$
$$\Lambda_x = -\frac{1}{2} \underline{Y}^2 U_{YY} > 0;$$

and

$$x_t = \log(Y_t{=}\underline{Y}) + (\underline{Y}\, U_{YY})^{-1} \left(U_{Y\xi} \widetilde{\xi}_t + U_{Yq} \tilde{\bar{\xi}}_t^d \right){:} \tag{78}$$

Thus we would obtain in this case the same form of loss function for monetary stabilization policy as in chapter 6 in Woodford (2003).

---

[17] We do not need to include a $\widetilde{Y}_t(q_t^u - {}^{-d}_{t})$ term, as this would be of at least third order, though we do have to inlcude a $\widetilde{Y}_t \,\tilde{\tilde{}}_t$ term.

[18] See equation (25) in Benigno and Woodford (2005).

[19] The first inequality below follows from (40), which implies $U_\Delta < 0$. The second inequality follows from the fact that $U_{YY} < 0$, as shown in appendix A.2.

Because the approximate welfare objective (77) then contains only purely quadratic terms, a second order approximation to welfare requires only a first order accurate solution for the evolution of inflation and output, and so to this order of approximation, there will be no effects of belief distortions on welfare. The characterization of optimal policy is then identical (to first order) to that under rational expectations, given for example in chapter 7 in Woodford (2003).

Belief distortion can matter to first order for the form of robustly optimal policy only to the extent that U_Y or U_q is non-zero. If the output subsidy falls short of its efficient level, in accordance with assumption 1, then $U_Y > 0$. In this case, a second order accurate solution for the evolution of $\widetilde{Y}_t$ is necessary in order to evaluate welfare to second order, and belief distortions matter. Specifically, belief distortions that exaggerate expected inflation shift the Phillips curve trade-off between inflation and output gap stabilization in an unfavorable direction, requiring a lower average value of $\widetilde{Y}_t$ for inflation stabilization, in a situation where output is already suboptimally low. Robustly optimal policy must then make it more difficult for near-rational belief distortions to exaggerate the expected inflation rate of inflation, as discussed in Adam and Woodford (2012). However, in the absence of a housing subsidy ($s^d = 0$), we would still have $U_q = 0$. In this case, the fact that mistaken beliefs may distort the housing price q_t^u is irrelevant for welfare (to second order), and the optimal target criterion continues to take the same form as that derived by Adam and Woodford (2012) for an economy without a housing market.

If instead housing is subsidized ($s^d > 0$), lemma 3 implies $U_q < 0$. In this case, first-order belief distortions can cause second order variation in q_t^u and so lower welfare. Robustly optimal policy must then seek to guard both against belief distortions that exaggerate expected inflation and those that exaggerate the expected future value of housing.

The impact of these two types of belief distortions on welfare can be seen by using quadratic approximations to the model structural equations to solve for the linear terms in (76) as explicit functions of the belief distortions and variables independent of belief distortions. We begin, as in Benigno and Woodford (2005), by using a second-order approximation to the structural equations to eliminate the linear terms $\widetilde{Y}_t$.

As shown in Appendix A.7, a second order approximation yields

$$
\begin{aligned}
&E_0 \sum_{t=0}^{\infty} \beta^t \underline{\Gamma}' \left(z(Y_t; q_t^u; \xi_t) + \alpha\beta m_{t+1} \Phi(Z_{t+1}) - Z_t \right) \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \underline{\Gamma}' \left(z_Y \widetilde{Y}_t + z_q (q_t^u - \bar{\xi}_t^d) + \frac{1}{2} z_{YY} \widetilde{Y}_t^2 + \widetilde{Y}_t \left(z_{Y\xi} \widetilde{\xi}_t + z_{Yq} \widetilde{\bar{\xi}}_t^d \right) \right) \\
&- E_0 \sum_{t=0}^{\infty} \beta^{t+1} \underline{\Gamma}_1 \left(\delta_1^\pi \widetilde{m}_{t+1} \pi_{t+1} + \delta_2^\pi \pi_{t+1}^2 \right) + \underline{\Gamma}_1 \delta_1^\pi \pi_0 + \underline{\Gamma}_1 \delta_3^\pi (\pi_0)^2 + t.i.p. + O(3) \quad (79)
\end{aligned}
$$

where the coefficients $\delta_1^\pi; \delta_2^\pi; \delta_3^\pi > 0$ are defined in the appendix. Structural equations

(42)-(43) imply that the left-hand side of (79) must equal zero in any DEE; hence the right-hand side must also equal zero to second order. We can thus add these terms to our second order approximation to the welfare objective and still have a second order approximation to (39) that must apply to any DEE.

Because the optimal steady state around which we compute our second order approximation satisfies $U_Y + \underline{\Gamma}' z_Y = 0$ (see equation (48)), the terms linear in $\widetilde{Y}_t$ cancel, and we obtain an approximation of the form

$$
\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t; \Delta_t; q_t^u; \xi_t) \\
= & - E_0 \sum_{t=0}^{\infty} \beta^t \left(\Lambda_\pi \pi_t^2 + \Lambda_x x_t^2 + \beta \underline{\Gamma}' \delta_1^\pi \widetilde{m}_{t+1} \pi_{t+1} - \delta \underline{\Psi} \left(q_t^u - \overline{\xi}_t^d \right) \right) \\
& + \underline{\Gamma}_1 \delta_1^\pi \pi_0 + \underline{\Gamma}_1 \delta_3^\pi (\pi_0)^2 + t.i.p. + O(3); \qquad (80)
\end{aligned}
$$

where now

$$
\begin{aligned}
\Lambda_\pi &\equiv -U_\Delta \sigma_\Delta + \underline{\Gamma}_1 \delta_2^\pi > 0 \\
\Lambda_x &\equiv -\frac{1}{2} \underline{Y}^2 \left(U_{YY} + \underline{\Gamma}' z_{YY} \right) > 0
\end{aligned}
$$

(generalizing the expressions given above to the case in which $\underline{\Gamma} \neq 0$), and x_t is defined as in section 5.2 (generalizing the expression given (78)). Appendix A.2 shows $U_{YY} < 0$ as long as the degree to which the output subsidy falls short of its efficient level is not too great. The same condition guarantees that the term $\underline{\Gamma}' z_{YY}$ will be close to zero, so that in this case we must again have $\Lambda_x > 0$.

The maximization of expected discounted utility then corresponds (to a second order approximation) to minimization of a discounted sum of losses of four types: squared inflation rates, squared output gaps, and two types of expectational errors. The losses due to expectational errors are terms proportional to

$$
E_t \widetilde{m}_{t+1} \pi_{t+1} = \widehat{E}_t \pi_{t+1} - E_t \pi_{t+1} \qquad (81)
$$

and terms that depend negatively on

$$
q_t^u - \overline{\xi}_t^d = \widehat{E}_t \left(\sum_{j=0}^{\infty} (\beta(1-\delta))^j \xi_{t+j}^d \right) - E_t \left(\sum_{j=0}^{\infty} (\beta(1-\delta))^j \xi_{t+j}^d \right). \qquad (82)
$$

Higher values of the former terms reduce welfare to the extent that $\underline{\Gamma}_1 > 0$, which holds if and only if the output subsidy is inefficiently low, by lemma 2. Higher values of the latter terms reduce welfare to the extent that $\underline{\Psi} < 0$, which holds if housing is subsidized and the inefficiency of the output subsidy is not too extreme, by lemma 3. As discussed above, belief distortion have no effects (to second order) on the π_t^2

and x_t^2 terms, so they matter for welfare only through their consequences for the expectational error terms (the ones proportional to expressions (81) and (82)).

We can further simplify (82), writing it in terms of one-period-ahead forecast errors, as in (81). It follows from (21) that

$$\begin{aligned} q_t^u &= \xi_t^d + \beta(1-\delta)E_t\left[m_{t+1}q_{t+1}^u\right] \\ &= \xi_t^d + \beta(1-\delta)E_t\left[\widetilde{m}_{t+1}q_{t+1}^u\right] + \beta(1-\delta)E_t\left[q_{t+1}^u\right]; \end{aligned}$$

which can be iterated to yield

$$q_t^u = \bar{\xi}_t^d + \sum_{j=0}^{\infty} (\beta(1-\delta))^{j+1} E_t\left[\widetilde{m}_{t+j+1}q_{t+j+1}^u\right]:$$

Because (5) implies $E_t\widetilde{m}_{t+1} = 0$, the previous equation can alternatively be written as

$$q_t^u - \bar{\xi}_t^d = \sum_{j=0}^{\infty} (\beta(1-\delta))^{j+1} E_t\left(\widetilde{m}_{t+j+1}\widetilde{q}_{t+j+1}^u\right); \tag{83}$$

from which it is evident that the excess valuation $q_t^u - \bar{\xi}_t^d$ is of second order, as asserted above.

We can then use (83) to write

$$\begin{aligned} &E_0\sum_{t=0}^{\infty}\beta^t\left(q_t^u - \bar{\xi}_t^d\right) \\ &= \frac{1-\delta}{\delta}E_0\sum_{t=0}^{\infty}\beta^{t+1}\widetilde{m}_{t+1}\widetilde{q}_{t+1}^u - \frac{1-\delta}{\delta}E_0\sum_{t=0}^{\infty}(\beta(1-\delta))^{t+1}\left(\widetilde{m}_{t+1}\widetilde{q}_{t+1}^u\right) && (84) \\ &= \frac{1-\delta}{\delta}E_0\sum_{t=0}^{\infty}\beta^{t+1}\widetilde{m}_{t+1}\widetilde{q}_{t+1}^u - \frac{1-\delta}{\delta}\left(q_0^u - \bar{\xi}_t^d\right): && (85) \end{aligned}$$

To second order, we can also write[20]

$$\widetilde{m}_{t+1}\widetilde{q}_{t+1}^u = \underline{\bar{\xi}}^d\widehat{m}_{t+1}\widehat{q}_{t+1}^u + O(3); \tag{86}$$

and similarly

$$\widetilde{m}_{t+1}\pi_{t+1} = \widehat{m}_{t+1}\pi_{t+1} + O(3): \tag{87}$$

Substituting (85)-(87) in (80), we obtain the alternative approximate welfare criterion

$$\begin{aligned} E_0\sum_{t=0}^{\infty}\beta^t U(Y_t;\Delta_t;q_t^u;\xi_t) = -E_0\sum_{t=0}^{\infty}\beta^t&\left(\Lambda_\pi\pi_t^2 + \Lambda_x x_t^2 + \beta\%_\pi\widehat{m}_{t+1}\pi_{t+1} + \beta\%_q\widehat{m}_{t+1}\widehat{q}_t^u\right) \\ &- (1-\delta)\underline{\Psi}\left(q_0^u - \bar{\xi}_t^d\right) + \underline{\Gamma}_1\delta_1^\pi\pi_0 + \underline{\Gamma}_1\delta_3^\pi(\pi_0)^2 \\ &+ t{:}i{:}p{:} + O(3); \end{aligned} \tag{88}$$

[20]This follows from $\widetilde{m}_{t+1} = \widehat{m}_{t+1} + O(2)$ and $\widetilde{q}_{t+1}^u = \underline{\bar{\ }}^d\widehat{q}_{t+1}^u + O(3)$.

where

$$\%_\pi \equiv \underline{\Gamma}' \delta_1^\pi > 0$$
$$\%_q \equiv -(1-\delta)\underline{\Psi}\underline{\bar{\xi}}^d > 0;$$

and where the asserted signs for hold $\%_\pi$ and $\%_q$ hold under the conditions stated in assumption 1.

