Combination Bidding in Multi-Unit Auctions*

Estelle Cantillon[†] and Martin Pesendorfer[‡]

April 28, 2002

Abstract

This paper considers the problem of identification and estimation in the first-price multi-unit auction. It is motivated by the auctions of bus routes held in London where, because of anticipated synergies, bidders are allowed to submit bids on combinations of routes as well as on individual routes. We show that equilibrium combination bidding does not require cost synergies and can instead serve as a tool to leverage market power across the different routes. As a result, the welfare consequences of allowing combination bidding in the first price auction are ambiguous, and depend on the importance of the cost synergies.

We provide conditions for identification in the multi-unit first price auction. In particular, we show that the presence of combination bids is a necessary condition for identification. We propose an estimation method to infer the multidimensional private information. The method consists of two stages. In the first stage, the distribution of bids is estimated parametrically. In the second stage, costs and the distribution of costs are inferred based on the first order conditions for optimally chosen bids.

We apply the estimation method to data from the London bus routes market. We quantify the magnitude of cost synergies and evaluate the welfare impacts of allowing combination bids in that market.

Preliminary and incomplete. Comments welcome.

*We are grateful to several o cials at London Transport Buses and at the London Omnibus Traction Society for their help in collecting the data, and to Ioannis Ioannou, Qinan Tang and Jay Cox for their excellent research assistance. We are indebted to Liran Einav, Ariel Pakes, Loic Sadoulet, Steven Stern, to seminar audiences at Harvard, Rutgers, UBC, Toronto, LSE, UTexas and Virginia, and to conference audiences at Gerzensee, Barcelona and Brussels for their helpful suggestions. Financial support was provided by the Division of Research at Harvard Business School and NSF Grant SES 9811134, respectively. Cantillon thanks the Cowles Foundation at Yale University for their hospitality and financial support during the academic year 2000-01.

[†]Harvard Business School and CEPR. Email: ecantillon@hbs.edu.

[‡]Economics Department, Yale University, CEPR and NBER. Email: martin.pesendorfer@yale.edu.

1 Introduction

This paper considers the problem of identification and estimation in the first-price multi-unit auction. It is motivated by the auctions held by the London Transportation authority to award contracts to service bus routes. Two special features of these auctions are that several bus routes are auctioned at the same time, and that bidders may submit combination bids in addition to stand-alone bids. In other words, the London bus routes market is an example of a combinatorial auction.

Combinatorial auctions - where bidders submit bids contingent on the final allocation - allow bidders to transmit richer information regarding the value they attach to the objects for sale. When the objects are not independent, for instance, because the bidders value bundles of objects differently than the sum of the constituent parts, allowing such contingent bids is a necessary condition for efficiency and optimality (Groves, 1973 and Clarke, 1971; Levin, 1997, Armstrong, 2000 and Avery and Hendershott, 2000).

This was well understood by the London bus procurement authorities. Indeed, two of the motivations for allowing combination bids in the London bus routes market were that they would allow bidders to pass on, through lower bids, some of the cost savings resulting from cost synergies between routes, and at the same time, that they would enhance the efficiency in the allocation of routes across bidders.

However, allowing combination bids in the first price auction may also hurt efficiency and costs. In section 2, we introduce a model of a private values multi-unit procurement auction that allows for cost synergies between objects. We highlight two distinct motivations for combination bidding. On the one hand, combination bidding gives rise to a strategic effect because bidders' stand-alone bids compete with their combination bids. As a result, bidders may find it profitable to inflate their stand-alone bids relative to their combination bids in order to favor the latter, even in the absence of any cost synergies. The reason is that combination bids allow bidders to link otherwise independent markets and leverage any advantage they may have in one market into the other. This effect may increase procurement costs and hurt efficiency. On the other hand, when cost synergies are important, the fact that combination bids do allow bidders to align their bids better on their costs, can help improve efficiency and lower costs. As a result, the welfare consequences of combination bidding depend on whether the leverage effect or the synergy effect dominates.

Section 3 provides conditions under which the unobserved costs and cost distribution are identified from bid data. We assume that bidders assess their winning probabilities correctly and choose bids to maximize their profits. We show that a necessary condition

for identification of the multidimensional private information based on bid data is that the auction permits bidders to submit a full set of combination bids, in addition to stand-alone bids. Moreover, as long as bidders do make use of all their bids, the cost distribution function is identified. The intuition is that identification requires the set of allowable bids – which represents the observed information – to be of the same dimension as the private information to infer. In particular, this means that the general first price multi-unit auction model with only stand-alone bids is not identified. Finally, we show that any constraint on bids, such as reserve prices, or the condition that combination bids must be lower than the sum of the constituent stand-alone bids, can reduce the dimensionality of the observed information and therefore introduce a level of underidentification. A cost range rather than a cost point is identified. We characterize this cost range.

Section 4 proposes our estimation method. It is based on Guerre, Perrigne and Vuong (2000)'s two stage estimation method for single unit auctions but extends it to multi-unit auction environments. The estimation proceeds in two stages. In the first stage, the multivariate joint distribution of bids for all units is estimated parametrically. In the second stage, the multivariate cost distribution is inferred using the first order conditions for optimal bids.

Section 5 describes the London bus routes market. This market is particularly well-suited for this kind of analysis. First, there is a common perception that synergies between routes are prevalent. Second, combination bids are permitted and play an important role in this market with about 30% of all routes won by combination bids. Thus, our method allows us to quantify the extent of cost synergies in this market, and therefore assess the relative importance of the leverage and synergy motivations for combination bids. Our very preliminary results are reported in section 6. We calculate the percentage mark-up of bids (relative to cost) for a selected sample of contracts for which stand-alone and combination bids are submitted. We find that the mark-up is about 26% on stand-alone bids and 35% on combination bids. Calculating the synergy effects we find that the cost of a combined route is on average 8% lower than the sum of the costs for the individual routes. However, we also find evidence that not all combination bids are motivated by underlying cost synergies.

Related literature. There is a growing literature on identification and estimation in auctions. Donald and Paarsch (1993), Laffont, Ossard and Vuong (1995), Guerre, Perrigne and Vuong (2000) and others propose identification results and estimation techniques to infer bidders' private information. The literature focuses to a large extent on the single-unit auction model and little is known about auctions in which multiple units are sold.

Exceptions include the sequential auctions analyzed by Donald, Paarsch and Robert (2001) and Jofre-Bonet and Pesendorfer (2001) and the discriminatory multi-unit auction analyzed by Hortacsu (2002). Donald et al. and Jofre-Bonet and Pesendorfer's approaches generalize previous estimation techniques to account for intertemporal linkages between auctions. Hortacsu (2002) studies the Turkish Treasury bill auctions. He shows that bidders' valuation schedules are identified from their observed demand schedules in the discriminatory multi-unit auction and proposes an estimation strategy based on resampling techniques.

There has also been a number of recent theoretical analyses of multi-unit auctions. Among these, Armstrong (2000) and Avery and Hendershott (2000) derive properties of the optimal multi-unit auction when types are multidimensional and objects may be substitutes or complements. A central question that these authors address is to what extent the auctioneer may benefit from bundling the objects (A seminal contribution to this question is Palfrey, 1983). Krishna and Rosenthal (1995) and Branco (1997) study the second price multi-unit auction with synergies. Milgrom (2000) highlights some perverse effects of combinatorial bidding in ascending auctions. Our analysis contributes to this literature by highlighting the motivations and consequences of combination bidding in the first price auction. Our leverage motivation is analogous to the bundling motivation in the (decision-theoretic) multi-dimensional screening literature (McAfee, McMillan and Whinston, 1989, Armstrong, 1996 and Armstrong and Rochet, 1999) but it had never been pointed out in the auction context.

Finally, the importance of synergies in multi-unit auctions has been emphasized by the recent experience in FCC spectrum auctions. Ausubel, Cramton, McAfee and McMillan (1997) and Moreton and Spiller (1998) use a regression analysis to measure synergy effects in these auctions.

2 The Model

This section introduces the model and highlights its key properties. The model integrates the salient features of the London bus routes market.

A seller simultaneously offers m contracts for sale to N risk neutral bidders. Each bidder i privately observes a cost draw, $c_s^i \in \mathbb{R}$, for each possible subset of the contracts, $s \subseteq S = \{1, \ldots, m\}$. Notice that there are a total of $2^m - 1$ possible subsets of contracts. We say that contracts s and t, with $s \cap t = \emptyset$, are *independent* from bidder i's perspective if $c_s^i + c_t^i = c_{s \cup t}^i$ where $c_{s \cup t}^i$ denotes bidder i's cost for the combination of contracts s and t.

They are *complements* if $c^i_{s\cup t} < c^i_s + c^i_t$ and *substitutes* if $c^i_{s\cup t} > c^i_s + c^i_t$.

Contract costs are ex-ante distributed according to the joint distribution $F((c^i_s)_{s\subseteq S,i=1,...N}|X)$ where $X = (x, w)$ denotes a vector of observable contract characteristics x and bidder characteristics w. We assume that F is common knowledge, and that it has a bounded and coordinate-wise convex support with a well defined strictly positive density everywhere. Notice that our formulation permits correlation in bidders' costs across bidders and contracts.

We compare two auction rules. The first auction rule replicates the rule used in the London bus routes market. Bidders may submit bids on all subsets of the set of contracts. Let b^i_s denote bidder i's bid on the subset of contracts $s \subseteq S$, and let $b^i = (b^i_1, ..., b^i_s, ...b^i_S) \in \mathbb{R}^{2^m-1}$. We sometimes use the symbol b^i_{-s} to denote the vector of bids by bidder i on all contracts except for s. Bidders pay the value of their winning bids and the auctioneer selects the winner(s) based on the allocation that minimizes her total payment. Formally, the last restriction requires that $b^i_{s\cup t} \leq b^i_s + b^i_t$ for all s, t such that $s \cap t = \emptyset$. A combination bid must be no greater than the sum of its constituent stand-alone bids. Otherwise the auctioneer would select $b^i_s + b^i_t$ and ignore the combination bid $b^i_{s\cup t}$.[1]

The second auction rule is the standard simultaneous first-price auction where bidders are allowed to submit bids on the individual contracts only. That is $b^i = (b^i_1, ..., b^i_m) \in \mathbb{R}^m$.

Fix bidder i, and for each contract s, define B^{-i}_s as the lowest bid submitted by bidder i's opponents on route (combination) s. By convention, let $B^{-i}_\emptyset = 0$ and let $B^{-i} = (B^{-i}_1, \ldots, B^{-i}_S)$.

When only stand-alone bids are allowed, bidder i's bid on an individual contract $s \in S$ competes only against the best bid of his opponents on that particular contract, B^{-i}_s. Nevertheless, his optimization problem departs from the optimization of bidders in a single unit auction because his costs depend on the final allocation of contracts. As a result, his bids on the individual contracts must take into account the possibility that he may also win other contracts.[2]

When combination bids are allowed, a different trade-off arises from the fact that bidders' own bids compete with one another. Formally, with m contracts, there are 2^m possible

[1] *Existence of an equilibrium* in the multi-unit auction with combination bidding is guaranteed. There always exists an equilibrium where all bidders submit a bid for the bundle S only. To see this, notice that given that i's opponents only submit a bid for the bundle, any bid by i on contract $s \subset S$ can only win together with i's own bid on $S \backslash s$. In other words, a bidder can only win a contract if he wins all of them. As a result, setting $b^i_s = \infty$ for all $s \neq S$ is *a* best response. Of course, other equilibria may also exist.