Finally, suppose that the central bank must choose a policy for dates $t \geq 0$ that will ensure a particular value for the quantity $\alpha\Gamma'_{-1}\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u$, as assumed in the upper bound problem (46) in section 4, where the values of the multipliers Γ_{-1} and Ψ_{-1} are chosen in a particular way (as functions of the history of shocks through period -1) so as to give the robustly optimal policy commitment a time-invariant form. We further suppose that $\Gamma_{-1} = (1+\zeta_{-1})\underline{\Gamma}$ for some scalar ζ_{-1} and that $\Psi_{-1} = \overline{\Psi} + \widetilde{\Psi}_{-1}$, where ζ_{-1} and $\widetilde{\Psi}_{-1}$ are of first order in the amplitude of the exogenous disturbances.[21] We show in Appendix A.7 that under these assumptions

$$\begin{aligned} \alpha\Gamma'_{-1}\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u = {} & -(1+\{_{-1})\underline{\Gamma}_1\delta_1^\pi\pi_0 + \underline{\Gamma}_1\left(\delta_2^\pi - \delta_3^\pi\right)\pi_0^2 \\ & + (1-\delta)\underline{\Psi}(q_0^u - \bar{\xi}_0^d) + t{:}i{:}p{:} + O(3){:} \end{aligned} \tag{89}$$

Since this expression must be independent of the policy decisions and of the belief distortions, it follows that we can write

$$\underline{\Gamma}_1\delta_1^\pi\pi_0 + \underline{\Gamma}_1\delta_3^\pi\pi_0^2 - (1-\delta)\underline{\Psi}(q_0^u - \bar{\xi}_0^d) = -\zeta_{-1}\%_\pi\pi_0 + \underline{\Gamma}_1\delta_2^\pi\pi_0^2 + t{:}i{:}p{:} + O(3){:}$$

Substituting this into (88), we find that maximization of (39) is equivalent to second order to minimization of

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} \beta^t \left(\Lambda_\pi\pi_t^2 + \Lambda_x x_t^2 + \beta\%_\pi\widehat{m}_{t+1}\pi_{t+1} + \beta\%_q\widehat{m}_{t+1}\widehat{q}_t^u\right) \\ & -\zeta_{-1}\%_\pi\pi_0 + \underline{\Gamma}_1\delta_2^\pi\pi_0^2 + t{:}i{:}p{:} + O(3){:} \end{aligned} \tag{90}$$

Moreover, (90) can be evaluated to second order on the basis of a first order solution for the dynamics of $\{\pi_t; x_t\}_{t=0}^\infty$, for any first order belief distortions $\{\widehat{m}_{t+1}\}_{t=0}^\infty$. Hence all that matters about the initial pre-commitment for the problem of constrained minimization of (90) is the constraint that it places on the first order approximate dynamics of inflation and the output gap. A first order approximation to (89) is simply

$$\alpha\Gamma'_{-1}\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u = -\underline{\Gamma}\delta_1^\pi\pi_0 + t{:}i{:}p{:} + O(2);$$

[21] One can show that the multipliers required in order for these initial precommitmenst to lead to the selection of a robustly optimal policy commitment at some date to continue the robustly optimal commitment solution chosen at an earlier date will satisfy these properties. Note that in the solution to the upper bound problem stated in section 4, $\Gamma_t = (1+\zeta_t)\underline{\Gamma}$ for all t, and $\zeta = O(1); \Psi_t = O(1)$ for all t.

so that to this order of approximation, the initial pre-commitment can be written in the form

$$\pi_0 = \overline{\pi}_0; \tag{91}$$

where $\overline{\pi}_0$ may depend on the history of shocks through period -1, but is independent of the policy chosen for periods $t \geq 0$ and of the belief distortions.

Given constraint (91), the π_0 and $(\pi_0)^2$ terms in (90) can also be subsumed among the "$t{:}i{:}p{:}$" terms. Hence maximization of (39) is equivalent (to second order) to minimization of a quadratic loss function

$$\Lambda = E_0 \sum_{t=0}^{\infty} \beta^t \left(\Lambda_\pi \pi_t^2 + \Lambda_x x_t^2 + \beta \%_\pi \widehat{m}_{t+1} \pi_{t+1} + \beta \%_q \widehat{m}_{t+1} \widehat{q}_t^u \right) : \tag{92}$$

This approximation to the central bank's objective function makes it clear in what sense policy should seek to reduce the extent to which over-estimates of either future inflation of future housing prices can occur as a result of "near-rational" belief distortions.

We can similarly approximate the penalty function (7) for belief distortions by

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1} m_{t+1} \log m_{t+1} = \underbrace{\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^{t+1} \widehat{m}_{t+1}^2}_{\equiv V(\widehat{m})} + O(3) \tag{93}$$

A linear quadratic approximation to the robustly optimal policy problem is then given by the problem

$$\min_{\widehat{c} \in \widehat{C}} \max_{\widehat{m} \in \widehat{M}} \Lambda(\widehat{c}; \widehat{m}) + \theta V(\widehat{m}) \tag{94}$$

where $\widehat{M}$ is the set of processes $\{\widehat{m}_{t+1}\}_{t=0}^{\infty}$ such that $E_t \widehat{m}_{t+1} = 0$ at all times and $V(\widehat{m})$ is finite; $\widehat{C}$ is the class of linear policy rules $\widehat{c}$ such that the log-linearized Phillips curves (61) have a solution $\{\pi_t; x_t\}_{t=0}^{\infty}$; and $\Lambda(\widehat{c}; \widehat{m})$ is the value of the loss function (92) evaluated for the processes $\{\pi_t; x_t\}_{t=0}^{\infty}$ determined by the policy $\widehat{c}$ and the belief distortions $\widehat{m}$. The linear policy $\widehat{c}$ solving (94) provides a linear approximation to the robustly optimal policy for the exact, nonlinear model.

The linear quadratic approximate problem (94) is simpler than the exact problem (3) in a number of respects. Notably, a first-order accurate approximation to the paths $\{\pi_t; x_t\}_{t=0}^{\infty}$ under any policy suffices in order to evaluate (92) to second order, under any belief distortion process $\{\widehat{m}_{t+1}\}_{t=0}^{\infty}$ of first order. This means that we can - without loss of generality - restrict attention to linear policy rules $\widehat{c}$, and approximate the equilibrium dynamics implied by any policy commitment by the solution to the linearized structural equations (61). Hence the equilibrium dynamics $\{\pi_t; x_t\}_{t=0}^{\infty}$ associated with a given policy are, to this order of approximation, independent of the

E_t where $\hat{m}_{t1}$

belief distortions $\hat{m}$. This makes it possible to replace the outer problem in (94) by a direct choice of the equilibrium paths $\{\pi_t; x_t\}_{t=0}^{\infty}$. We can thus alternatively consider the problem

$$\min_{\hat{x}\in\hat{X}} \max_{\hat{m}\in\widehat{M}} L(\hat{x}; \hat{m}) - \theta V(\hat{m}); \tag{95}$$

where $\hat{X}$ is the set of processes $\hat{x} = \{\pi_t; x_t\}_{t=0}^{\infty}$ (specified as linear functions of the history of exogenous disturbances) such that

$$E_0 \sum_{t=0}^{\infty} \beta^t \pi_t^2 < \infty \text{ and } E_0 \sum_{t=0}^{\infty} \beta^t x_t^2 < \infty;$$

and that are consistent with the structural relations (61) and the initial pre-commitment (91); and $L(\hat{x}; \hat{m})$ is the value of (92) for the processes $\hat{x}$ and $\hat{m}$.

An advantage of this approximate problem is that it is possible to directly solve the inner problem for arbitrary outcome processes $\{\pi_t; x_t\}_{t=0}^{\infty}$, and using this result to directly solve for the equilibrium outcomes $\hat{x}$ under a robustly optimal policy commitment, rather than having to characterize upper bound dynamics (as in sections 4 and 5

t g^{∞}

$t=0$.

as our characterization in proposition 1 of the local approximation to the upper bound dynamics. In terms of the coefficients of the loss function (92), we observe that the coefficients in (62) are given up to an arbitrary multiplicative factor by

$$\xi_\pi = \Lambda_\pi > 0, \ \lambda = \frac{\Lambda_x}{\kappa} > 0; \xi_m = \frac{\%_\pi}{2} > 0:$$

Alternatively, we can obtain an objective function for the central bank by explicitly solving for the maximized value of (92), substituting the worst-case belief distortions given by (63) for the $\widehat{m}_{t+1}$ terms. This maximized value is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\Lambda_\pi \pi_t^2 + \Lambda_x x_t^2 + \beta \theta^{-1} E_t \left[(w_{t+1} - E_t(w_t))^2 \right] \right); \tag{97}$$

where

$$w_{t+1} = \%_\pi \pi_{t+1} + \%_q \widehat{q}^u_{t+1}.$$

The outer problem in (95) can then be equivalently stated as the choice of processes $\{\pi_t; x_t\}_{t=0}^{\infty}$ consistent with the structural relations (61) and the initial precommitment (91), so as to minimize the quadratic loss function (97), given the process $\{\widehat{q}^u_t\}_{t=0}^{\infty}$ that (to first order) is determined by exogenous housing demand disturbances.

A concern for robustness to possible belief distortions, i.e., a finite value for θ, therefore requires the central bank to seek to minimize the variance of surprise variations in the composite variable w_{t+1}, in addition to its usual stabilization objectives (minimization of the squared inflation and output gap terms). The reason is that greater surprise variability in this variable makes it easier for the private sector to maker larger errors in its estimated of the conditional mean of this variable; an a higher average value of $\widehat{E}_t w_{t+1}$ is the type of expectations error that reduces welfare, as can be seen from (92).

The variance of surprise variations in w_{t+1} depends not only on the variance of inflation surprises, but also on the correlation of inflation surprises with housing price surprises. Reductions of the distortions associated with worst-case belief distortions requires that the covariance of inflation surprises with housing price surprises be reduced (or even be negative); hence the robustly optimal policy involves "leaning against" housing price surprises, as concluded in section 6.