[2] This is the standard "exposure problem."

winning allocations. Either $b_s^i + B_{S\backslash s}^{-i}$ beats any other alternative bid combination $b_t^i + B_{S\backslash t}^{-i}$ for $t \neq s$ and B_S^{-i}, in which case bidder i wins exactly the set of contracts s with cost c_s^i, or $B_S^{-i} < b_s^i + B_{S\backslash s}^{-i}$ for all s, in which case bidder i does not win anything. This yields the following payoff function for bidder i (ignoring ties):

$$\begin{cases} b_s^i - c_s^i & \text{if } b_s^i + B_{S\backslash s}^{-i} < \min\{b_t^i + B_t^{-i} \text{ for } t \neq s, B_S^{-i}\} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Consider the set of contracts s. Holding the distribution of the opponents' best bids $(B_1^{-i}, ..., B_S^{-i})$ fixed, decreasing b_s^i increases bidder i's chance to win exactly set of contracts s by lowering the price of allocation $b_s^i + B_{S\backslash s}^{-i}$ relative to the others. However, this may come at the expense of winning potentially more profitable contract combinations if for instance $b_s^i - c_s^i < b_t^i - c_t^i$ for some t.

This is a standard price discrimination trade-off and it is analogous to the pricing problem of the multi-product monopolist in the multi-dimensional screening literature. To make this analogy more transparent, suppose that there are only two contracts, 1 and 2. Figure 1 represents bidder i's bid $(b_1^i, b_2^i, b_{1\cup 2}^i)$ in the (B_1^{-i}, B_2^{-i}) space (in that space, combination bid $b_{1\cup 2}^i$ can be represented by a line with slope -1).

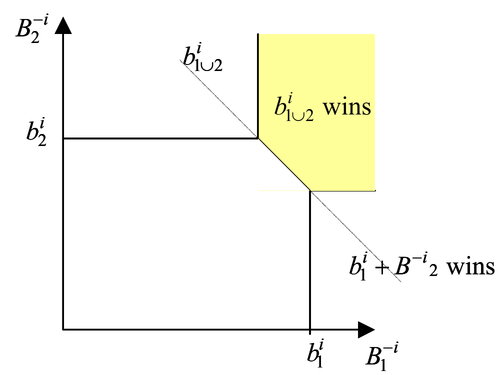

Figure 1:

Ignoring $B_{1\cup 2}^{-i}$, bidder i wins contract 1 only when $b_1^i + B_2^i < \min\{b_{1\cup 2}^i, B_1^{-i} + b_2^i\}$, that is whenever the realization of (B_1^{-i}, B_2^{-i}) falls in the lower right quadrant of figure 1. Similarly, bidder i wins contract 2 only when (B_1^{-i}, B_2^{-i}) is in the upper left quadrant. In the upper right truncated quadrant (shaded), $b_{1\cup 2}^i$ beats every other bid combination so bidder i wins both goods. He wins none in the lower left quadrant.

Replacing the distribution of the opponents' best bids (B_1^{-i}, B_2^{-i}) by the distribution of consumers reserve prices for good 1 and good 2, and reinterpreting b_1^i, b_2^i and $b_{1\cup 2}^i$ as

the prices for goods 1, 2 and the bundle of the two respectively, yields the standard multiproduct pricing problem. McAfee, McMillan and Whinston (1989) use figure 1 to derive a *sufficient* condition for bundling to be profitable for the monopolist with additive costs. In particular, they find that when demand is independent across goods – in our setting, whenever (B_1^{-i}, B_2^{-i}) are independently distributed – submitting a price for the bundle that is lower than the sum of the individual prices is optimal for the monopolist. Armstrong and Rochet (1999) solve for the global maximization of the multi-product monopolist. Their analysis confirms that bundling (the equivalent of submitting a non trivial combination bid $b_{1\cup 2} < b_1 + b_2$) is profitable unless there is strong correlation across buyers' reservation values. See also Armstrong (1996 and 2000).

The following example suggests that the strategic effect to which combination bidding gives rise may have perverse consequences on welfare.

Example: The Leverage Motivation

Consider two independent contracts. Types are one-dimensional and independently distributed. Let $c_s^i(\theta^i)$ be bidder i's cost for contract s, for the realization of type $\theta^i \in \mathbb{R}$. Without loss of generality, we take θ^i to be *i.i.d* uniform on [0,1] since any asymmetry among bidders can be captured by the c_s^i functions. There are 3 bidders: A, B and C. Bidders A and B are "local" bidders in the sense that bidder A is only interested in contract 1, $c_1^A(\theta) < \infty$ but $c_2^A(\theta) = \infty$ for all θ and bidder B is only interested in contract 2. By contrast, bidder C is interested in both contracts and he is a "global" bidder. Furthermore, assume that $c_1^A(\theta) = c_1^C(\theta) = c_1(\theta)$ and $c_2^B(\theta) = c_2^C(\theta) = c_2(\theta)$ for all θ,[3] costs are increasing in θ, $c_1'(.)$ and $c_2'(.) > 0$, and that there is no synergy between the two contracts for the global bidder, $c_{1\cup 2}^C = c_1^C + c_2^C$.

It is instructive to first consider the scenario where combination bidding is not allowed. Since the allocation in each market is independent of the outcome and the bids in the other market, and given that bidders are symmetric at the individual market level, the unique equilibrium is symmetric and in strictly increasing strategies. As a result, the outcome is efficient. Moreover, conditional on the optimal reserve price and the usual regularity condition, this simple auction format minimizes procurement costs.

Now suppose that bidders are allowed to submit combination bids, and towards a contradiction suppose there is an equilibrium where bidders only submit stand-alone bids. From the perspective of the global bidder, $(b_1^A, b_2^B) = (B_1^{-C}, B_2^{-C})$ are independently distributed. Therefore, the analysis of McAfee et al. (1989) applies and bidder C will find it advanta-

[3] This means that bidders are symmetric at each object level.

geous to submit a non trivial combination bid at equilibrium, $b^C_{1\cup 2} < b^C_1 + b^C_2$. This means that we can rule out the equilibrium where bidders only submit stand-alone bids (with the "trivial" combination bid $b^C_{1\cup 2} = b^C_1 + b^C_2$). *Combination bidding must take place in any equilibrium.*

In this example, combination bidding hurts efficiency and cost. Efficiency is hurt because whether the local bidder A wins contract 1 or not, no longer depends on bidder A's and bidder C's signals only but also on bidder B's signal (through the combination bid of the global bidder). By the revenue equivalence theorem, and given that the optimal auction is efficient, procurement cost is also higher.[4] The example can be generalized and we have the following result.

Proposition 1 *Suppose that the contracts are independent and that competition in the local markets is symmetric. Then allowing combination bids can only increase procurement costs and hurt efficiency.*

Proposition 1 suggests a class of environments where allowing combination bidding hurts both costs and efficiency. Nevertheless, we have also constructed examples, especially when high synergy levels are present, in which equilibrium bidding with combination bids results in both higher efficiency *and* lower procurement costs relative to the equilibrium outcome of the game in which combination bidding is not allowed.

There are three lessons from this analysis. First, observing a combination bid lower than the sum of bids for the stand-alone constituent units is no guarantee that there are underlying synergies. Submitting a combination bid can be profitable exactly for the same reason why the multi-product monopolist finds price discrimination profitable.

Second, correlation in the environment is an important determinant of combination bidding. In fact, we can show that there is no independent role for combination bidding if bidders' private information is unidimensional (that is, if costs are perfectly correlated across contracts) and bidders are not too asymmetric.

Third, understanding the costs and benefits of combination bidding is an important policy question. One benefit of combination bidding is that it allows bidders to better

[4]The intuition is that the combination bid pools the two markets together and it allows the global bidder to leverage any advantage he has in one market into the other. Indeed, suppose that bidder A has a very high cost realization for contract 1 that is, the global bidder has an advantage in market 1. Then, if the global bidder only submits a combination bid, it reduces the toughness of the competition he faces in the second market because bidder B needs to have a really low bid to compensate bidder A's high bid and have a chance to win. This mechanism, market linkage through combination bidding, is analogous to the leverage theory in industrial organization (Whinston, 1989).

align bids on costs. In fact, combination bidding is a necessary condition for efficiency and optimality (Groves, 1973 and Clarke, 1971; Levin, 1997, Armstrong, 2000 and Avery and Hendershott, 2000). However, our analysis shows that combination bidding may also have perverse effects in the first price auction. This suggests that the question is ultimately an empirical one because the answer depends on the nature and extent of synergies present in the market. This motivates the next section.

3 Identification

This section describes our idenfication results for the multi-unit first price auction. We observe data on bids, contract characteristics and bidder characteristics. Our goal is to infer costs, which we do under two assumptions: (1) the observed data on bids, contract characteristics and bidder characteristics can be used to correctly infer bidders' beliefs about the winning chances of their bids, and (2) bidders choose bids to maximize the interim expected payoff.

The bidding model is identified if the distribution of costs can be uniquely inferred from the observed data. In this section, we provide a new positive result for non-parametric identification in the multi-unit first price auction and show that the model is generically identified when combination bids are allowed. We also illustrate how to obtain identification bounds for the cost parameters when additional restrictions are placed on the set of allowable bids.

Guerre, Perrigne and Vuong (2000) prove non-parametric identification in the one dimensional independent private values setting for single object first-price auctions.[5] Hortacsu (2002) studies the homogeneous multi-unit discriminatory auction for Treasury bills. His identification problem is closer to ours since it is intrinsically multi-dimensional: he observes (a distribution of) demand schedules and his goal is to infer the (distribution of) marginal valuation curves. Extending Guerre et al., Hortacsu proves non-parametric identification in the case where bidders submit demand functions. In practice, his data is coarser (for example, bidders submit a vector of bid quantity pairs instead of continuous demand schedules) and he introduces additional identification assumptions in order to infer the parameters of interest. The difference between our setting with heterogeneous goods and Hortcasu's model of homogeneous and divisible goods is that demand is identified by a *vector* of costs

[5] Other results for single unit auctions include La ont and Vuong (1996) and Li, Perrigne and Vuong (2000) who extend Guerre et al.'s identification result to a liated private values, and Athey and Haile (2001) who analyze the identification problem when bid observations are missing.

$(c_1, ..., c_S) \in R^{2^m-1}$ in our setting whereas it is identified by a marginal valuation *function* in his. This leads to different mathematical structures.

3.1 Identification Conditions

Fix bidder i. We make the following assumption on the distribution of the low bids by bidder i's opponents, B^{-i}:

Assumption 1: B^{-i} has a compact, convex support with full dimension in $\mathbb{R}^{2^m-1}$, and it is distributed continuously on that support.