8 Conclusion

Monetary policymakers concerned about whether private-sector expectations will necessarily coincide with those implied by their own model, that they use to understand the economy and choose their policy commitment, may find it desirable to include housing prices in the set of variables that they must track in order to verify that policy is on course, alongside the traditional "target variables" of inflation and a suitably

defined measure of the output gap. This can be the case even under circumstances where an optimal policy commitment could be formulated purely in terms of a desired relationship between the paths of inflation and the output gap, if one could be confident that one's policy would result in a rational-expectations equilibrium. We have illustrated this in the context of a standard New Keynesian model extended to include a housing sector, where we find that robustly optimal policy can be characterized by a linear "target criterion," but this must involve housing price surprises in addition to the paths of inflation and the output gap. In the presence of a housing subsidy, this requires monetary policy to be tighter (less tight) following unexpected increases (decreases) in the housing price than in the case in which the policymaker can rely on the private sector to have the same expectations as herself.

Of course, our analysis does not pretend to provide a complete analysis of the problem of the desirable policy response to housing booms and busts. In our simple model, mis-pricing of housing due to expectational errors matters for welfare only because of its consequences for the degree to which productive resources are drawn into the housing sector; hence the dependence of our results on the degree to which there is already an inefficient over-supply of housing in the steady state, owing to housing subsidies. We believe that this is one reason why housing booms are harmful, but it probably is not the only one. Central banks' concern to "lean against" housing booms is often based on the fear that both household and bank balance sheets may be impaired in the event of a subsequent collapse of housing prices, as a result of the increased household borrowing often observed during a housing boom. Our model does not address this issue, as for simplicity we abstract both from household borrowing and from the existence of banks. The exercise must therefore be viewed more as an illustration of our proposed approach than as a complete treatment of a policy issue. It should, however, suffice to indicate that conclusions about the need to include asset prices among the target variables based on a rational-expectations analysis need not be robust to an allowance for even modest departures from rational expectations.

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A Appendix

A.1 Steady State Results

This appendix proves a number of claims made in section 5.1. Using (31) and (32) on can write (55) more explicitly as

$$\lambda\phi\frac{\bar{H}^{-\nu}}{\underline{A}^{1+\omega}}\underline{Y}^{\omega} = \frac{\eta-1}{\eta}(1-\underline{\tau})\bar{C}^{\tilde{\sigma}^{-1}}C\left(\underline{Y};\underline{q}^{u};\underline{\xi}\right)^{-\tilde{\sigma}^{-1}}: \tag{98}$$

Since the left-hand side is increasing and the right-hand side decreasing in $\underline{Y}$ (as $C_Y > 0$), there is a unique value for $\underline{Y}$ solving this equation, as claimed.

Using the definitions of k and f and (98), we have

$$\begin{aligned}
f_Y &= (1-\tau)\bar{C}^{\tilde{\sigma}^{-1}}C(Y;q^u;\xi)^{-\tilde{\sigma}^{-1}} \\
&\quad - \tilde{\sigma}^{-1}(1-\tau)\bar{C}^{\tilde{\sigma}^{-1}}Y\ C(Y;q^u;\xi)^{-\tilde{\sigma}^{-1}-1}C_Y(Y;q^u;\xi) \\
k_Y &= \frac{\eta}{\eta-1}\lambda\phi\frac{\bar{H}^{-\nu}}{A^{1+\omega}}(1+\,!\,)Y^{\omega} \\
&= (1+\,!\,)(1-\underline{\tau})\bar{C}^{\tilde{\sigma}^{-1}}\underline{C}\left(\underline{Y};\underline{q}^{u};\underline{\xi}\right)^{-\tilde{\sigma}^{-1}};
\end{aligned}$$

so that from $C_Y > 0$ and $! > 0$ we get

$$\begin{aligned}
k_Y - f_Y &= (1-\underline{\tau})\bar{C}^{\tilde{\sigma}^{-1}}\underline{C}\left(\underline{Y};\underline{q}^{u};\underline{\xi}\right)^{-\tilde{\sigma}^{-1}} \\
&\quad \cdot\left(\,!\,+\tilde{\sigma}^{-1}Y\,C(Y;q^u;\xi)^{-1}C_Y(Y;q^u;\xi)\right) \\
&\quad > 0:
\end{aligned} \tag{99}$$

From (40) we get

$$\begin{aligned}
U_Y(Y_t;\Delta_t;q_t^u;\xi_t) &= \overline{C}_t^{\tilde{\sigma}^{-1}}C(Y_t;q_t^u;\xi_t)^{-\tilde{\sigma}^{-1}}C_Y(Y_t;q_t^u;\xi_t) \\
&\quad - \frac{\lambda}{1+\nu}(1+\,!\,)\frac{\bar{H}_t^{-\nu}}{A_t^{1+\omega}}Y_t^{\omega}\Delta_t \\
&\quad + \frac{A_t^d\bar{\xi}_t^d}{\tilde{\alpha}}\Omega(q_t^u;\xi_t)^{\tilde{\alpha}}\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}\right)C(Y_t;q_t^u;\xi_t)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1}C_Y(Y_t;q_t^u;\xi_t):
\end{aligned} \tag{100}$$

Using (98), $1+\,!\,=\phi(1+\nu)$ and evaluating at the steady state we have

$$\begin{aligned}
U_Y(\underline{Y};1;\underline{q}^u;\underline{\xi}) &= \overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}C_Y(\underline{Y};\underline{q}^u;\underline{\xi}) \\
&\quad - \frac{\eta-1}{\eta}(1-\underline{\tau})\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}} \\
&\quad + \frac{A^d\bar{\xi}^d}{\tilde{\alpha}}\underline{\Omega}^{\tilde{\alpha}}\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}\right)C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1}C_Y(\underline{Y};\underline{q}^u;\underline{\xi}):
\end{aligned}$$

Using the fact that at the steady state

$$C_Y\left(\underline{Y};\underline{q}^u;\underline{\xi}\right)=\frac{1-g}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}} \tag{101}$$

and (24) we have

$$\begin{aligned}
U_Y(\underline{Y};1;\underline{q}^u;\underline{\xi}) =&\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\\
&-\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}\frac{\eta-1}{\eta}(1-\underline{\tau})\\
&+\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\underline{\Omega}\frac{1}{1+s^d}\left(\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\right)C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}\\
=&\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}\\
&\cdot\left(-\frac{\eta-1}{\eta}(1-\underline{\tau})+C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\frac{1+s^d+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}}{1+s^d}\right)\\
=&\overline{C}^{\tilde{\sigma}^{-1}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}}\left(\frac{1-g}{1+s^d}-\frac{\eta-1}{\eta}(1-\underline{\tau})+\frac{s^d}{1+s^d}C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\right):
\end{aligned} \tag{102}$$

The efficient output subsidy $1-\underline{\tau}^{eff}(s^d)$ is the one giving rise to $U_Y(\underline{Y};1;\underline{q}^u;\underline{\xi})=0$ and is implicitly defined as

$$\begin{aligned}
1-\underline{\tau}^{eff}(s^d)&=\frac{\eta}{\eta-1}\left(\frac{1-g+s^dC_Y(\underline{Y};\underline{q}^u;\underline{\xi})}{1+s^d}\right)\\
&=\frac{\eta}{\eta-1}(1-g)\left(\frac{1+\frac{s^d}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y},\underline{q}^u,\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}}}{1+s^d}\right):
\end{aligned}$$

From $\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}>0$ follows that $\partial\left(1-\underline{\tau}^{eff}(s^d)\right)/\partial s^d<0$. Using (101) we can express the terms in the last parenthesis in (102), which determine the sign of U_Y whenever τ deviates from $\underline{\tau}^{eff}(s^d)$, as

$$\begin{aligned}
&\frac{1-g}{1+s^d}-\frac{\eta-1}{\eta}(1-\underline{\tau})+\frac{s^d}{1+s^d}C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\\
&=\frac{1-g}{1+s^d}-\frac{\eta-1}{\eta}(1-\underline{\tau})+\frac{s^d}{1+s^d}\frac{1-g}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}}:
\end{aligned} \tag{103}$$

The derivative of the r.h.s of (103) w.r.t. $(1-\underline{\tau})$ is given by

$$-\frac{\eta-1}{\eta}-\frac{s^d}{1+s^d}\frac{1-g}{\left(1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}\right)^2}C_Y(\underline{Y};\underline{q}^u;\underline{\xi})\frac{\partial\underline{Y}}{\partial(1-\underline{\tau})}$$

and is strictly negative because $C_Y > 0$ and because (98) implies $\frac{\partial \underline{Y}}{\partial(1-\underline{\tau})} > 0$. Since $U_Y = 0$ for $\underline{\tau} = \underline{\tau}^{eff}(s^d)$ this shows that $U_Y < 0$ whenever $1 - \underline{\tau} > 1 - \underline{\tau}^{eff}(s^d)$ and $U_Y > 0$ whenever $1 - \underline{\tau} < 1 - \underline{\tau}(s^d)$, as claimed in lemma 2.

A.2 Proof of Lemma 1

We establish the claim for the case of an efficient output subsidy. By continuity it also holds for values of $1 - \underline{\tau}$ sufficiently close to $1 - \underline{\tau}^{eff}(s^d)$. We have

$$\begin{aligned} U_Y(Y_t; \Delta_t; q_t^u; \xi_t) = {} & \overline{C}_t^{\tilde{\sigma}^{-1}} C(Y_t; q_t^u; \xi_t)^{-\tilde{\sigma}^{-1}} C_Y(Y_t; q_t^u; \xi_t) \\ & - \frac{\lambda}{1+\nu} (1 + \omega) \frac{\bar{H}_t^{-\nu}}{A_t^{1+\omega}} Y_t^{\omega} \Delta_t \\ & + \frac{A_t^d \bar{\xi}_t^d}{\tilde{\alpha}} \Omega(q_t^u; \xi_t)^{\tilde{\alpha}} \left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}} \tilde{\sigma}^{-1} \right) C(Y_t; q_t^u; \xi_t)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1} C_Y(Y_t; q_t^u; \xi_t). \end{aligned} \tag{104}$$

Differentiating once more w.r.t. Y_t and evaluating at the steady state, one obtains

$$\begin{aligned} U_{YY} = {} & -\tilde{\sigma}^{-1} \underline{\overline{C}}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}-1} C_Y C_Y \\ & + \underline{\overline{C}}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} C_{YY} \\ & - \frac{\lambda}{1+\nu} (1 + \omega)\, \omega \frac{\underline{\bar{H}}^{-\nu}}{\underline{A}^{1+\omega}} \underline{Y}^{\omega-1} \\ & + \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} (1 + \chi) \left(\chi - \tilde{\sigma}^{-1} \right) \underline{C}^{\chi - \tilde{\sigma}^{-1}-1} C_Y C_Y \\ & + \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} (1 + \chi) \underline{C}^{\chi - \tilde{\sigma}^{-1}} C_{YY}, \end{aligned}$$

where $\chi = \frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}} - 1$. Using $\underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} = \frac{\underline{\Omega}}{1+s^d} \underline{\overline{C}}^{\tilde{\sigma}^{-1}}$ we have

$$\begin{aligned} U_{YY} = {} & -\frac{\lambda}{1+\nu} (1 + \omega)\, \omega \frac{\underline{\bar{H}}^{-\nu}}{\underline{A}^{1+\omega}} \underline{Y}^{\omega-1} \\ & + \underline{\overline{C}}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} \begin{pmatrix} -\tilde{\sigma}^{-1} \underline{C}^{-1} C_Y C_Y \\ + C_{YY} \\ + \frac{1}{1+s^d} (1 + \chi) \left(\chi - \tilde{\sigma}^{-1} \right) \underline{\Omega} \underline{C}^{\chi-1} C_Y C_Y \\ + \frac{1}{1+s^d} (1 + \chi) \underline{\Omega} \underline{C}^{\chi} C_{YY} \end{pmatrix}. \end{aligned}$$

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Using also

$$C_Y = \frac{1-\underline{g}}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}} = \frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} > 0$$

1

$$C_{YY} = -\frac{1-\underline{g}}{\Big(1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}C}$$

)

$1+ \chi)\, C$

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A.3 Proof of Lemma 3

From (40) we obtain using (24)

$$\begin{aligned}
U_q(\underline{Y};\underline{q}^u;\underline{\xi}) &= \overline{C}^{\tilde{\sigma}^{-1}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}} C_q(\underline{Y};\underline{q}^u;\underline{\xi}) \\
&+ \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} \; \frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1} C_q(\underline{Y};\underline{q}^u;\underline{\xi}) \\
&+ \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}-1} \; C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}} \left(\frac{1}{\underline{q}^u} \frac{\Omega}{1-\tilde{\alpha}} \right) \\
&= \overline{C}^{\tilde{\sigma}^{-1}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{-\tilde{\sigma}^{-1}} C_q(\underline{Y};\underline{q}^u;\underline{\xi}) \left(1 + \; \frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}} \underline{\Omega} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1} \frac{1}{1+s^d} \right) \\
&+ \frac{\underline{A}^d}{1-\tilde{\alpha}} \underline{\Omega}^{\tilde{\alpha}} \; C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}}
\end{aligned}$$

Implicitly differentiating (23) we obtain

$$C_q(\underline{Y};\underline{q}^u;\underline{\xi}) = \frac{-\frac{\partial\Omega(\underline{q}^u,\underline{\xi})}{\partial q} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}}}{1+\chi^*} \tag{107}$$

where $\chi^* = \frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}} \underline{\Omega} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1} > 0$ and $\frac{\partial\Omega(\underline{q}^u,\underline{\xi})}{\partial q} = \frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}}\underline{\Omega}$. Using this we can write

$$\begin{aligned}
U_q(\underline{Y};\underline{q}^u;\underline{\xi}) &= -\overline{C}^{\tilde{\sigma}^{-1}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\alpha}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}} \frac{1+\frac{\chi^*}{1+s^d}}{1+\chi^*} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega} \\
&+ \frac{\underline{A}^d}{1-\tilde{\alpha}} \underline{\Omega}^{\tilde{\alpha}} \; C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}} \\
&= -\overline{C}^{\tilde{\sigma}^{-1}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\alpha}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega} \left(\frac{1+\frac{\chi^*}{1+s^d}}{1+\chi^*} - \frac{1}{1+s^d} \right) \underline{\Omega} \\
&= -\overline{C}^{\tilde{\sigma}^{-1}} C(\underline{Y};\underline{q}^u;\underline{\xi})^{\frac{\alpha}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega} \left(\frac{s^d}{(1+\chi^*)(1+s^d)} \right) \underline{\Omega}
\end{aligned}$$

proving that $U_q < 0$ for $s^d > 0$, $U_q > 0$ for $s^d < 0$, and $U_q = 0$ for $s^d = 0$. For $\underline{\tau}$ sufficiently close to $\underline{\tau}^{eff}(s^d)$ we furthermore have from lemma 2 that U_Y is sufficiently close to zero, so that (57) implies that $\underline{\Gamma}_1$ is also sufficiently to zero, so that from (59) it follows that $\underline{\Psi}$ has the same sign as U_q, whenever $\underline{\tau} \neq \underline{\tau}^{eff}$. Furthermore, for $\underline{\tau} = \underline{\tau}^{eff}$ we have $U_Y = \underline{\Gamma}_1 = 0$ and also $U_q = 0$, so that from (59) we obtain $\underline{\Psi} = 0$.

A.4 Proof of Proposition 1

With initial price dispersion satisfying $\Delta_{-1} = 0 + O(2)$ it follows from equation (45) that $\Delta_{-1} = 0 + O(2)$ for all $t \geq 0$. Furthermore, equations (5) and (21) jointly imply

$$\widehat{q}_t^u = \widehat{\bar{\xi}}_t^d + O(2); \tag{108}$$

so that $\widehat{q}^u_t$ is determined to first order purely by exogenous disturbances.

From $1 - \underline{\tau} < 1 - \underline{\tau}^{eff}(s^d)$ and lemma 2 follows that

$$\underline{\Gamma}_1 > 0:$$

Furthermore, we have from lemma 3

$$\begin{aligned} \underline{\Psi} &< 0 \quad \text{if } s^d > 0 \\ \underline{\Psi} &> 0 \quad \text{if } s^d < 0: \end{aligned}$$

Using the fact that $\widehat{\Delta}_t$ and $\widehat{q}^u_t - \widehat{\bar{\xi}}^d_t$ are all zero to first order accuracy, a linearization of (48) delivers

$$\underline{Y}U_{YY}\widehat{Y}_t + U_{Yq}\tilde{\bar{\xi}}_t + U_{Y\xi}\tilde{\xi}_t + \underline{Y\,\Gamma}'z_{YY}\widehat{Y}_t + \underline{\Gamma}'z_{Yq}\tilde{\bar{\xi}}_t + \underline{\Gamma}'z_{Y\xi}\tilde{\xi}_t + (f_Y - k_Y)\,\tilde{\Gamma}_{1t} = 0 \quad (109)$$

where $\tilde{\bar{\xi}}_t = \bar{\xi}_t - \underline{\bar{\xi}}$ and $\tilde{\xi}_t = \xi_t - \underline{\xi}$. Using a first order approximation to (60) one can rewrite (109) as

$$\tilde{\Gamma}_{1t} = -\lambda_x x_t \quad (110)$$

where

$$\lambda_x = \frac{\underline{Y}U_{YY} + \underline{Y}\;\underline{\Gamma}'z_{YY}}{f_Y - k_Y} > 0$$

The last inequality follows from $f_Y - k_Y < 0$, $U_{YY} < 0$ and the fact that for $\underline{\tau}$ sufficiently close to $\underline{\tau}^{eff}(s^d)$ we have that $\underline{\Gamma}'$ is approximately zero.

From the proof of proposition 2 in Adam and Woodford (2012) follows that log-linearization of (49)-(50) delivers

$$b(\widehat{K}_t - \widehat{F}_t) - \tilde{\Gamma}_{1t} + \tilde{\Gamma}_{1,t-1} + \underline{\Gamma}_1\widehat{m}_t = 0 \quad (111)$$

where

$$b = -\left(\frac{\underline{Y}}{\underline{K}}\frac{1-\alpha}{\alpha}\frac{\eta(1+!)}{1+!\eta} + \alpha c\right) > 0$$

and

$$c = \underline{\Gamma}_1\frac{\underline{F}}{\underline{K}}\left(-\frac{(1-\alpha)}{\alpha}\frac{\eta(1+!)}{1+!\eta} - \left(\frac{(1-\alpha)}{\alpha}\right)^2\frac{\eta(1+!)}{1+!\eta}\right) < 0$$

Log-linearization of (36) delivers

$$\pi_t = \frac{1-\alpha}{\alpha}\frac{1}{1+!\eta}(\widehat{K}_t - \widehat{F}_t): \quad (112)$$

Using the previous equation and (110) to substitute for $\widehat{K} - \widehat{F}$ and $\widehat{\Gamma}$ in (111) one obtains the targeting rule (62), where

$$\begin{aligned} \xi_\pi &= b\frac{\alpha}{1-\alpha}(1+!\eta) > 0 \\ \xi_m &= \underline{\Gamma}_1 > 0 \end{aligned}$$

Log-linearization of the constraints (42) and (43) together with (112) delivers (61) where the 'cost-push' disturbance is defined as

$$u_t = \kappa_\xi \widetilde{\xi}_t + \kappa_Y \widehat{Y}_t^* \tag{113}$$

and

$$\kappa_Y = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)} \frac{\underline{Y}}{\underline{k}} (k_Y - f_Y) > 0 \tag{114}$$

$$\kappa_\xi = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)} \frac{1}{k} (k'_\xi - f'_\xi) \tag{115}$$

The linearization of (52) delivers

$$\theta \widehat{m}_{t+1} + \alpha\Phi(\underline{Z})\widetilde{\Gamma}_t + \alpha\underline{K}\ \underline{\Gamma}' D(1)\widehat{Z}_{t+1} + (1-\delta)\underline{\Psi}\underline{q}^u\widehat{q}^u_{t+1} + (1-\delta)\underline{q}^u\widetilde{\Psi}_t + \widetilde{\ }_t = 0:$$

Applying the expectations operator E_t to the previous equation, subtracting the result from it, and using $\alpha\bar{\Gamma}' D(1) = \bar{\Gamma}'$ yields

$$\theta \widehat{m}_{t+1} = \underline{K}\ \underline{\Gamma}_1 \left(\widehat{K}_{t+1} - \widehat{F}_{t+1} - E_t \left(\widehat{K}_{t+1} - \widehat{F}_{t+1} \right) \right) - (1-\delta)\underline{\Psi}\underline{q}^u \left(\widehat{q}^u_{t+1} - E_t \widehat{q}^u_{t+1} \right):$$

Using (112) and shifting back by one period, we get (63) with

$$\lambda_m = \frac{\underline{K}\ \underline{\Gamma}_1}{\theta} \frac{\alpha}{1-\alpha} (1+!\eta) > 0 \tag{116}$$

$$\lambda_q = -\frac{(1-\delta)}{\theta} \underline{\Psi}\underline{q}^u \tag{117}$$

The sign of λ_q depends on the sign of $\underline{\Psi}$, which can be determined from lemma 3.