Let $G_s : R^{2^m-1} \to [0,1]$ denote the (correctly inferred) probability that bidder i's bid vector b^i wins exactly route (combination) s. Let $G_s^t(b^i)$ be its partial derivative with respect to bidder i's bid on contract t. A direct consequence of assumption 1 (proved in the appendix), is that $G_s(b^i)$ is continuous everywhere, and continuously differentiable almost everywhere.[6]

In the general (unconstrained) multi-unit auction, bidders submit bids on all subsets of routes and therefore solve a $2^m - 1$ dimensional problem (in b_s^i, $s \subseteq S$):

$$\max_{(b_s^i)_{s\subseteq S}} \sum_{s\subseteq S} (b_s^i - c_s) G_s(b^i) \tag{2}$$

At any point where this objective function is differentiable, the optimal bid vector by bidder i must satisfy the first order conditions:

$$G_t(b^i) + \sum_{s\subseteq S} (b_s^i - c_s) G_s^t(b^i) = 0 \qquad\qquad t \subseteq S$$

or, in matrix notation:

$$\nabla G(b^i)[b^i - c] = -G(b^i) \tag{3}$$

where the $(2^m - 1)$ by $(2^m - 1)$ matrix ∇G is defined by $\nabla G_{t,s}(b^i) = G_s^t(b^i)$ for $s, t \subseteq S$ and $G(b^i)$ is a $2^m - 1$ x 1 vector with $G_s(b^i)$ as components.

The first order conditions define a *system of linear equations* in the unknown cost parameters, c_s, $s \subseteq S$. Identification of a cost realization then reduces to the question of existence and uniqueness of a solution (the $[b^i - c]$ vector) to this system.

[6] In addition, the left and right derivatives of G_s are always well-defined (by convention, we will use the left hand-side derivative).

Proposition 2 (Identification) *Suppose that assumption 1 holds and define the* $(2^m - 1)$ *by* $(2^m - 1)$ *matrix* $\nabla G(b^i)$ *by* $\nabla G_{t,s}(b^i) = G_s^t(b^i)$. *A sufficient condition for identification in the first price multi-unit auction is that* $\nabla G(b^i)$ *is invertible for all* i *and all* b^i.

Proof. Proposition 2 follows directly from the first order conditions (3). Since (3) defines a system of linear equations in the unknown cost parameters (the $[b^i - c]$ vector), a necessary and sufficient condition for a unique solution $c^i = \phi^i(b^i) \in \mathbb{R}^{2^m-1}$ for all b^i is that the matrix $\nabla G(b^i)$ is invertible. Identification of the distribution of costs F follows directly. Indeed, let $h(b^1, ..., b^N)$ be the equilibrium density of bids. If the system in (3) admits a unique solution $\phi^i(b^i)$ for all b^i and all i, then $h(b^1, ..., b^N)$ defines implicitly a unique distribution of costs $F(c^1, ..., c^N) = \int_{\{(b^1,...,b^N)|\phi^i(b^i)\leq c^i \text{ for all } i\}} h(b)db$. ■

To provide sufficient conditions for the matrix ∇G to be invertible, the following definition will be useful:

Definition: Bid b_s^i by bidder i is *irrelevant* if $G_s(b_s^i, b_{-s}^i) = 0$ and there exists $\varepsilon > 0$ such that $G_s(b_s^i - \varepsilon, b_{-s}^i) = 0$.

Irrelevant bids never win. Nevertheless, irrelevant bids can be optimal from a bidder's perspective because of the leverage motivation for combination bids: submitting a bid that never wins on a contract ensures that this bid does not compete with any other, potentially more profitable, bid. Such bids are problematic for inference. Indeed, suppose bidder i submitted an irrelevant bid on contract s. Then, $(\widehat{b}_s^i, b_{-s}^i)$ for $\widehat{b}_s^i > b_s^i$ would have been equally optimal for bidder i, and therefore equally informative. More concretely, the definition of irrelevant bids implies that $G_t^s(b^i) = 0$ (small changes in b_s^i do not affect the probability that bidder i wins contract s or any other contracts) and $G_s^t(b^i) = 0$ for all t. This means that the row and column corresponding to contract s in matrix ∇G are all zeros. Therefore ∇G cannot be inverted.

Nevertheless, we can still use the information that winning contract s was not profitable for bidder i. Define $b_s^{eff} < b_s^i$ as the highest bid on contract s that would make it just relevant. That is, b_s^{eff} is such that $G_s(b_s^{eff}, b_{-s}^i) = 0$ but $G_s(b_s^{eff} - \varepsilon, b_{-s}^i) > 0$ for all $\varepsilon > 0$. Since payoffs are continuous in bids, bidder i is actually indifferent between submitting (b_s^i, b_{-s}^i) or (b_s^{eff}, b_{-s}^i). In addition, since bidder i found it profitable not to win contract s, it must be that:

$$\frac{\partial}{\partial b_s^i} \sum_t (b_t^i - c_t) G_t(b_s^i, b_{-s}^i) = \delta_s \geq 0 \tag{4}$$

for some nonnegative value of δ_s.[7] We shall see in the next section how this inequality can be used to provide bounds on the unobserved parameters. In the meantime, we prove the following properties of the elements of matrix ∇G:

Lemma 1 *Suppose that assumption 1 holds. Consider matrix $\nabla G_{t,s} = G_s^t$ evaluated at any optimal bid vector b^i by bidder i. Then:*

(1) $G_t^t \leq 0$ for all t

(2) $G_s^t \geq 0$ for all $t \neq s$.

(3) $\sum_s G_s^t \leq 0$ for all t

In addition, if b^i does not contain an irrelevant bid, then (1) *is always strict and (3) is strict for at least one t. As a consequence, ∇G is invertible and the determinant of any submatrix made from removing some rows and the corresponding columns of ∇G has sign (-1)^r where r is the number of remaining rows/columns.*

The proof of lemma 1 can be found in the appendix. Proposition 3 follows directly.

Proposition 3 (Invertibility of ∇G) *Suppose that assumption 1 holds. Then matrix $\nabla G(b^i)$ is invertible at all b^i as long as all bids are relevant.*

Proposition 3 has a number of direct implications for identification in multi-unit auctions:

Corollary 1 *(i) Suppose that assumption 1 holds, then the model is identified if all bids are relevant. (ii) If only stand-alone bids are permitted, then the model is not identified. (iii) If contracts are independent and only stand-alone bids are permitted, then the model is identified.*

Corollary 1 is proved in the appendix. A general lesson is that identification requires the dimensionality of the observed information to match that of the information to be inferred. Specifically, in the multi-unit auction model with multi-dimensional types, the underlying private information to infer (the costs c_s) is $2^m - 1$ dimensional. On the other hand, the set of bids determines the dimensionality of the *observed* information. When bidders make full use of all their bids, as in corollary 1(i), the observed information is $2^m - 1$ dimensional, and identification follows. By contrast, when only stand-alone bids are permitted, the observed

[7] The critical value b_s^{eff} corresponds to a potential point of non-di erentiability in the payo function (see remark 1 following propostion 5 in the appendix). Unless indicated otherwise, we use left-handside derivatives.

information is m dimensional and there is no hope to infer costs, unless the dimensionality of private information is also m.[8]

3.2 Incorporating Restrictions of the London Bus Market

In the previous section, we have seen that the presence of irrelevant bids may be problematic for identification. In addition, real-life auctions include various restrictions on the set of allowable bids which may also lead to the violation of the conditions in corollary 1.

In this section, we show how to extend our identification results and derive identification bounds in these cases. We illustrate our approach by considering two types of restrictions present in our data. First, the rule of the auction imposes that bids on combination of contracts must be no greater than the sum of the constituent bids. Second, London Transport Buses imposes a (secret) reserve price. In addition, bidders are not obliged to submit bids on all routes and some bidders indeed submit bids only on a subset of the routes auctioned in any particular tranche. Our interpretation is that it was not profitable for these bidders to submit a bid that would have had a positive chance of winning.

Together, these restrictions mean that the optimization problem that any typical bidder faces becomes:

$$\max_{b^i} \sum_{s \subseteq S} (b^i_s - c_s) G_s(b^i) \tag{5}$$

subject to a combination bid constraint:

$$b^i_s \leq b^i_w + b^i_t \text{ for all } s, t, w \subseteq S \text{ such that } t \cap w = \emptyset \text{ and } t \cup w = s \tag{6}$$

and a reserve price constraint in the event bidder i does submit a bid on any given contract:

$$b^i_s \leq R_s \text{ for all } s \subseteq S \tag{7}$$

(by convention, we will write $b^i_s = \infty$ if bidder i does not submit a bid on contract s).

The fact that bidders do not need to submit a bid on all contracts introduces a discontinuity in the payoff function. Moreover, because of the multi-unit nature of the auction, we cannot transform problem (5) subject to (6) and (7) into an optimization problem of

[8] In addition, if the inference ignores the multi-unit nature of the auction (by treating each object as a separate auction), it can be shown that the cost estimates are biased downward if contracts are complements and upward if they are substitutes.

a *continuous function* over a *compact set* (for example, by imposing compulsory bidding) such that any solution to the original problem is a solution to the transformed problem.[9]

Nevertheless, we can still use the logic of the standard approach to optimization under constraints and use first order conditions to infer costs. First, whenever a bidder submits a relevant bid on all contracts, the Kuhn-Tucker conditions for the optimization problem (5) subject to (6), (7) *and* compulsory bidding provide a proper description of the bidder's optimization problem.

Second, whenever one of these bids, say b_s^i, is irrelevant, the first order condition with respect to that particular contract can be evaluated at (b_s^{eff}, b_{-s}^i) where $b_s^{eff} < b_s^i$ makes bidder i's bid just relevant, as suggested in the previous section.

Finally, first order conditions can also be used for contracts on which bidder i did not submit a bid. Indeed, suppose that $b_s^i = \infty$. This means that bidder i's expected payoff from not submitting a bid on contract s is greater or equal to the expected payoff he would get from submitting a bid equal to the reserve price. Formally,

$$\sum_{t \neq s}(b_t^i - c_t)G_t(b_s^i = \infty, b_{-s}^i) \geq \sum_{t \neq s}(b_t^i - c_t)G_t(R_s, b_{-s}^i) + (R_s - c_s)G_s(R_s, b_{-s}^i)$$

Since bidder i's probability of winning contract $t \neq s$ is the same whether bidder i submits a bid equal to the reserve price on contract s or does not submit a bid on that contract, we have $G_t(b_s^i = \infty, b_{-s}^i) = G_t(R_s, b_{-s}^i)$ for $t \neq s$, and the expression above reduces to:

$$c_s \geq R_s \tag{8}$$

In turn, this implies that $(b_s^i - c_s)G_s(b_s^i, b_{-s}^i)$ is increasing on $b_s^i \leq R_s$. Together with the fact that $G_t^s(b^i) \geq 0$ for all $t \neq s$, expected payoff is increasing in b_s^i and in particular:[10]

$$\frac{\partial}{\partial b_s^i} \sum_t (b_t^i - c_t)G_t(b_s^i, b_{-s}^i)|_{b_s^i = R_s} = \mu_s \geq 0$$

[9]To see this, suppose that at the optimum $b^*$of (5), bidder i submits a relevant bid on contract s but not on contract t. Then, his first order condition with respect to b_s^i takes into account the fact that b_s^i does not have any externality on contract t: $G_t^s(b^*) = 0$. However, this property does not necessarily hold if all bids satisfy the reserve price constraint. As a result, b_s^* may no longer satisfy the first order condition in the transformed problem.