A.5 Proof of Proposition 2

From (113) we have

$$\begin{aligned} u_t &= \kappa_\xi \widetilde{\xi}_t + \kappa_Y \widehat{Y}_t^* \\ &= \kappa_{A^d} \underline{A}^d \widehat{A}_t^d + \kappa_{\bar{\xi}^d} \underline{\bar{\xi}}^d \widehat{\bar{\xi}}_t^d + \kappa_Y \widehat{Y}_t^* + n{:}h{:}s{:}; \end{aligned} \tag{118}$$

where $n{:}h{:}s{:}$ denotes the effects of non-housing shocks. From (115) and noting that the function $k(\cdot)$ is independent of the housing sector shocks A_t^d and $\bar{\xi}_t^d$ and that the function $f(\cdot)$ depends only via $q_t^u = \widehat{\bar{\xi}}_t^d + O(2)$ on $\widehat{\bar{\xi}}_t^d$, we get

$$\begin{aligned} \kappa_{A^d} \underline{A}^d \widehat{A}_t^d + \kappa_{\bar{\xi}^d} \underline{\bar{\xi}}^d \widehat{\bar{\xi}}_t^d = &- \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)} \frac{1}{k} f_{A^d} \underline{A}^d \widehat{A}_t^d \\ &- \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)} \frac{1}{k} f_q \underline{\xi}^d \widehat{\xi}_t^d \end{aligned}$$

Using (31) we get

$$f_{A^d} = -\widetilde{\sigma}^{-1}(1-\underline{\tau})\underline{\overline{C}}^{\widetilde{\sigma}^{-1}}\underline{Y}\ \underline{C}^{-\widetilde{\sigma}^{-1}-1}C_{A^d} > 0; \qquad (119)$$

where C_{A^d} denotes the derivative of the function $C(Y; q^u; \xi)$ w.r.t. A^d, evaluated at the steady state, and where (23) implies $C_{A^d} < 0$. We thus have

$$\kappa_{A^d}\underline{A}^d\widehat{A}^d_t + \kappa_{\bar{\xi}^d}\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t = -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}f_{A^d}\underline{A}^d\widehat{A}^d_t$$

Next, we determine the effects of A^d_t and $\bar{\xi}^d_t$ on Y^*_t. From (60) we get

$$U_Y(Y^*_t; 1; \bar{\xi}^d_t; \xi_t) + \underline{\Gamma}_1\left(f_Y(Y^*_t; \bar{\xi}^d_t; \xi_t) - k_Y(Y^*_t; \xi_t)\right) = 0$$

The linear approximation to this w.r.t. Y^*_t and all housing sector shocks delivers

$$\begin{aligned} &U_{YY}\underline{Y}\widehat{Y}^*_t + U_{Yq}\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t + U_{YA^d}\underline{A}^d\widehat{A}^d_t + U_{Y\bar{\xi}^d}\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t \\ +&\underline{\Gamma}_1(f_{YY}\underline{Y}\widehat{Y}^*_t + f_{Yq}\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t + f_{YA^d}\underline{A}^d\widehat{A}^d_t - k_{YY}\underline{Y}\widehat{Y}^*_t) = 0 + n{:}h{:}s{:}; \end{aligned} \qquad (120)$$

where we used the fact that $k(\cdot)$ is independent of A^d and $\bar{\xi}^d$, and subsumed the effects of non-housing shocks into $n{:}h{:}s{:}$ For the **special case with an efficient output subsidy**, lemma 2 shows that $\underline{\Gamma}_1 = 0$, so that then

$$\widehat{Y}^*_t = -\frac{1}{U_{YY}\underline{Y}}\left(\left(U_{Yq} + U_{Y\bar{\xi}^d}\right)\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t + U_{YA^d}\underline{A}^d\widehat{A}^d_t\right) + n{:}h{:}s{:};$$

so that

$$\begin{aligned} u_t =& -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}f_{A^d}\underline{A}^d\widehat{A}^d_t - \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}f_q\underline{\xi}^d\widehat{\xi}^d_t \\ &-\frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}\frac{(k_Y - f_Y)}{U_{YY}}\left(\left(U_{Yq} + U_{Y\bar{\xi}^d}\right)\underline{\bar{\xi}}^d\widehat{\bar{\xi}}^d_t + U_{YA^d}\underline{A}^d\widehat{A}^d_t\right) \\ =&\kappa^{\bar{\xi}^d}\widehat{\bar{\xi}}^d_t + \kappa^{A^d}\widehat{A}^d_t; \end{aligned}$$

where

$$\begin{aligned} \kappa^{\bar{\xi}^d} &= -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}\underline{\bar{\xi}}^d\left(f_q + \frac{(k_Y - f_Y)}{U_{YY}}(U_{Yq} + U_{Y\bar{\xi}^d})\right) \\ \kappa^{A^d} &= -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+!\eta)}\frac{1}{k}\underline{A}^d\left(f_{A^d} + \frac{(k_Y - f_Y)}{U_{YY}}U_{YA^d}\right) \end{aligned}$$

Due to the way in which $q_t^u; \bar{\xi}_t^d$ and A^d enter the functions $f(\cdot)$ and $U(\cdot)$, we have that $\underline{\bar{\xi}}^d f_q = \underline{A}^d f_{A^d}$ and $\underline{\bar{\xi}}^d (U_{Yq} + U_{Y\bar{\xi}^d}) = \underline{A}^d U_{YA^d}$, so that

$$\kappa_u \equiv \kappa^{\bar{\xi}^d} = \kappa^{A^d}.$$

We now derive an explicit expression for $\kappa^{\bar{\xi}^d}$, so as to determine its sign. From (99) we have

$$k_Y - f_Y = (1 - \underline{\tau}) \overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} \left(\omega + \tilde{\sigma}^{-1} \underline{Y} \frac{C_Y}{\underline{C}} \right) > 0. \tag{121}$$

Furthermore, lemma 1 insures $U_{YY} < 0$, as long as s^d is not too negative. From (100) we get

$$\begin{aligned} U_Y(Y_t; \Delta_t; q_t^u; \xi_t) &= \overline{C}_t^{\tilde{\sigma}^{-1}} C(Y_t; q_t^u; \xi_t)^{-\tilde{\sigma}^{-1}} C_Y(Y_t; q_t^u; \xi_t) \\ &\quad - \frac{\lambda}{1 + \nu} (1 + \omega) \frac{\bar{H}_t^{-\nu}}{A_t^{1+\omega}} Y_t^{\omega} \Delta_t \\ &\quad + A_t^d \bar{\xi}_t^d \Omega(q_t^u; \xi_t)^{\tilde{\alpha}} (1 + \chi) C(Y_t; q_t^u; \xi_t)^{\chi - \tilde{\sigma}^{-1}} C_Y(Y_t; q_t^u; \xi_t). \end{aligned} \tag{122}$$

where $\chi = \tilde{\sigma}^{-1}/(1 - \tilde{\alpha}) - 1$. Taking derivatives w.r.t. q_t^u and $\bar{\xi}_t^d$ and evaluating at the steady state delivers

$$\begin{aligned} U_{Yq} &= -\tilde{\sigma}^{-1} \overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}-1} C_q C_Y + \overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} C_{Yq} \\ &\quad + \underline{A}^d \underline{\bar{\xi}}^d \tilde{\alpha} \underline{\Omega}^{\tilde{\alpha}-1} \frac{\partial \Omega}{\partial q} (1 + \chi) \underline{C}^{\chi - \tilde{\sigma}^{-1}} C_Y \\ &\quad + \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} (1 + \chi) \left(\chi - \tilde{\sigma}^{-1} \right) \underline{C}^{\chi - \tilde{\sigma}^{-1} - 1} C_q C_Y \\ &\quad + \underline{A}^d \underline{\bar{\xi}}^d \underline{\Omega}^{\tilde{\alpha}} (1 + \chi) \underline{C}^{\chi - \tilde{\sigma}^{-1}} C_{Yq} \\ U_{Y\bar{\xi}^d} &= \underline{A}^d \underline{\Omega}^{\tilde{\alpha}} (1 + \chi) \underline{C}^{\chi - \tilde{\sigma}^{-1}} C_Y \end{aligned}$$

where

$$\frac{\partial \Omega}{\partial q} = \frac{1}{\underline{q}^u} \frac{\underline{\Omega}}{1 - \tilde{\alpha}}.$$

From (107) we get

$$C_q(\underline{Y}; \underline{q}^u; \underline{\xi}) = \frac{-\frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega C}^{\chi+1}}{1 + (\chi + 1) \underline{\Omega C}^{\chi}}; \tag{123}$$

and from (101)

$$
\begin{aligned}
C_Y &= \frac{1-\underline{g}}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}} = \frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} > 0 \\
C_{A^d} &= -\frac{\frac{1}{1-\tilde{\alpha}}\frac{1}{\underline{A}^d}\underline{\Omega C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}}}{1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}} = -\frac{\frac{1}{1-\tilde{\alpha}}\frac{1}{\underline{A}^d}\underline{\Omega C}^{1+\chi}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} < 0 \qquad (124) \\
C_{YY} &= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}\right)^2}\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\left(\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1\right)\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-2}C_Y \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi-1}C_Y \\
C_{Yq} &= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\left(\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi-1}C_q + \left(1+\chi\right)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{q}^u}\right) \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\left(\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi-1}\frac{-\frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}}\underline{\Omega C}^{\chi+1}}{1+\left(\chi+1\right)\underline{\Omega C}^{\chi}} + \left(1+\chi\right)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{q}^u}\right) \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{q}^u} \\
C_{YA^d} &= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}\right)^2}\left(\underline{\Omega}\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\left(\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1\right)\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-2}C_{A^d} + \frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}\underline{C}^{\frac{\tilde{\sigma}^{-1}}{1-\tilde{\alpha}}-1}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{A}^d}\right) \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\left(\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi-1}C_{A^d} + (1+\chi)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{A}^d}\right) \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\left(-\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi-1}\frac{\frac{1}{1-\tilde{\alpha}}\frac{1}{\underline{A}^d}\underline{\Omega C}^{1+\chi}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} + (1+\chi)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{A}^d}\right) \\
&= -\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\left(1+\chi\right)\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{A}^d}\left(\frac{-\underline{\Omega}\chi\underline{C}^{2\chi}+\underline{C}^{\chi}+\underline{\Omega}\left(1+\chi\right)\underline{C}^{2\chi}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}\right) \\
&= -\frac{\left(1+\chi\right)}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\frac{1-\underline{g}}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{A}^d}\underline{C}^{\chi}\left(1+\underline{\Omega C}^{\chi}\right):
\end{aligned}
$$

We thus have

$$
\begin{aligned}
f_q &= -\tilde{\sigma}^{-1}(1-\underline{\tau})\overline{\underline{C}}^{\tilde{\sigma}^{-1}}\underline{Y\,C}^{-\tilde{\sigma}^{-1}-1}C_q \\
&= \tilde{\sigma}^{-1}(1-\underline{\tau})\overline{\underline{C}}^{\tilde{\sigma}^{-1}}\underline{Y\,C}^{-\tilde{\sigma}^{-1}}\frac{\underline{\Omega C}^{\chi}}{1+\left(\chi+1\right)\underline{\Omega C}^{\chi}}\frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}};
\end{aligned}
$$

and can furthermore express

$$
\begin{aligned}
&U_{Yq} + U_{Y\bar{\underline{\xi}}^d} \\
&= -\tilde{\sigma}^{-1}\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}-1}C_qC_Y \\
&+ \overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}C_{Yq} \\
&+ \underline{A}^d\bar{\underline{\xi}}^d\tilde{\alpha}\underline{\Omega}^{\tilde{\alpha}-1}\frac{\partial\underline{\Omega}}{\partial \underline{q}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}C_Y \\
&+ \underline{A}^d\bar{\underline{\xi}}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\left(\chi-\tilde{\sigma}^{-1}\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}-1}C_qC_Y \\
&+ \underline{A}^d\bar{\underline{\xi}}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\underline{C}^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1}C_{Yq} \\
&+ \underline{A}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\underline{C}^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\tilde{\sigma}^{-1}-1}C_Y \\
&= -\tilde{\sigma}^{-1}\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}-1}\frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}\frac{-\frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}}\underline{\Omega C}^{\chi+1}}{1+\left(\chi+1\right)\underline{\Omega C}^{\chi}} \\
&- \overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{q}^u} \\
&+ \underline{A}^d\bar{\underline{\xi}}^d\tilde{\alpha}\underline{\Omega}^{\tilde{\alpha}-1}\frac{1}{\underline{q}^u}\frac{\underline{\Omega}}{1-\tilde{\alpha}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} \\
&+ \underline{A}^d\bar{\underline{\xi}}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\left(\chi-\tilde{\sigma}^{-1}\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}-1}\frac{-\frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}}\underline{\Omega C}^{\chi+1}}{1+\left(\chi+1\right)\underline{\Omega C}^{\chi}}\frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} \\
&- \underline{A}^d\bar{\underline{\xi}}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\underline{C}^{\chi}\frac{1}{1-\tilde{\alpha}}\frac{\underline{\Omega}}{\underline{q}^u} \\
&+ \underline{A}^d\underline{\Omega}^{\tilde{\alpha}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{1-\underline{g}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}
\end{aligned}
$$

Using

$$\underline{A}^d\bar{\underline{\xi}}^d\underline{\Omega}^{\tilde{\alpha}} = \frac{\Omega}{1+s^d}\overline{C}^{\tilde{\sigma}^{-1}}$$

and $\bar{\underline{\xi}}^d = \underline{q}^u$ we can write this as

$$
\begin{aligned}
&U_{Yq} + U_{Y\bar{\xi}^d} \\
&= +\tilde{\sigma}^{-1}\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}-1}\frac{1}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\underline{\Omega C}^{\chi+1}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}} \qquad (125)\\
&-\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}\frac{1}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\underline{\Omega C}^{\chi}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\\
&+\tilde{\alpha}\frac{\underline{\Omega}}{1+s^d}\overline{C}^{\tilde{\sigma}^{-1}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{1}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\\
&-\frac{\underline{\Omega}}{1+s^d}\overline{C}^{\tilde{\sigma}^{-1}}\left(1+\chi\right)\left(\chi-\widetilde{\sigma}^{-1}\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{\underline{\Omega C}^{\chi}}{\left(1+\left(\chi+1\right)\underline{\Omega C}^{\chi}\right)^2}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\\
&-\frac{\underline{\Omega}}{1+s^d}\overline{C}^{\tilde{\sigma}^{-1}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^3}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\\
&+\frac{\underline{\Omega}}{1+s^d}\overline{C}^{\tilde{\sigma}^{-1}}\left(1+\chi\right)\underline{C}^{\chi-\tilde{\sigma}^{-1}}\frac{1-\widetilde{\alpha}}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\\
&=\frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\widetilde{\alpha}}\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}\underline{\Omega C}^{\chi}\underbrace{\left(\begin{array}{c}
\tilde{\sigma}^{-1}\frac{1}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^2}\\
-\frac{1}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^3}\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)\\
+\tilde{\alpha}\frac{1}{1+s^d}\left(1+\chi\right)\frac{1}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}}\\
-\frac{1}{1+s^d}\left(1+\chi\right)\left(\chi-\widetilde{\sigma}^{-1}\right)\underline{\Omega C}^{\chi}\frac{1}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^2}\\
-\frac{1}{1+s^d}\left(1+\chi\right)\underline{\Omega C}^{\chi}\frac{\left(1+\underline{\Omega C}^{\chi}\right)\left(1+\chi\right)}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^3}\\
+\frac{1}{1+s^d}\left(1+\chi\right)\frac{1-\widetilde{\alpha}}{1+\left(1+\chi\right)\underline{\Omega C}^{\chi}}
\end{array}\right)}_{\equiv C_1} \qquad (126)
\end{aligned}
$$

From (105), (98) and $1 + ! = \phi(1+\nu)$we get

$$
\begin{aligned}
U_{YY} = &-\underline{\bar{C}}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}\,!\,\frac{\eta-1}{\eta}(1-\underline{\tau})\underline{Y}^{-1}\\
&+\overline{\underline{C}}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}-1}C_Y\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\underbrace{\left(\begin{array}{c}
-\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)\\
-\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi}\\
+\frac{1}{1+s^d}\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)\left(1+\chi\right)\left(\chi-\widetilde{\sigma}^{-1}\right)\underline{\Omega C}^{\chi}\\
-\frac{1}{1+s^d}\left(1+\chi\right)\underline{\Omega C}^{\chi}\underline{\Omega}\left(1+\chi\right)\chi\underline{C}^{\chi}
\end{array}\right)}_{\equiv C_0}: \qquad (127)
\end{aligned}
$$

We can thus express

$$
\begin{aligned}
&f_q + \frac{(k_Y - f_Y)}{U_{YY}}(U_{Yq} + U_{Y\overline{\xi}^d}) \\
&= \tilde{\sigma}^{-1}(1-\underline{\tau})\overline{C}^{\tilde{\sigma}^{-1}}\underline{Y}\,\underline{C}^{-\tilde{\sigma}^{-1}}\frac{\underline{\Omega C}^{\chi}}{1+(\chi+1)\,\underline{\Omega C}^{\chi}}\frac{1}{\underline{q}^u}\frac{1}{1-\tilde{\alpha}} \\
&+ \frac{(1-\underline{\tau})\left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)}{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau})\underline{Y}^{-1} + \frac{C_Y}{\underline{C}}\frac{1-\underline{g}}{(1+\underline{\Omega}(1+\chi)\underline{C}^{\chi})^2}C_0} \\
&\cdot \frac{1}{\underline{q}^u}\frac{1-\underline{g}}{1-\tilde{\alpha}}\overline{C}^{\tilde{\sigma}^{-1}}\underline{C}^{-\tilde{\sigma}^{-1}}\underline{\Omega C}^{\chi}C_1: \qquad (128)
\end{aligned}
$$

From (106)

$$
\begin{aligned}
C_0 &= -\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right)\left(1+\frac{1}{1+s^d}\,(1+\chi)\,\underline{\Omega C}^{\chi}\right) + \left(\frac{1}{1+s^d}-1\right)(1+\chi)\,\chi\underline{\Omega C}^{\chi} \\
&= -\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right)\frac{1}{1+s^d}\left(1+s^d+(1+\chi)\,\underline{\Omega C}^{\chi}\right) - \frac{s^d}{1+s^d}(1+\chi)\,\chi\underline{\Omega C}^{\chi} \\
&= \frac{1}{1+s^d}\left(-\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right)\left(1+s^d+(1+\chi)\,\underline{\Omega C}^{\chi}\right) - s^d(1+\chi)\,\chi\underline{\Omega C}^{\chi}\right) \\
&= \frac{1}{1+s^d}\left(-\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right)^2 - s^d\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right) - s^d(1+\chi)\,\chi\underline{\Omega C}^{\chi}\right); \qquad (129)
\end{aligned}
$$

and from (126) we get

$$
\begin{aligned}
C_1 &= \tilde{\sigma}^{-1} \frac{1}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} \\
&- \frac{1}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^3} \left(1+\underline{\Omega C}^{\chi}\right)(1+\chi) \\
&- \frac{1}{1+s^d}(1+\chi)\left(\chi-\tilde{\sigma}^{-1}\right)\underline{\Omega C}^{\chi} \frac{1}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} \\
&- \frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi} \frac{\left(1+\underline{\Omega C}^{\chi}\right)(1+\chi)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^3} \\
&+ \frac{1}{1+s^d}(1+\chi)\frac{1}{1+(1+\chi)\,\underline{\Omega C}^{\chi}} \\
&= \tilde{\sigma}^{-1} \frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} \\
&- \left(1+\underline{\Omega C}^{\chi}\right)(1+\chi)\frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^3} \\
&- \frac{1}{1+s^d}\frac{(1+\chi)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2}\left(\chi\underline{\Omega C}^{\chi}-1-(1+\chi)\,\underline{\Omega C}^{\chi}\right) \\
&= \tilde{\sigma}^{-1} \frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} \\
&- \left(1+\underline{\Omega C}^{\chi}\right)(1+\chi)\frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^3} \\
&+ \frac{1}{1+s^d}\frac{(1+\chi)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} \\
&= \tilde{\sigma}^{-1} \frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} + \frac{(1+\chi)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2}\left(\frac{1}{1+s^d}-\frac{1+\frac{1}{1+s^d}(1+\chi)\,\underline{\Omega C}^{\chi}}{1+(1+\chi)\,\underline{\Omega C}^{\chi}}\right) \\
&= \frac{1}{1+s^d}\left(\tilde{\sigma}^{-1}\frac{1+s^d+(1+\chi)\,\underline{\Omega C}^{\chi}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} + \frac{(1+\chi)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2}\left(1-\frac{1+s^d+(1+\chi)\,\underline{\Omega C}^{\chi}}{1+(1+\chi)\,\underline{\Omega C}^{\chi}}\right)\right) \\
&= \frac{1}{1+s^d}\left(\frac{\tilde{\sigma}^{-1}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)} + \frac{s^d\tilde{\sigma}^{-1}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2} - \frac{(1+\chi)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^2}\frac{s^d}{1+(1+\chi)\,\underline{\Omega C}^{\chi}}\right) \\
&= \frac{1}{1+s^d}\left(\frac{\tilde{\sigma}^{-1}}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)} + s^d\frac{\tilde{\sigma}^{-1}\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)-(1+\chi)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+(1+\chi)\,\underline{\Omega C}^{\chi}\right)^3}\right)
\end{aligned}
\tag{130}
$$