[10]If R_s is an irrelevant bid, the derivative should be evaluated at the bid b_s^{eff} that makes a bid on contract s just relevant.

To summarize, given any observed bid vector b^i, we propose to evaluate bidder i's first order condition with respect to contract s at (b_s^{eff}, b_{-s}^i) where b_s^{eff} is the effective bid associated with b_s^i and it satisfies the following definition:

Definition: Consider any bid b_s on contract s. Its associated *effective bid*, b_s^{eff} is the highest value of a bid on contract s such that b_s^{eff} is relevant and satisfies the reserve price, subject to the condition that $b_s^{eff} \leq b_s$.

Clearly, effective and actual bids only differ in the cases of non submitted bids or irrelevant bids. These are also the cases where, as we have argued, the first order condition needs to be evaluated at the value of the bid that makes it just relevant. In addition, since $G_s^t(b_s^i, b_{-s}^i) = 0$ for all $t \neq s$, and for all s such that b_s^i is irrelevant, these adjusted first order conditions can still be expressed as a system of linear equations of the form

$$\nabla G(b^i)[b^{eff} - c] = D \tag{9}$$

where the ∇G matrix is now defined as

$$\nabla G_{t,s}(b^i) = G_s^t(b_s^{eff}, b_{-s}^i),$$

the $[b^{eff} - c]$ vector is now evaluated at the associated effective bid vector, and the D column vector collects the G functions with the Lagrangian multipliers:

$$D_s(b^i; \delta, \mu, \lambda) = -G_s(b_s^{eff}, b_{-s}^i) - \delta_s - \mu_s - \textstyle\sum_{r,t} \lambda_{s=r\cup t} + \sum_{r,t} \lambda_{t=s\cup r}$$

According to our arguments above, δ_s is positive if b_s^i is irrelevant, μ_s is positive if bidder i did not submit a bid on contract s, and $\lambda_{s=r\cup t} \geq 0$ is the Lagrangian multiplier for the constraint $b_s^i \leq b_r^i + b_t^i$ for $s = r \cup t$ and $r \cap t = \emptyset$.

The same arguments as in lemma 1 can be used to prove that matrix ∇G with rows evaluated at (b_s^{eff}, b_{-s}^i) is invertible and negative definite. This means that

$$[b^{eff} - c] = \nabla G^{-1}(b^i) D(b^i; \delta, \mu, \lambda) \tag{10}$$

is the solution to (9). Expression (10) is important. It says that, given any fixed values for the Lagrangian multipliers, costs are uniquely identified from the bid observation. In particular, if no constraint is binding, all the Lagrangian multipliers are equal to zero and costs are identified: we are back to the special case of corollary 1. By contrast, any binding constraint introduces a degree of underidentification because we do not observe the value of the Lagrangian multiplier that solved bidder i's optimization problem when he (optimally)

chose to submit bid vector b^i. The only thing we know from the previous discussion is that the multiplier of the binding constraint is positive. The next proposition characterizes more precisely the extent of the underidentification in the cost parameters:

Proposition 4 (Identification bounds) *Any binding constraint introduces a one-dimensional degree of underidentification in the cost vector c with the following properties: (1) c_s depends positively and linearly on the value of the multipliers μ_s and δ_s; but it is independent of μ_t and δ_t for all $t \neq s$; (2) c_s depends positively on $\lambda_{s=t\cup w}$, and negatively on $\lambda_{t=s\cup w}$ for all s, t and w.*

The proof of proposition 4 can be found in the appendix. It uses Cramer's rule and the properties of determinants to sign how the solution to (10) varies with the value of the Lagrangian multipliers.

There are several important elements to note in proposition 4. First, irrelevant bids and non submitted bids only affect the identification of the cost parameter of the associated contract. This is somewhat remarkable in this multi-unit auction setting where costs are a priori jointly determined as the solution to a system of equations. The reason is that, in the case of irrelevant or non submitted bids, bids on *other* contracts do not affect the probability that bidder i wins the contract on which he either did not submit a bid, or submitted an irrelevant bid (it remains zero). Concretely, let contract s be such an "irrelevant" contract. Then $G_s^t(b_t^{eff}, b_{-t}^i) = 0$ for all $t \neq s$. This removes the interdependency between all the equations as far as the identification of costs c_s and c_t for $t \neq s$ is concerned. A direct consequence is that, in the event of non submitted bids, we can infer that the costs of the contract was higher that the reserve price – exactly as in the single unit first price auction (see (8) and Guerre et al., 2000).

Second, proposition 4 allows us to derive bounds on the value of the cost parameters that can rationalize the observed bids. For example, suppose that constraint $b_s^i \leq b_t^i + b_w^i$ is binding. The solution to (10) evaluated at $\lambda_{s=t\cup w} = 0$ provides a lower bound to the cost parameter c_s. Of course, given the analysis in section 3, a parameter of greater interest still is the extent of synergies between contracts t and w. The next result is proved in the appendix.

Corollary 2 *Consider any 2 disjoint contracts, t and w. If the combination bid constraint for these contracts is binding at the optimum, an upper bound to the synergy involved between these two contracts is given by the solution $c_t + c_w - c_{t\cup w}$ of the system in (10) when the Lagrangian multiplier $\lambda_{\{t\cup w\}=t\cup w}$ is set equal to zero.*

3.3 Identification of the Cost Distribution Function [to be added]

Proposition 4 characterizes the extent to which costs are underidentified from the bid observation. This does not mean that the *distribution* of costs is underidentified.

4 Estimation Method

This section describes our estimation approach. Section 4.1 describes our parametric density specification. Section 4.2 describes a simulated method of moments estimator for the bid density function. Section 4.3 takes the bid density function as given and describes our numerical method to infer costs.

We observe data on a cross section of auctions $t = 1, \ldots T$. Let b^{it} denote the bid vector of bidder i submitted for the contracts in auction t and let $X^t = (x^t, w^{1t}, \ldots, w^{Nt})$ denote the route and bidder characteristics on auction t. $w^{-i,t}$ denotes the vector of characteristics for bidders other than bidder i and we sometimes write $X^{it} = (x^t, w^{it}, w^{-i,t})$ where the i superscript indicates that bidder characteristics are evaluated from bidder i's perspective. In this section, we make the following further assumption on the data generation process.

Assumption. The cost realizations for bidder i are stochastically independent of the cost realizations of bidder j, for all $j \neq i$, conditional on characteristics X^t.

Furthermore, we assume that characteristics X^t are observable to the bidders and the econometrician. We do not consider bidder or contract heterogeneity that is not observed to the econometrician.

4.1 A Multivariate Bid Density Function

We specify the density function of (latent) bids b^* of bidder i in auction t as a multivariate log-normal density function in which the parameters are a (linear) function of bidder and auction characteristics X^{it}. Given our independence assumption on the cost draws across bidders conditional on X^{it}, the bids by bidder i are stochastically independent from the bids of bidder j conditional on characteristics X^{it}.

The statistical model for latent bids by bidder i on auction t is:

$$\begin{bmatrix} \ln(\frac{b_1^*}{IC_1} - \gamma) \\ \vdots \\ \ln(\frac{b_S^*}{IC_S} - \gamma) \end{bmatrix} = \mu(X^{it}) + \Lambda(X^{it}) \cdot \varepsilon \tag{11}$$

In (11), latent bids are normalized by an internal cost estimate IC_s, μ denotes the $(2^m - 1)$ dimensional vector of means, Λ is a $(2^m - 1)$ by $(2^m - 1)$ matrix and ε is a $(2^m - 1)$ dimensional standard normal random variable: $\varepsilon \sim N(0, I)$ with 0 denotes the null vector and I denotes the identity matrix. We assume that the mean μ_s and Λ_{st} are linear functions of characteristics,

$$\mu_s(X^i) = \beta_s X^{it} \quad \text{and} \quad \Lambda_{s\tau}(X^{it}) = \alpha_{s\tau} X^{it}$$

Thus, $\beta = [\beta_1, \ldots, \beta_S]$ is a $(2^m - 1) \times k$ dimensional parameter matrix where k denotes the number of explanatory variables in X and $\alpha = [\alpha_{s\tau}]$ is a $(2^m - 1) \times (2^m - 1)$ dimensional matrix. Each element $\alpha_{s\tau}$ is a parameter vector of dimension k. Notice, that our specification implies that the variance-covariance matrix of log bids is given by $\Sigma = \Lambda\Lambda'$.

Latent bids above the reserve price are not observed. As in Laffont, Ossard and Vuong (1995), we define the observed bid as equal to the reserve price, R_s^t, when the latent bid is not observed:

$$b_s^{it} = b(b^*|X^{it}) = b_s^{*it} \cdot 1_{\{b_s^{*it} < R_s^t\}} + R_s^t \cdot 1_{\{b_s^{*it} \geq R_s^t\}}$$

Henceforth, we use this convention and restrict attention to observed bids.

There is a large literature on estimation methods of the parameters of a lognormal density function (see Griffiths (1980)). Proposed methods include maximum likelihood and method of moments. Assuming that the lower bound γ is known, the regularity conditions of maximum likelihood are satisfied, and maximum likelihood yields consistent and efficient estimates of the parameters (α, β) as the number of auctions T gets large. If the lower bound γ is not known and is to be estimated, then a regularity condition of maximum likelihood is violated. The method of moments provides an alternative estimation method that yields consistent estimates for the parameter vector $\theta = (\alpha, \beta, \gamma)$, as the number of auctions T increases (see Hansen (1982)). Numerical calculation of the moment conditions can be computationally intensive as our density is multivariate. Simulation estimators (McFadden (1989) and Pakes and Pollard (1989)) provide an elegant solution to this problem. The next section describes our estimator.