Combining (128), (129) and (130) we have

$$
\begin{aligned}
& f_q + \frac{(k_Y - f_Y)}{U_{YY}} (U_{Yq} + U_{Y\bar{\xi}^d}) \\
& = \tilde{\sigma}^{-1}(1-\underline{\tau})\overline{C}^{\tilde{\sigma}^{-1}} \underline{Y\, C}^{-\tilde{\sigma}^{-1}} \frac{\underline{\Omega C}^{\chi}}{1+(\chi+1)\,\underline{\Omega C}^{\chi}} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \\
& + \frac{(1-\underline{\tau})\left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)}{- ! \,\frac{\eta-1}{\eta}(1-\underline{\tau})\underline{Y}^{-1} + \frac{C_Y}{\underline{C}} \frac{1-\underline{g}}{(1+\underline{\Omega}(1+\chi)\underline{C}^{\chi})^2} C_0} \\
& \cdot \frac{1}{\underline{q}^u} \frac{1-\underline{g}}{1-\tilde{\alpha}} \overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} \underline{\Omega C}^{\chi} C_1 \\
& = (1-\underline{\tau})\overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} \underline{Y} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega C}^{\chi} \left(\begin{array}{c} \frac{\tilde{\sigma}^{-1}}{1+(\chi+1)\underline{\Omega C}^{\chi}} \\ + \frac{(1-g)\left(\omega + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)}{-\omega \frac{\eta-1}{\eta}(1-\underline{\tau}) + \underline{Y}\frac{C_Y}{\underline{C}} \frac{1-\underline{g}}{(1+\underline{\Omega}(1+\chi)\underline{C}^{\chi})^2} C_0} C_1 \end{array} \right) \\
& = (1-\underline{\tau})\overline{C}^{\tilde{\sigma}^{-1}} \underline{C}^{-\tilde{\sigma}^{-1}} \underline{Y} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \underline{\Omega C}^{\chi} \\
& \cdot \left(\frac{\tilde{\sigma}^{-1} \left(\begin{array}{c} - ! \,\frac{\eta-1}{\eta}(1-\underline{\tau}) \\ + \underline{Y}\frac{C_Y}{\underline{C}} \frac{1-\underline{g}}{(1+\underline{\Omega}(1+\chi)\underline{C}^{\chi})^2} C_0 \end{array} \right) + (1-g)\left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)(1+(\chi+1)\,\underline{\Omega C}^{\chi})\, C_1}{\underbrace{(1+(\chi+1)\,\underline{\Omega C}^{\chi})}_{>0} \underbrace{\left(- ! \,\frac{\eta-1}{\eta}(1-\underline{\tau}) + \underline{Y}\frac{C_Y}{\underline{C}} \frac{1-\underline{g}}{\left(1+\underline{\Omega}\,(1+\chi)\,\underline{C}^{\chi}\right)^2} C_0 \right)}_{\propto U_{YY} \text{ and } U_{YY}<0 \text{ for } s^d \text{ not too negative from lemma 1}}} \right)
\end{aligned}
\tag{131}
$$

So that the sign of $-\left(f_q + \frac{(k_Y - f_Y)}{U_{YY}}(U_{Yq} + U_{Y\bar{\xi}^d})\right)$ is identical to the sign of

$$
\begin{aligned}
&\tilde{\sigma}^{-1}\left(-!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) + \underline{Y}\frac{C_Y}{\underline{C}}\frac{1-\underline{g}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}C_0\right) \\
&+(1-g)\left(!\,+\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)\left(1+\left(\chi+1\right)\underline{\Omega C}^{\chi}\right)C_1 \\
&=\tilde{\sigma}^{-1}\left(\begin{array}{c} -!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) + \underline{Y}\frac{C_Y}{\underline{C}}\frac{1-g}{(1+\underline{\Omega}(1+\chi)\underline{C}^{\chi})^2} \\ \cdot\frac{1}{1+s^d}\left(-\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2 - s^d\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right) - s^d\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}\right)\end{array}\right) \\
&+(1-g)\left(!\,+\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)\left(1+\left(\chi+1\right)\underline{\Omega C}^{\chi}\right) \\
&\cdot\frac{1}{1+s^d}\left(\frac{\tilde{\sigma}^{-1}}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)} + s^d\frac{\tilde{\sigma}^{-1}\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right) - \left(1+\chi\right)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^3}\right) \\
&=-\tilde{\sigma}^{-1}!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) \\
&+\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)\frac{1}{1+s^d}\left(-\tilde{\sigma}^{-1} - s^d\tilde{\sigma}^{-1}\frac{1}{1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}} - s^d\frac{\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}\right) \\
&+(1-g)\left(!\,+\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)\frac{1}{1+s^d}\left(\tilde{\sigma}^{-1} + s^d\frac{\tilde{\sigma}^{-1}}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)} - s^d\frac{\left(1+\chi\right)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^2}\right) \\
&=!\left(\begin{array}{c} -\tilde{\sigma}^{-1}\frac{\eta-1}{\eta}(1-\underline{\tau}) \\ +(1-g)\frac{1}{1+s^d}\left(\tilde{\sigma}^{-1} + s^d\frac{\tilde{\sigma}^{-1}}{(1+(1+\chi)\underline{\Omega C}^{\chi})} - s^d\frac{(1+\chi)(1+\underline{\Omega C}^{\chi})}{(1+(1+\chi)\underline{\Omega C}^{\chi})^2}\right)\end{array}\right) \\
&-2\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)\frac{s^d}{1+s^d}\frac{\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2} \\
&=-\tilde{\sigma}^{-1}!\left(\frac{\eta-1}{\eta}(1-\underline{\tau}) - \frac{1-g}{1+s^d}\right) + !\,(1-g)\frac{s^d}{1+s^d}\left(\frac{\tilde{\sigma}^{-1}}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)} - \frac{\left(1+\chi\right)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^2}\right) \\
&-2\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)\frac{s^d}{1+s^d}\frac{\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2} \\
&=-\tilde{\sigma}^{-1}!\left(\frac{\eta-1}{\eta}(1-\underline{\tau}) - \frac{1-g}{1+s^d}\right) + !\,\tilde{\sigma}^{-1}\frac{(1-g)}{1+s^d}\frac{s^d}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)} - !\,(1-g)\frac{s^d}{1+s^d}\frac{\left(1+\chi\right)\left(1+\underline{\Omega}\right.}{\left(1+\left(1+\chi\right)\underline{\Omega}\right.} \\
&-\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)\frac{s^d}{1+s^d}2\frac{\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2} \\
&=-\tilde{\sigma}^{-1}!\underbrace{\left(\frac{\eta-1}{\eta}(1-\underline{\tau}) - \frac{1-g}{1+s^d} - \frac{(1-g)}{1+s^d}\frac{s^d}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)}\right)}_{=0\text{ at the efficient output subsidy where }U_Y=0\text{ from (100)}} \\
&-\frac{s^d}{1+s^d}\underbrace{!\,(1-g)\frac{\left(1+\chi\right)\left(1+\underline{\Omega C}^{\chi}\right)}{\left(1+\left(1+\chi\right)\underline{\Omega C}^{\chi}\right)^2}}_{>0} \\
&-\frac{s^d}{1+s^d}\underbrace{\tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)2\frac{\left(1+\chi\right)\chi\underline{\Omega C}^{\chi}}{\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2}}_{>0}
\end{aligned}
$$

We thus have

$$
\begin{aligned}
-(f_q+\frac{(k_Y-f_Y)}{U_{YY}}(U_{Yq}+U_{Y\bar{\xi}^d}))&=0 \ \& \ \kappa_u=0 \text{ for } s^d=0\\
-(f_q+\frac{(k_Y-f_Y)}{U_{YY}}(U_{Yq}+U_{Y\bar{\xi}^d}))&<0 \ \& \ \kappa_u<0 \text{ for } s^d>0\\
-(f_q+\frac{(k_Y-f_Y)}{U_{YY}}(U_{Yq}+U_{Y\bar{\xi}^d}))&>0 \ \& \ \kappa_u>0 \text{ for } s^d<0
\end{aligned}
$$

.

In the special case with $s^d=0$ we have from (106)

$$C_0=-\tilde{\sigma}^{-1}\left(1+\underline{\Omega}\left(1+\chi\right)\underline{C}^{\chi}\right)^2; \tag{132}$$

and for $s^d=0$ we also have, as show below

$$C_1=\frac{\tilde{\sigma}^{-1}}{1+\left(1+\chi\right)\underline{\Omega}\underline{C}^{\chi}}>0; \tag{133}$$

so that

$$
\begin{aligned}
& f_q + \frac{(k_Y - f_Y)}{U_{YY}}(U_{Yq} + U_{Y\bar{\xi}^d}) \\
& = \tilde{\sigma}^{-1}(1-\underline{\tau})\overline{\underline{C}}^{\tilde{\sigma}^{-1}}\, \underline{Y\, C^{-\tilde{\sigma}^{-1}}}\, \frac{\underline{\Omega C^{\chi}}}{1+(\chi+1)\,\underline{\Omega C^{\chi}}} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \\
& + \frac{(1-\underline{\tau})\left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)}{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau})\underline{Y}^{-1} + -\tilde{\sigma}^{-1}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)} \\
& \cdot \frac{1}{\underline{q}^u} \frac{1-\underline{g}}{1-\tilde{\alpha}} \overline{\underline{C}}^{\tilde{\sigma}^{-1}}\, \underline{C}^{-\tilde{\sigma}^{-1}}\, \underline{\Omega C^{\chi}} \frac{\tilde{\sigma}^{-1}}{1+(1+\chi)\,\underline{\Omega C^{\chi}}} \\
& = \tilde{\sigma}^{-1}(1-\underline{\tau})\overline{\underline{C}}^{\tilde{\sigma}^{-1}}\, \underline{C}^{-\tilde{\sigma}^{-1}}\, \underline{Y} \frac{1}{\underline{q}^u} \frac{1}{1-\tilde{\alpha}} \frac{\underline{\Omega C^{\chi}}}{1+(1+\chi)\,\underline{\Omega C^{\chi}}} \\
& \cdot \left(1 + \frac{\left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)}{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) - \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)} \left(1-\underline{g}\right)\right) \\
& = \frac{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) - \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right) + \left(! + \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\right)\left(1-\underline{g}\right)}{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) - \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)} \\
& = \frac{-!\left(\frac{\eta-1}{\eta}(1-\underline{\tau}) + \left(1-\underline{g}\right)\right)}{-!\,\frac{\eta-1}{\eta}(1-\underline{\tau}) - \tilde{\sigma}^{-1}\underline{Y}\frac{C_Y}{\underline{C}}\left(1-\underline{g}\right)} \\
& = 0
\end{aligned}
$$

where the last equality follows from the fact that $-\frac{\eta-1}{\eta}(1-\underline{\tau}^{eff}(0)) + \left(1-\underline{g}\right) = 0$ when the output subsidy is efficient, as has been assumed.

A.6 Proofs of Propositions 3 and 4

We start with the proof of proposition 3. From equation (21) follows that the dynamics of $\widehat{q}_t^u$ are to first order independent of policy choices and belief distortions (with the latter being of first order), so that equations (61) and (72) jointly determine to first order a unique path for $\{x_t; \pi_t\}_{t=0}^{\infty}$. Since the dynamics of Δ_t is to first order independent of policy decisions and belief distortions, see (44), and since (33) and (34) determine to first order F_t and K_t, given the first order solution for x_t and π_t; this determines to first order the dynamics of $\{q_t^u; Y_t; F_t; K_t; \Delta_t\}_{t=0}^{\infty}$. Under the assumed policy commitment, belief distortions thus affect the evolution of endogenous variables at most to second order. The second (or higher) order effects of beliefs that differ slightly from the worst-case beliefs implied by the upper bound dynamics can be computed using the first order accurate paths for (which are already determined), the considered deviations from the worst-case beliefs, and the second (or higher) order approximations to equations (21) and (33)-(45). The interest rate required to

support this outcome can be determined from equation (41), using the solution for the other variables.

Next, we establish proposition 4. The problem of choosing worst-case beliefs under a policy commitment to the target criterion (72) can be written as

$$\min_{\{m_{t+1},q_t^u,Y_t,F_t,K_t,\Delta_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{l} U(Y_t;\Delta_t;q_t^u;\xi_t) + \theta\beta m_{t+1} \log m_{t+1} \ldots \\ +\gamma_t \left(\tilde{h}(\Delta_{t-1};K_t/F_t) - \Delta_t\right) \\ \\ \\ \end{array} \right.$$

A.7 Details of the Linear-Quadratic Approximation

We start by deriving the second-order approximation in equation (79):

$$
\begin{aligned}
& E_0\underline{\Gamma}'\left(z(Y_t; q_t^u; \xi_t) + \alpha\beta m_{t+1}\Phi(Z_{t+1}) - Z_t\right) \\
&= E_0\left[\begin{array}{c}
\underline{\Gamma}'\left(\begin{array}{c} z_Y\widetilde{Y}_t + z_q(q_t^u - \overline{\xi}_t^d) + \alpha\beta D(1)\widetilde{Z}_{t+1} - \widetilde{Z}_t \\ +\frac{1}{2}z_{YY}\widetilde{Y}_t^2 + \widetilde{Y}_t\left(z_{Y\xi}\widetilde{\xi}_t + z_{Yq}\widetilde{\overline{\xi}}_t^d\right)\end{array}\right) \\
+\frac{1}{2}\underline{K}\alpha\beta c\left(\widehat{F}_{t+1} - \widehat{K}_{t+1}\right)^2 + \underline{K}\alpha\beta\underline{\Gamma}'D(1)\widetilde{m}_{t+1}\left(\begin{array}{c}\widehat{F}_{t+1} \\ \widehat{K}_{t+1}\end{array}\right)
\end{array}\right] \\
&\quad + t{:}i{:}p{:} + O(3); \qquad (135)
\end{aligned}
$$

where $\underline{\Gamma}' = (\underline{\Gamma}_1; -\underline{\Gamma}_1)$ and where we used the fact that $\underline{K}$ times the Hessian matrix of second partial derivatives of the function $\underline{\Gamma}'\Phi(Z_{t+1})$ is of the form

$$
c\left(\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right);
$$

with

$$
c = \underline{\Gamma}_1\frac{\underline{F}}{\underline{K}}\left(-\frac{(1-\alpha)}{\alpha}\frac{\eta(1+!)}{1+!\eta} - \left(\frac{(1-\alpha)}{\alpha}\right)^2\frac{\eta(1+!)}{1+!\eta}\right);
$$

and $c < 0$ whenever steady state output falls short of its first best level, as then $\underline{\Gamma}_1 > 0$ (see appendix A.2 in Adam and Woodford (2012) for further details).

A linear approximation to (36) delivers

$$
\widehat{F}_{t+1} - \widehat{K}_{t+1} = -\frac{\alpha}{1-\alpha}(1+!\eta)\pi_t + O(2) \qquad (136)
$$

Using this and

$$
a\underline{\Gamma}'D(1) = \underline{\Gamma}'
$$

we can write (135) as

$$
\begin{aligned}
& E_0\underline{\Gamma}'\left(z(Y_t; q_t^u; \xi_t) + \alpha\beta m_{t+1}\Phi(Z_{t+1}) - Z_t\right) \\
&= E_0\left[\begin{array}{l}
\underline{\Gamma}'\left(\begin{array}{c} z_Y\widetilde{Y}_t + z_q(q_t^u - \overline{\xi}_t^d) + \beta\widetilde{Z}_{t+1} - \widetilde{Z}_t \\ +\frac{1}{2}z_{YY}\widetilde{Y}_t^2 + \widetilde{Y}_t\left(z_{Y\xi}\widetilde{\xi}_t + z_{Yq}\widetilde{\overline{\xi}}_t^d\right)\end{array}\right) \\
+\frac{1}{2}\underline{K}\alpha\beta c\left(\frac{\alpha}{1-\alpha}(1+!\eta)\right)^2\pi_{t+1}^2 - \underline{K}\beta\underline{\Gamma}_1\frac{\alpha}{1-\alpha}(1+!\eta)\widetilde{m}_{t+1}\pi_{t+1}
\end{array}\right] \\
&\quad + t{:}i{:}p{:} + O(3) \\
&= E_0\left[\begin{array}{l}
\underline{\Gamma}'\left(\begin{array}{c} z_Y\widetilde{Y}_t + z_q(q_t^u - \overline{\xi}_t^d) + \beta\widetilde{Z}_{t+1} - \widetilde{Z}_t \\ +\frac{1}{2}z_{YY}\widetilde{Y}_t^2 + \widetilde{Y}_t\left(z_{Y\xi}\widetilde{\xi}_t + z_{Yq}\widetilde{\overline{\xi}}_t^d\right)\end{array}\right) \\
-\beta\underline{\Gamma}_1\delta_2^\pi\pi_{t+1}^2 - \beta\underline{\Gamma}_1\delta_1^\pi\widetilde{m}_{t+1}\pi_{t+1}
\end{array}\right] \\
&\quad + t{:}i{:}p{:} + O(3); \qquad (137)
\end{aligned}
$$

where the last line uses the definitions

$$\delta_1^\pi = \underline{K}\frac{\alpha}{1-\alpha}(1+!\eta) > 0$$
$$\delta_2^\pi = \frac{1}{2}\underline{F}\frac{\alpha}{1-\alpha}\eta(1+!)(1+!\eta) > 0$$

and the fact that $\underline{K} = \underline{F}$ at point around which we approximate. Multiplying by β^t and summing over all $t \geq 0$ delivers

$$E_0 \sum_{t=0}^{\infty} \beta^t \underline{\Gamma}' \left(\beta \widetilde{Z}_{t+1} - \widetilde{Z}_t \right) = -\underline{\Gamma}_1 \underline{F} \left(\widehat{F}_0 - \widehat{K}_0 \right) \tag{138}$$

A second order approximation to (36) allows to express $\widehat{F}_t = \log F_t{=}F$ and $\widehat{K}_t = \log K_t{=}K$ in terms of $\pi_t = \log \Pi_t$:

$$\widehat{F}_t - \widehat{K}_t = -\frac{\alpha}{1-\alpha}(1+!\eta)\pi_0 - \frac{1}{2}\alpha(1+!\eta)\frac{(\eta-1)}{(1-\alpha)^2}(\pi_0)^2 + O(3) \tag{139}$$

so that

$$\begin{aligned}
& -\underline{\Gamma}_1 \underline{F} \left(\widehat{F}_0 - \widehat{K}_0 \right) \\
&= -\underline{\Gamma}_1 \underline{F} \left(-\frac{\alpha}{1-\alpha}(1+!\eta)\pi_0 - \frac{1}{2}\alpha(1+!\eta)\frac{(\eta-1)}{(1-\alpha)^2}(\pi_0)^2 \right) + O(3) \\
&= \underline{\Gamma}_1 \delta_1^\pi \pi_0 + \underline{\Gamma}_1 \frac{1}{2}\underline{F}\frac{\alpha}{1-\alpha}(1+!\eta)\frac{(\eta-1)}{1-\alpha}(\pi_0)^2 + O(3) \\
&= \underline{\Gamma}_1 \delta_1^\pi \pi_0 + \underline{\Gamma}_1 \delta_3^\pi (\pi_0)^2 + O(3);
\end{aligned} \tag{140}$$

where

$$\delta_3^\pi = \frac{1}{2}\underline{F}\frac{\alpha}{1-\alpha}(1+!\eta)\frac{(\eta-1)}{1-\alpha};$$

and where the second to last equality uses the definition of δ_1^π and the fact that $\underline{F} = \underline{K}$ in the steady state. Combining (137) with (138) and (140) delivers (79) in the main text.

Next, we derive (89):

$$
\begin{aligned}
&\alpha\Gamma'_{-1}\Phi(Z_0) + \Psi_{-1}(1-\delta)q_0^u \\
&= \alpha(1+\tilde{\zeta}_{-1})\underline{\Gamma}'D(1)\widetilde{Z}_0 + \frac{1}{2}\alpha c\underline{K}\left(\widehat{F}_0 - \widehat{K}_0\right)^2 + (1-\delta)\underline{\Psi}(q_0^u - \overline{\xi}_0^d) \\
&+ t.i.p. + O(3) \\
&= (1+\tilde{\zeta}_{-1})\underline{\Gamma}_1\underline{K}(\widehat{F}_0 - \widehat{K}_0) + \frac{1}{2}\alpha c\underline{K}\left(\widehat{F}_0 - \widehat{K}_0\right)^2 + (1-\delta)\underline{\Psi}(q_0^u - \overline{\xi}_0^d) \\
&+ t.i.p. + O(3) \\
&= (1+\tilde{\zeta}_{-1})\underline{\Gamma}_1\underline{K}(-\frac{\alpha}{1-\alpha}(1+\,!\,\eta)\,\pi_0 - \frac{1}{2}\alpha(1+\,!\,\eta)\frac{(\eta-1)}{(1-\alpha)^2}(\pi_0)^2) \\
&+ \frac{1}{2}\alpha c\underline{K}\left(-\frac{\alpha}{1-\alpha}(1+\,!\,\eta)\,\pi_0\right)^2 + (1-\delta)\underline{\Psi}(q_0^u - \overline{\xi}_0^d) \\
&+ t.i.p. + O(3) \\
&= -(1+\tilde{\zeta}_{-1})\underline{\Gamma}_1\delta_1^\pi\pi_0 \\
&\underbrace{-\underline{\Gamma}_1\underline{K}\frac{1}{2}\left(\alpha(1+\,!\,\eta)\frac{(\eta-1)}{(1-\alpha)^2} + \alpha\frac{c}{\underline{\Gamma}_1}\left(\frac{\alpha}{1-\alpha}(1+\,!\,\eta)\right)^2\right)\pi_0^2}_{=\underline{\Gamma}_1(\delta_2^\pi - \delta_3^\pi)} \\
&+ (1-\delta)\underline{\Psi}(q_0^u - \overline{\xi}_0^d) \\
&+ t.i.p. + O(3);
\end{aligned}
$$

where the third equality uses (136) and (139).