4.2 A Simulated Method of Moments Estimator

Consider the difference between the observed and the (conditional) expected kth moment of the bid vector:

$$v^{itk} = (b^{it})^k - E\left[b^k | X^{it}, \theta\right]$$

The symbol $(b^{it})^1$ denotes the vector of first moments $(b_1^{it}, \ldots, b_S^{it})$, the symbol $(b^{it})^2$ denotes the vector of second moments $((b_1^{it} b_1^{it}), b_1^{it} b_2^{it}, \ldots, (b_S^{it} b_S^{it}))$ and so on. Notice, that the difference v^{itk} when evaluated at the true parameter value θ^*, is mean independent of the exogenous data:

$$E\left[v^{itk} | X^{it}, \theta = \theta^*\right] = 0$$

Given this condition together with some standard regularity conditions, we can adopt the method of moments estimator described in Hansen (1982). The estimating equation is given by

$$(b^{it})^k = E\left[b^k | X^{it}, \theta\right] + v^{itk}$$

Unfortunately, $E\left[b^k | X^{it}, \theta\right]$ is the kth moment of a truncated multi-variate normal random variable, which is numerically time-consuming to calculate. We solve the integration problem by replacing the difficult to calculate expected value with a simulated, unbiased estimate. The expected kth order moment of the observed bid can be written as:

$$E\left[b^k | X^{it}, \theta\right] = \int \cdots \int \left[(b(b^*(\varepsilon | X^{it}, \theta) | X^{it}, \theta)^k \frac{\phi(\varepsilon)}{\varphi(\varepsilon)}\right] \varphi(\varepsilon) d\varepsilon_1 \cdots d\varepsilon_S$$

by multiplying and dividing the integrand by the importance function $\varphi(\cdot)$ and where ϕ denotes the multivariate standard normal density for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_S)$, and the function b^* is implicitly defined in equation (11).[11]

Given a fixed set of random draws, ε_{it}, for each bidder and auction, we can calculate an estimate, $\widehat{b^k}$, with the property:

[11]

$$b^*(\varepsilon_1, \ldots, \varepsilon_S | X^{it}) = IC \cdot \left[\exp\left(\mu(X^{it}) + \Lambda(X^{it}) \cdot \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_S \end{pmatrix}\right) + \begin{pmatrix} \gamma \\ \vdots \\ \gamma \end{pmatrix}\right]$$

$$E_\varepsilon \widehat{b^k}(X^{it},\theta,\varepsilon) = E\left[b^k|X^{it},\theta\right]$$

The estimate leads to a new estimating equation

$$b^{itk} = \widehat{b^k}(X^{it},\theta,\widehat{\varepsilon}) + \widehat{v}^{itk} \tag{12}$$

Since $\widehat{b^k}$ is an unbiased estimator of $E\left[b^k|X^{it},\theta\right]$, the new prediction error, $\widehat{v}^{itk}$, is also mean independent of the exogenous data at the true parameter values. This suggests a method of moments technique may still be appropriate to estimate θ. Under regularity conditions which are satisfied here, McFadden (1989) and Pakes and Pollard (1989) show that this suggestion is correct.

Our estimator of the above form is found by taking, for each bidder and auction, independent draws from the multivariate importance function φ of errors $\varepsilon = (\varepsilon^{11},\ldots,\varepsilon^{NT})$. As importance function φ we use the standard multivariate normal denisty ϕ. A simulated moment estimator is defined as the average across multiple simulated draws to reduce the variance of the estimate while preserving unbiasedness. This is, we take L draws, $\varepsilon = ($ $\varepsilon^1,\ldots,\varepsilon^L)$ and calculate

$$\widehat{b^k}(X^{it},\theta,\widehat{\varepsilon}) = \frac{1}{L}\sum_{l=1}^{L}\left[(b(b^*(\varepsilon_1^{itl},\ldots,\varepsilon_S^{itl}|X^{it},\theta)|X^{it},\theta)^k \frac{\phi(\varepsilon_1^{itl},\ldots,\varepsilon_S^{itl})}{\varphi(\varepsilon_1^{itl},\ldots,\varepsilon_S^{itl})}\right]$$

This equation for $\widehat{b^k}$ can then be substituted into the estimating equation (12). Let $\widehat{v}^{it}$ denote the resulting vector moment prediction errors for bidder i on auction t. It consists of the (2^m-1) vector of bids for individual route combinations stacked on top of the $(2^m-1)(2^m)/2$ elements of cross products of bids for individual route combinations, and so on. We denote a typical element of the vector by $\widehat{v}^{itl}$.

Let W^{itk} be a matrix of instruments for the kth moment prediction error $(k=1,\ldots,K)$. We can write the instrument matrix for bidder i on auction t, W^{it}, as a block diagonal matrix:

$$W^{it} = \begin{bmatrix} W^{it1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W^{itK} \end{bmatrix}$$

The method of moment estimation technique is based on the restriction that $\widehat{v}^{it}$ is uncorrelated with the exogenous data W^{it}. With NT independent observations, the sample analog of this restriction involves the sample correlation

$$\overline{g}(\theta) = \sum_i \sum_t \sum_l g_{itl}(\theta) = \sum_i \sum_t \sum_l \widehat{v}^{itl}(\theta) \otimes W^{itl}$$

where the vector of instruments W^{itl} has at least as many elements as there are parameters in θ. An estimate $\widehat{\theta}$ is chosen to minimize a quadratic distance measure

$$G(\theta) = \overline{g}(\theta)' A \overline{g}(\theta)$$

for some positive definite matrix A. A preliminary estimate $\widehat{\theta}_1$ is obtained by setting A equal to the identity matrix. Then, a new weighting matrix is calculated as the sample variance of the individual moment conditions, $g_{itl}(\widehat{\theta}_1)$. A second and final estimate, $\widehat{\theta}$, is then obtained from the use of this moment condition. The variance of $\widehat{\theta}$ is estimated by the formula given, for example in Pakes and Pollard (1989).

A Monte Carlo study revealed that the estimator is well behaved even for small number of observations. Moreover, the first two moments, $k = 1, 2$, are sufficient to identify the parameter vector θ.

4.3 Inference of the Cost Distribution Function

This section describes our numerical approximation technique to infer the cost distribution function based on estimates of the bid density.

Let $h(b_1^i, ..., b_s^i, ..., b_S^i | X^i)$ denote the estimated probability density function of bids by bidder i on all the subsets of S conditional on X^i. The probability that bidder i's bid (vector) b^i wins exactly route (combination) s conditional on X^i, $G_s(b^i|X^i)$, can be written as a function of the density of the lowest bids by i's opponents conditional on X^i, which we denote as $h_{(1)}(B_1^{-i}, ..., B_S^{-i} | X^i)$. Notice further that the density $h_{(1)}(B_1^{-i}, ..., B_S^{-i} | X^i)$ can be expressed directly as a function of the bid density $h(.|.)$. The analytical expressions involves multi-dimensional integrals which are complex to calculate numerically.

Following Judd (1998), we solve the described integration problem using Monte Carlo integration methods. The method is based on the law of large numbers and can be explained as follows: For each bidder $j \neq i$ we draw a bid (vector) from the density $h(.|X^j)$ conditional on the characteristics X^j. We then determine the low competitors'

bids $(B^{-i}|X^i) = (B_1^{-i}, ..., B_S^{-i}|X^i)$. We repeat this exercise L times by repeatedly drawing bids and determining the low competitors' bids. The "pseudo data" of competitors' low bids, $(B_l^{-i}|X^i)_{l=1}^L$, is then used directly to approximate the probability that bidder i wins exactly route s with the bid b^i, $G_s(b^i|X^i)$. The empirical frequency of this event is given by:

$$G_s(b^i|X^i) = \frac{\sum_l 1_{\{\text{bid } b^i \text{ wins exactly route } s| \ (B_l^{-i}|X^i)\}}}{L},$$

where $1_{\{x\}} = 1$ if x is true and 0 otherwise. By the law of large numbers, the approximation error vanishes as L increases. The derivative G_s^t, can be calculated numerically by using one sided differences with $\varepsilon(L)$ appropriately chosen. The numerical difference yields,

$$G_s^t(b^i|X^i) = \frac{G_s(b_1^i, ...b_t^i, ..., b_S^i|X^i) - G_s(b_1^i, ...b_t^i - \varepsilon, ..., b_S^i|X^i)}{\varepsilon}.$$

Next, we describe how we determine the cost density and distribution function. We approximate the cost density function using a step function on a specified grid. We partition the bid (and cost) space into M intervals, $C_s = \{[\frac{(i-1)R_s}{M-1}, \frac{iR_s}{M-1})\}_{i=1}^{M-1} \cup [R_s, \infty)$.[12] The first $M-1$ intervals are of length $\frac{R_s}{M-1}$ each. The last interval, $[R_s, \infty)$, accounts for unobserved bids. We take the Cartesian product of the partitions in each dimension and specify the grid in the bid space as $C = X_{s \in S} C_s$. Observe that the grid C consists of $n = M^{2^m - 1}$ cubes and each cube is of dimension $2^m - 1$. The fineness of the grid is determined by the number M, and our numerical approximation becomes more accurate as we increase M. We denote a typical element of C by C_l. The probability of the cube C_l in the bid space is denoted by h_l. It is given by:

$$h_l(X^i) = \int_{C_l} h(b|X^i) db.$$

In order to determine the associated probability of the cube C_l in the cost space, we conduct the following calculation: For each $l = 1 \ldots n$, we select a bid vector $b_l \in C_l$. Without loss of generality we chose b_l as the midpoint of the cube C_l. The previous section describes how to obtain the cost range associated with b_l based on the Kuhn-Tucker conditions for optimal bids. We denote the associated cost range with $\phi_l(b_l|X^i)$. Notice, that

[12] For simplicity of exposition, we assume that the cost space equals the bid space. This assumption is not required and can be relaxed.

the cost range can be either a singleton, or a path or a higher dimensional area. Moreover, the cost range can be calculated by varying the Lagrange multiplier(s) between 0 and ∞. The value of the cost density function on the cube C_l equals the frequency with which cost observations fall into the cube. If the cost range is a singleton, then the frequency is exactly determined, while if the cost range is not a singleton, then this translates into an upper and a lower bound on the cost distribution function only.

We specify the density function of the cube C_l for a lower and an upper bound of the cost distribution function using the componentwise maximum and minimum of the cost range,

$$\begin{aligned} f_l^{\inf} &= \frac{1}{n}\sum_{k=1}^{n} 1_{\{\sup(\phi_k(b_k|X^i))\subset C_l\}} \cdot h_k(X^i), \\ f_l^{\sup} &= \frac{1}{n}\sum_{k=1}^{n} 1_{\{\inf(\phi_k(b_k|X^i))\subset C_l\}} \cdot h_k(X^i). \end{aligned}$$

where $\sup(\phi)$ (and $\inf(\phi)$) are the componentwise largest (and smallest) element in the set ϕ.

Our estimator of the cost distribution function is then the cumulative distribution function associated with the lower and upper bound on the empirical frequency densities. They are given by:

$$\begin{aligned} F^{\inf}(c_1, ..., c_S|X^i) &= \sum_{l=1}^{n} f_l^{\inf} \cdot 1_{\{C_l \subset \{c'|c_1' \leq c_1, ..., c_S' \leq c_S\}\}}, \\ F^{\sup}(c_1, ..., c_S|X^i) &= \sum_{l=1}^{n} f_l^{\sup} \cdot 1_{\{C_l \subset \{c'|c_1' \leq c_1, ..., c_S' \leq c_S\}\}}. \end{aligned}$$

We can make three observations: First, our numerical approximation entails an error and the error becomes negligible as M and L increase. Second, we calculate the standard errors for cost estimates using the delta method. Third, if no constraint is binding, then the upper and lower bound on the cost distribution function coincide.

5 The London Bus Market

This section describes the London bus market and provides descriptive summaries of our data.

The London bus market represents about 800 routes serving an area of 1,630 square kilometers and more than 3.5 million passengers per day. It is valued at 600 million Pounds per year (US $900 million). Deregulation was introduced by the London Regional Transport Act of 1984. Prior to deregulation, bus services in the Greater London area were provided by the publicly-owned London Buses Limited. The Transport Act created London Transport Buses as the authority responsible for bus service procurement, planning, development and operations of bus stations and network-wide operational support and maintenance. In order to enhance competition, the Act split the formerly unitary London Buses into 12 operational subsidiaries. These were later privatized in 1994. In practice, the introduction of route tendering was very gradual and it was not until 1995 that the whole network was tendered at least once.[13] Since then, tendering has reached its steady state regime with 15-20% of the network tendered every year.

The procurement process: About every two weeks London Transport Buses issues an invitation to tender which provides a detailed description of upcoming contracts for sale. The invitation simultaneously covers several routes, usually in the same area of London (the set of routes that corresponds to an invitation is called a *tranche*). For each route, the invitation provides a complete description of the service for tender including the routing, service frequency and vehicle type. Contract length is typically five years. A set of pre-qualified operators may submit sealed bids for individual routes. In addition, operators may submit a bid for route combinations within the tranche. A bid specifies an annual price at which the operator is willing to provide the service.[14] There is a period of two months between the invitation to tender and the tender return date, and another two months before contracts are awarded. The official award criterion is *best economic value* and the process follows EU law for fair competition.[15] In practice, this means that the contract is awarded to the low bidder but deviations at the margin are possible to account for operator quality

[13]Non-tendered routes remained operated by the subsidiaries of London Buses Limited under a negotiated block grant. The private operators and the subsidiaries competed for the tendered services.

[14]London Transport Buses has experimented with di erent contractual forms. The majority of contracts are so called gross cost contracts, in which the revenues collected on the buses accrue to London Transport Buses and the operator receives a fixed fee for the service. Some contracts are net cost contracts, in which the operators take responsibility for the revenues. The price for the operator service then consists of those revenues plus a transfer from (or payment to) London Transport Buses. Finally, net cost contracts may contain a provision that limits the risk the operator takes in case the revenues were too di erent from the forecast. If bidders are risk neutral, which we assume in our analysis, all three contracts forms are equivalent.

[15]EEC directive 93/38.

for instance.[16] To allow winning operators to reorganize and order new buses if necessary, contracts start 8 to 10 months after the award date.

The market for bus operators. Bus operators tend to be organized in groups with operational subsidiaries active in local areas of London. As of November 2000, there were 64 pre-qualified bus companies in the London area. Because operational companies that belong to the same group don't bid against one another, we define the bidding entity at the group level, and refer to it as a "bidder" or an "operator." This yields 51 independent pre-qualified bidders in the market. After the privatization of the London Buses subsidiaries in 1994, a substantial reorganization and consolidation of the industry took place. Since then, the market has stabilized with a C4 ratio around 70% between late 1996 and 2000.

Description of the data.[17] We have collected data on 176 tranches consisting of a total of 674 routes offered to operators between December 1995 and May 2001. For each tranche and for each route in the tranche, the data include the following information: (1) contract duration and planned start of the contract (2) route characteristics including the route start and end points; route type (day route, night route, school service, mobility route); annual mileage; bus type including single deck, midibuses, double deck or routemaster; and the peak vehicle requirement;[18] (3) the identity of bidders and all their submitted bids (including bids for combinations of the routes in the tranche). For the auctions held starting in May 2000, the data also contain an internal cost estimate for every route which was generated by London Transport Buses.

The majority of the auctions consist of few routes. A total of 116 tranches have 1, 2 or 3 routes.[19] In the following we omit all data for tranches with 4 or more routes and consider only tranches with 1, 2 or 3 routes.

Table 1 illustrates that on average 3.7 bidders submit at least one bid on a tranche. The number of bidders ranges between 1 and 12. Fewer bids are submitted on individual routes. On average 2.9 operators submit a bid for an individual route. The number of bids per route ranges between 1 and 7. A total of 40 bidders submit at least one bid during the

[16] The empirical analysis revealed no systematic patterns in these considerations that we could model explicitly. We interpret the considerations at the margin as noise in the awarding process.

[17] Appendix A provides further details concerning the sources of the data.

[18] The peak vehicle requirement depends on the frequency of service. It determines how many buses the winning operator will need to commit to the contract.

[19] The distribution of routes across tranches is the following: 48 tranches consist of a single route, 35 tranches have two routes, 33 tranches have 3 routes, 12 tranches have 4 routes, 12 have 5 routes, 25 tranches have between 6 and 10 routes, and 11 tranches have more than 10 routes.

Table 1 Descriptive Summary of the Data*

Variable	Obs	Mean	Std	Min	Max
No-Bidders-per-Tranche	113	3.69	1.82	1	12
No-Bidders-per-Route	210	2.91	1.55	1	7
No-Garages-per-Bidder	40	3.45	4.13	1	21
Log-Stand-Alone-Bid	614	13.10	1.28	9.47	15.87
Log-Combination-Bid	77	14.47	0.74	11.75	15.89
Money-Left-on-the-Table	169	13.9	0.21	0	1.6

* We consider tranches with 1,2 and 3 routes.

sample period. Of those, 29 win a contract. The average bidder submits a bid from 3.5 garages.

We have also collected data on the industry. For each bus operator active in the tendered bus services in London, we have a complete history of its depots (openings/first time use for the tendered market and closings, location) since deregulation, as well as its committed fleet for the tendered market (on a monthly basis, by bus type). Depots are leased on a long term basis or bought, and a typical garage has capacity for 50-100 buses and serves about 8 routes. For our sample period, 31 independent groups have garages serving London bus routes, for a total of 109 garages.[20]

There is considerable asymmetry among bidders. A total of 15 operators have one garage, 4 operators have two garages, 4 operators have 3 garages, one operator has 5 garages, one operator has 6 garages, one operator has 8 garages, two operators have 9 garages, one operator has 11 garages, one operator has 17 garages and 1 operator has 21 garages. The asymmetry is also reflected in the distribution of market shares in the value of awarded contracts in our sample. The largest operator won 20.5% of the market, the second largest won 16.5%, the third largest won 14.8%, the fourth won 13.0%, the fifth won 9.9%, the sixth won 7.5%. The remaining 23 operators won less than 3% each.

Operators submit a total of 614 stand-alone bids. The distribution of stand-alone bids resembles a log-normal distribution. The average stand-alone bid equals 13.1 in logarithm which amounts to about 490,000 Pounds.

Combination bids: On the tranches with two and three routes a total of 77 non trivial combination bids are submitted.[21] Non trivial combination bids account for 17% of all

[20]In addition, some of the bidders in our sample are "entrants" who do not have an established garage.

[21]Because bidders are committed by their bids, stand-alone bids define implicitly a combination bid (with value equal to the sum of the stand-alone bids). We call a combination bid "non trivial" when it is less than

submitted bids on these tranches. The average combination bid equals 14.47 in logarithm which amounts to about 1,500,000 Pounds. A total of 50 routes of 163 awarded routes were won by combination bids.

On average, the discount of a combination bid relative to the sum of stand-alone bids by the same company equals 4.5%.[22] The discount amounts to 3.9% with two-route bids, 7.7% with 3-route bids. Reasons invoked by operators to offer discounts for combinations of bids include the possibility to share spare vehicles and depot overhead costs in general, and more efficient organization and coordination of working schedules.

Contract heterogeneity. There are many dimensions along which the routes in our sample vary including annual mileage and peak vehicle requirement. Route characteristics affect costs and, ultimately, participation and bids. A monetary measure of contract heterogeneity is the internal cost estimate prepared by London Transport Buses for each route since May 2000. We find that the internal cost estimate is an accurate assessment of the final cost. We considered a regression of the log of low bids on the internal cost estimate. The regression explains about 95% in the variation of low bids. Below, we use the internal cost estimate to account for contract heterogeneity. This helps us keep the number of explanatory variables in our empirical specification small.

The determinants of uncertainty and bidder participation. We next assess the amount of uncertainty in bids. A measure of uncertainty is the relative difference between the lowest and second lowest stand-alone bid, or money left on the table. It equals 13.9%. Thus, stand-alone bids overpay by about 68,000 Pounds on average. This suggests that there is considerable uncertainty about the competitors' bids.

As the number of bidders increases, the amount overpaid decreases. The money left on the table equals 21.3% when two bids are submitted, 11.9% when three bids are submitted, 8.9% when four bids are submitted and 8.5% when five or more bids are submitted. Thus, even as the number of bids submitted increases, the uncertainty does not vanish. With five or more bidders the amount overpaid for the average contract equals almost 41,600 Pounds.

What determines the uncertainty in bids? At the operator level, costs are determined in part by the actual expenses in capital, labor and fuel incurred in carrying out the contract. But they also depend on the opportunity of using these resources, especially capital, in

the sum of the component stand-alone bids.

[22]We omit bids for which no combination bid was submitted. This exaggerates the amount of the combination bid discount since we only observe a combination bid, if it is less than the sum of stand-alone bids. On the other hand, we also omit from the statistic two bid observations for which not all stand-alone bids of constituent units were submitted.

other ways. There is probably little uncertainty among operators concerning the expected cost of labor or fuel (there are well functioning markets for these), but opportunity costs may not be known to other operators. Thus, *uncertainty in this market may be best viewed as stemming from private information about (opportunity) costs.*

An important question for modelling bidding behavior in the London bus routes market is to determine whether cost uncertainty arises at the firm, tranche, or route level. In other words, does the opportunity cost vary at the firm level, the garage level or route level? To examine these questions we decompose the variation in the bid submission decision. We examine how much of the variation in the decision variable is explained by tranche-depot fixed effect, route fixed effect as well as dead mileage (closest distance from the route to the garage).

As described above there is considerable asymmetry between bidders. Small bidders operate locally and submit only a small number of bids. For this reason, we focus on bidders with at least 20 bids which leaves us with the largest 13 bidders.[23] Due to the large number of explanatory variables, we consider the linear probability model and estimate it using OLS. The empirical model is:

$$y = Z\lambda + u$$

where $y = 1$ if a bid is submitted and zero otherwise, Z denotes a vector of explanatory variables and u denotes the residual.

Table 2 reports the R-squared and the adjusted R-squared for five specifications of the decision to submit a bid. The individual specifications gradually add more variables to X. We determine for each specification what fraction of the variation in the dependent variable is explained. We interpret the increase in the fraction as a measure of the importance of the added variables.

Model (1) considers dead mileage (the distance from the garage to the route) as the only explanatory variable. Dead mileage (linear and quadratic) explains 24% of the variation in bid submissions and the adjusted R-squared equals 0.24.

Model (2) adds tranche fixed effects to the explanatory variables. In total we add 66 dummies. The explanatory variables explain 30% of the variation in bid submission and the adjusted R-squared equals 0.28. We test the hypothesis that tranche fixed effects are

[23]To conduct our analysis we omit data points according to the following criteria. First, we omit mobility routes (29 observations). Second, we consider only tranches with two and three routes. We omit tranches with one route (570 observations). We are left with 2,241 observations.

Table 2 Variance Decomposition of the Bid Submission Decision*

Model	Variables Included	No of Variables	R-Squared	Adj. R-Squared
(1)	Dead Mileage (DM)	3	0.24	0.24
(2)	DM+Tranche Fixed Effects (TF)	68	0.30	28
(2')	DM+TF+Route Fixed Effects	163	0.33	0.28
(3)	DM+TF+ Operator Fixed Effects	80	0.31	0.29
(4)	DM+TF+Depot Fixed Effects	148	0.42	0.38

* We consider tranches with 2 and 3 routes.

equal to zero. The test statistic is an F-distributed random variable with (65,1535) degrees of freedom. It equals 3.14 and we can reject the null of no significant tranche effects.

Model (2') considers route fixed effects instead of tranche fixed effects. The specification includes 161 route dummies. The variables explain 33% of the variation in bids, but the adjusted R-squared remains at 0.28 as in model (2). This evidence suggests that there is little uncertainty common to all bidders at the route level. We test whether route fixed effects are significantly different from zero when tranche fixed effects are present. The test statistic is an F-distributed random variable with (96,1395) degrees of freedom. It equals 1.04. We cannot reject the null that route fixed effects are zero.

Model (3) adds 13 bidder fixed effects to model (2). We find that 31% in the variation is explained and the adjusted R-squared increases to 0.29. We test whether bidder fixed effects are significantly different from zero. The test statistic is an F-distributed random variable with (12,2161) degrees of freedom. It equals 2.28. We can reject the null that bidder fixed effects are zero.

Model (4) decomposes the bidder fixed effect into depot fixed effects constant across tranches. The total number of explanatory variables increases to 147. We find 42% of the variation explained with an adjusted R-squared of 0.38. Again, we can reject the null that garage fixed effects are zero. The F-statistic equals 5.36 with (80,2093) degrees of freedom.

According to the adjusted R-squared in model (4) about 58% of the variation are unexplained. The unexplained part comes from the remaining uncertainty as to whether a bidder submits a bid on a given route after controlling for dead mileage, bidder-garage fixed effects and tranche fixed effects. We may interpret this uncertainty as a bidder specific idiosyncracy arising at the route and tranche level. Notice also, that the order in which we add variables may affect the contribution to the R-squared. We looked at permutations of the order and found no major differences.

The empirical evidence suggests the following origins for the cost uncertainty: First, there is no evidence of cost shocks common to all bidders at the route level after controlling for tranche fixed effects. This suggest that there is no correlation in route costs across bidders' within a tranche. Second, a substantial part of the uncertainty in bidders' decisions is explained by bidder asymmetry captured by dead mileage, bidder fixed effects and depot fixed effects. Thus, bidder asymmetry is important in this market. Third, there is considerable residual uncertainty for each bidder arising at the route and tranche level.

Private versus common values. Most uncertainty in the bidding environment comes from the privately observed opportunity costs of bidders. Opportunity costs depend on used capacity in depots, possibilities for extension, alternative use of the fleet, etcetera. Given this, knowing an opponent's opportunity cost is unlikely to affect a bidder's view on his own opportunity cost. Thus, a private value model may be an appropriate representation of the environment. In addition, we did not find any evidence of common shocks at the route level which lends further support to this hypothesis.

Summary and conclusions. The London bus routes market is best characterized as a multi-unit auction, with the *tranche as the proper unit of analysis*. First, combination bidding at the tranche level is prevalent, both in terms of occurrence as well as in terms of award outcomes, so we cannot view each route as a separate auction. Second, combination bidding is motivated in part by (local) cost synergies among routes, but different tranches tend to cover different geographical areas. Moreover, the London bus market is a mature and stable market. Hence, dynamic issues (inter-tranche effects) should not play an important role in this market. Finally, our analysis of bidder participation found that the cost uncertainty arises at the route level for each bidder and there is little evidence of shocks common to bidders. Thus, the informational environment is best viewed as multi-dimensional private values independent across bidders.

6 Estimation Results [preliminary and incomplete]

This section describes our estimates.

Our data do not contain reserve prices. To account for the reserve price in the estimation, we presume that reserve prices follow a specific functional form that is linear in the internal cost estimate. A lower bound on the reserve price is then the highest ratio of accepted bid to the reserve price. In our data this ratio equals 1.45. We use this number for our analysis. We vary this number to assess the robustness of our estimates to changes in the reserve

price rule. Specifically, lowering the number should not affect the estimates qualitatively.

As instruments for the moment condition any of the exogenous data are admissable. These include all bidder and auction specific variables on each auction and the powers of these variables. The total number of instruments has to exceed the number of parameters. For the mean of route combination s on auction t for bidder i we select the following eight instruments: internal cost estimate, the number of routes in the tranche, dead mileage of bidder i to route (combination) s, the number of garages of bidder i, a dummy that equals one if the route combination s consists of routes with identical bus types[24], the number of firms with a garage within 5 miles of one route on the tranche, a constant, the square of the internal cost estimate. For the second moment of bids of bidder i on route combination $s \cup t$ we select the internal cost estimate for the route combination $s \cup t$, the total number of routes within the tranche, dead mileage of bidder i to route combination $s \cup t$, a dummy that equals one if the route combination $s \cup t$ consists of routes with identical bus types and a constant. The total number of instruments exceeds the number of parameters in the model which guarantees identification.

Potential bidders on auction t include all bidders with a garage within 5 miles of at least one route within auction t. Bidders who submit a bid and have a garage further than 5 miles from all routes on auction t are classified as fringe bidders. We assume that it is common knowledge that fringe bidders submit a bid on at least one route on auction t and include fringe bidders in the set of potential bidders.

Parameter Specification: There are a number of natural restrictions to impose on the way the parameters of the bid density depends on bidder and auction characteristics $X^{it} = (x^t$

the number of bidders with at least one garage within 5 mile of one of the routes of the tranche, the number of garages of bidder i within 5 miles of one route on the tranche, the dead mileage of bidder i. This yields the following specification for μ_s^i:

$$\begin{aligned}\mu_s^i &= \beta_1 + \beta_2 \mathrm{IC}_s + \beta_3 \text{NO-GARAGES-TOTAL} + \beta_3 \text{NO-GARAGES-i} \\ &\quad + \beta_4 \text{DEAD-MILEAGE-i} + \beta_5 \text{IDENTICAL-BUS-TYPES} + \beta_{6s} \text{NO-ROUTES-IN-THE-TRANCHE}\end{aligned}$$

We specify the elements $\alpha_{s,\tau}$ of the matrix Λ as follows:

$$\alpha_{s,\tau} = \lambda_1 1_{\{s=\tau\}} + \lambda_2 1_{\{s\neq\tau \text{ and } s\cap\tau=\emptyset\}} + \lambda_3 1_{\{s\neq\tau \text{ and } s\cap\tau\neq\emptyset\}} + \lambda_4 1_{\{\text{routes } s \text{ and } \tau \text{ have the same bus type}\}}$$

The first constant accounts for diagonal elements in Λ, while the second and third constant account for off diagonal elements. We distinguish two off-diagonal effects depending on whether route s and τ have a non-empty interesection or not. The last constant accounts for synergy effects that may arise if the bus types required in routes s and τ are the same.

Table 3 provides the parameter estimates of the bid distribution function. Standard errors are in parenthesis. The effect on the mean bid is the following. The bid tends to be lower on tranches with more routes. Routes with less expected synergy effects have larger differences between combination bids and stand-alone bids although the effect is relatively small in magnitude. A bid on a large contract is closer to the internal cost estimate than a bid on a small contract. Finally, large bidders expect less competition than small bidders.

The parameter estimates of the variance are more difficult to interpret because the variance equals $\Lambda\Lambda'$. Calculating the variance covariance matrix reveals the following: There is less variation in bids on larger contracts. Bids between routes are positively correlated. The correlation is stronger when synergy effects are stronger (ROUTE-DISTANCE is small). The correlation coefficient between two stand-alone bids within the same tranche equals about 0.55 on average for tranches with high synergy effects (below median ROUTE-DISTANCE). The correlation coefficient decreases to about 0.26 for tranches with low synergy effects (above median ROUTE-DISTANCE). The correlation coefficient between a stand-alone bid and a combination bid equals about 0.74 for tranches with high synergy effects and falls to about 0.46 with low synergy effects.

(Figure 3 about here)

bus type. The regression explains 95% of the variation in the internal cost estimate.

Figure 3 illustrates the estimated bidding functions for the stand-alone and combination bid in the absence of synergies. The calculations assume two equal sized contracts each with internal cost estimate of 1,322,065 and a bidder with 21 garages which is the largest bidder. Further, we assume that the costs for the two routes are equal and there are no synergies (the cost of both routes equals the sum of the costs of the two routes separately). The figure depicts the combination bid and the sum of stand-alone bids varying the cost for the two routes combined. Both bids are monotone increasing in costs and for both bid functions the difference between the bid and the cost is decreasing. Furthermore, in the figure, the sum of stand-alone bids exceeds the combination bid. Moreover, the difference is increasing in the cost. The combination bid discount equals 5% on average for the illustrated range of costs.

7 Conclusions [To be written]

8 Appendix A: Data sources and coding issues

8.1 Data sources:

London Buses' tendering program: For each tranche and route in the tranche, this document provides the tender issue date, the tender return date, the planned start of the contract, the contract duration, together with the start and end point of the routes in the tranche.

Bid evaluation documents: These are London Transport Buses internal documents assessing the bids received for one to several routes in a tranche. These documents provide information on all route characteristics, including the identity of the incumbent when this is an existing route, the bids received (including combination bids), the identity of the bidders and, most of the time, the garage from which they plan to operate the route.[26] These documents analyze the bids received and make an award recommendation. When this recommendation deviates from the lowest price criterion, the criterion used is detailed and justified.

Route history: History of all the London Bus routes since 1934, compiled by the London Omnibus Traction Society (LOTS). For each route, this data contains information on the

[26] Missing values for the garage locations were completed using the bidder's closest garage to any of the end points of the route at the time of the tender return.

identity of the bus operator, the garage from which operation is carried out, the bus type and peak vehicle requirements (PVR) for weekdays, Saturdays and Sundays. For our analysis we have used weekdays PVR.

Depot history: Document compiled by the London Omnibus Traction Society (LOTS) since the deregulation in 1985. Provides information on openings, closings and transfers of bus depots used for London bus routes. This document is also our primary source of information for entry, mergers and acquisition (secondary sources included London Buses internal memos, companies' websites and LOTS' London Bus and Tram fleetbook publications).

8.2 Coding issues:

Route Alternatives: London Transport Buses sometimes specifies alternative specifications for a route (different bus types, frequencies or routing, for example). By convention, we have coded only the bid information related to the awarded service specification.

Age of vehicle: Vehicle age is the only dimension of the offer, besides price, that is not specified by London Buses. Hence, operators often submit different bid - vehicle age combinations. In the data, we have coded the bids for both existing and new buses. However, we did not find evidence that would suggest a trade-off between age and bid levels in the award decision. Rather, London Transport Buses seems to evaluate bids holding the age dimension constant, and award decisions are in practice indistinguishable from the award decisions of a contracting authority that would randomize between the age category it prefers, and then selects the best bid within that category. As a result, strategic interactions between the bids along the age dimension can be ignored, and in our main regressions, we have focused on the bids submitted for the age category that has attracted bids from the greatest number of bidders.

Tranches: By definition, a tranche is a set of routes auctioned at the same time. For our analysis, we have split several of the original tranches into independent subtranches when the following criteria were satisfied: (1) The two subsets of routes were in distinct geographical areas of London, (2) No combination bids were submitted across the two subsets of routes, *and* (3) The bids received on the two different subsets of routes originate from two different sets of bidders, or at least from two different sets of garages.

9 Appendix B: Proofs

Proposition 5 (Continuity and differentiability of G_s) *Given assumption 1, bidder i's probability of winning contract s, $G_s(b^i)$, is continuous at all b^i on its domain for all s, and it is a.e. continuously differentiable in b^i_t for all t.*

Proof: We consider the standard combinatorial auction. Given m contracts, there are 2^m possible winning allocations among bidder i and his opponents: Either $b^i_s + B^{-i}_{S\backslash s}$ is the bidder-bid combination most advantageous to the auctioneer, or B^{-i}_S is. Bidder i can win contract s if

$$b^i_s + B^{-i}_{S\backslash s} \leq \min\{b^i_t + B^{-i}_{S\backslash t} \text{ for all } t \neq s; B^{-i}_S\} \tag{13}$$

that is, if

$$\begin{aligned} B^{-i}_{S\backslash s} - B^{-i}_{S\backslash t} &\leq b^i_t - b^i_s \text{ for all } t \neq s \text{ and} \\ B^{-i}_{S\backslash s} - B^{-i}_S &\leq -b^i_s \end{aligned}$$

Because all the B^{-i}_s are continuously distributed on their support, so is their differences.[27] Let x_t denote the realization of $B^{-i}_{S\backslash s} - B^{-i}_{S\backslash t}$ for $t \subseteq S \cup \emptyset$, $t \neq s$, and let x the vector of such realizations. Define $f_s(x)$ as the density of x on its domain. Then $f_s(x)$ is continuous on that support. Set $f_s(x) = 0$ elsewhere. Then,

$$G_s(b^i) = \int \int^{x_t \leq b^i_t - b^i_s} .... \int^{x_\emptyset \leq -b^i_s} f_s(x)dx.$$

is well-defined and is continuous in b^i. In addition, if $f_s(x)$ is continuous in x_t at $x_t = b^i_t - b^i_s$ (for $t \neq s$) (or at $x_\emptyset = -b^i_s$ for $t = \emptyset$), then $G^t_s(b^i)$ exists and is continuous.[28] Finally, since B^{-i} has a convex support with full dimension in $\mathbb{R}^{2^m-1}$, the support of x is also convex with full dimension. Therefore the only potential points of discontinuity of f_s are at the boundaries of the support of x, G_s is a.e. differentiable and everywhere left and right differentiable. QED

Remark 1 *Points of non-differentiality of G_s. The only points where G_s may fail to be (continuously) differentiable are at the boundaries of the support of x, where f goes from something positive to zero. This is related to the distinction between relevant and irrelevant bids.*

[27]This is always true. To see this. Consider the continuous distribution $g(x, y, z)$. Then the distribution of $d = x - y$, is given by $f(d) = \int \int g(x, x - d, z)dxdz$. Since g is continuous in y, then f is continuous in d on its domain.

[28]For example, for $t \neq s$, $G^t_s(b^i) = \int ... \int^{x_\emptyset \leq -b^i_s} f_s(b^i_t - b^i_s, x_{-t})dx_{-t}$.

Remark 2 *The result is proved for the general multi-unit auction case. However, it also extends to the London Bus auction. Under the auction rule adopted by London Transport Buses, we need to add the set of competing bidder-bid combinations* $\{b_w^i + b_{t\backslash w}^i + B_{S\backslash t}^{-i}$ *for all* $t \neq s$ *and for all* $w \subset t\}$ *in the right hand side of* (13). *In addition, the definition of* B_s^{-i} *should be slightly amended to account for the fact that the auctioneer can go for combination of stand-alone bids from the same bidder if this ends up yielding a cheaper package. However, on the domain of acceptable bids* $b_t^i \leq b_w^i + b_{t\backslash w}^i$, *these terms do not play any role, and* $G_s(.)$ *remains continuous. Reserve prices simply limit the domain on which* $G_s(.)$ *is defined. It does not affect its properties.*

Proof of lemma 1: Given m contracts, there are 2^m possible allocations of these contracts between bidder i and his opponents: Either $b_s^i + B_{S\backslash s}^{-i}$ is the best bid combination submitted, in which case bidder i wins exactly subset s of the contracts (there are $2^m - 1$ such possibilities), or B_S^{-i} is the best bid combination, in which case bidder i does not win anything.

By raising b_t^i, bidder i increases the price of one of his possible "winning allocations" relative to the other ones. This cannot increase the probability that he wins bundle t, so (1) must hold.

Next, since the increase in b_t^i does not affect the relative ranking among $b_s^i + B_{S\backslash s}^{-i}$ for $s \neq t$ and B_S^{-i}, the probability that any of these competing bid combination wins cannot decrease, so (2) and (3) must hold ((3) uses the fact that one of the beneficiaries of the increase in b_t^i is bid combination B_S^{-i}, hence the aggregate probability that bidder i wins anything, $\sum_s G_s$, must be decreasing).

When b^i does not include any irrelevant bid, decreasing b_t^i must strictly increase the chance that $b_t^i + B_{S\backslash t}^{-i}$ wins, otherwise it could not have been an equilibrium.

To prove that inequality (3) holds strictly for at least one t, we exploit the analogy between the bidder's optimization problem in our setting and the standard multidimensional screening problem. Armstrong (1996) shows that, under some general conditions on the set of private information of potential buyers, exclusion is optimal in multi-dimensional screening problems. Armstrong's assumptions are satisfied by our assumption 1 and his result translates in our auction setting as bid B_S^{-i} having a strictly positive probability of winning.[29] Hence, B_S^{-i} effectively competes with at least one of the other possible $b_t^i + B_{S\backslash t}^{-i}$ bid combination for some t. This means that decreasing b_t^i must strictly hurt bid B_S^{-i} hence the

[29] A complete translation of Armstrong's proof to our setting is available upon request.

probability of bidder i winning anything must be strictly increasing in b_t^i, that is, inequality (3) holds strictly for t.

Finally, we introduce the following definition:

Definition: The set of contracts $\Omega \subseteq S \cup \emptyset$ forms a connected chain of substitutes if for all s and s' in Ω, either $G_s^{s'} > 0$ or there exist $w_1, ..., w_n \in \Omega$ such that $G_s^{w_1} > 0, G_{w_1}^{w_2} > 0, ..., G_{w_n}^{s'} > 0$.

From now on, we adopt the convention that $b_\emptyset^i = 0$ and say that bidder i wins contract $\emptyset$ when B_S^{-i} is the winning bid combination (so that bidder i wins nothing). With this notation, $\sum_{s \subseteq S \text{ or } s=\emptyset} G_s = 1$.

Claim 1: If all bids on the set of contracts $S' \subseteq S$ are relevant, then $\Omega = S' \cup \emptyset$ forms a connected chain of substitutes.

Proof: First, since $G_s^s < 0$, any contract must be connected with at least one other contract. Second, we argue that if two contracts in Ω are not connected according to definition 1, they must exist at least two disjoint sets of contracts in Ω, with no contract in the first set connected with a contract in the other set. Towards a contradiction, suppose that set $\{s, t\}$ and the rest form two disjoint sets of contracts (the focus on a set of two contracts is without loss of generality). Consider the following continuous random variables (the randomness comes from the B^{-i}'s), $\min\{b_t^i + B_{\Omega\backslash t}^{-i},\ b_s^i + B_{\Omega\backslash s}^{-i}\}$ and $\min_{w \neq t,s}\{b_w^i + B_{\Omega\backslash w}^{-i}\}$. Since all bids are relevant, sometimes $\min\{b_t^i + B_{\Omega\backslash t}^{-i},\ b_s^i + B_{\Omega\backslash s}^{-i}\} \leq \min_{w \neq t,s}\{b_w^i + B_{\Omega\backslash w}^{-i}\}$ (bidder i wins contract s or t) and sometimes $\min\{b_t^i + B_{\Omega\backslash t}^{-i},\ b_s^i + B_{\Omega\backslash s}^{-i}\} \leq \min_{w \neq t,s}\{b_w^i + B_{\Omega\backslash w}^{-i}\}$. By continuity, $\min\{b_t^i + B_{\Omega\backslash t}^{-i},\ b_s^i + B_{\Omega\backslash s}^{-i}\} = \min_{w \neq t,s}\{b_w^i + B_{\Omega\backslash w}^{-i}\}$ must happen so that at least s or t must compete directly with some w in the other set, i.e. G_w^t or $G_w^s > 0$. A contradiction.

In particular claim 1 implies that the set of all contracts forms a connected chain when all bids are relevant. We can now prove that $\det \nabla G < 0$ (so that ∇G is invertible). The proof is by induction. Since (3) holds strictly for at least one contract we relabel the rows and columns of matrix ∇G such that the sum of the elements in the first row is strictly negative (this does not change the value of the determinant):

$$\sum_s G_s^1 < 0 \tag{14}$$

Now, consider the linear transformation, L_1 on the columns of ∇G that adds to column $s \neq 1$, α_{1s} times column 1 such that $G_s^1 + \alpha_{1s} G_1^1 = 0$ for $s \neq 1$ (notice, $\alpha_{1s} \geq 0$ and $\sum \alpha_{1s} < 1$ given (14)). This leaves the first row of matrix ∇G with all zeros except in the first position. Denote the resulting matrix by $L_1 \nabla G$ and let $[L_1 \nabla G]$ be matrix $L_1 \nabla G$

from which the first row and the first column have been removed. Since determinants are invariant to linear transformations, $\det \nabla G = \det L_1 \nabla G = G_1^1 \det[L_1 \nabla G]$.

We claim that the resulting 2^m x 2^m matrix $[L_1 \nabla G]$ satisfies properties (1) to (3) of the original matrix, including the strict inequalities. Property (1): The diagonal elements of matrix $[L_1 \nabla G]$ are equal to $G_s^s + \alpha_{1s} G_1^s$. Since the G_t^s elements satisfy properties (1) to (3) and $\alpha_{1i} < 1$, we have $G_s^s + \alpha_{1s} G_1^s < 0$. Property (2): The off-diagonal of the new matrix are equal to $G_t^s + \alpha_{1t} G_1^s \geq 0$ since it is a sum of positive elements. Property (3): The sum of the row elements of the $[L_1 \nabla G]$ matrix is equal to $\sum_{s \neq 1} G_s^t + G_1^t \sum_{s \neq 1} \alpha_{1s} \leq 0$ since $\sum G_s^t \leq 0$ and $\sum_{s \neq 1} \alpha_{1s} < 1$. To show that this inequality holds strictly for at least one row of the new matrix $[L_1 \nabla G]$, we need to consider two cases. First, if any of the elements G_1^s of the first column of the original matrix was strictly positive, then since $\sum_{s \neq 1} \alpha_{1s} < 1$, there exists a row in the new matrix such that condition (3) holds strictly. If all the elements $G_1^s = 0$ for $s \neq 1$, contract 1 is directly connected only to contract $\emptyset$. But then by claim 1, it must be that one of the remaining contracts, say t, is connected to $\emptyset$. This means that $\sum_s G_s^t < 0$ in the original matrix, and in the new matrix.

Repeating the argument leads to $sign(\det \nabla G) = sign(-1)^{2^m - 1} < 0$.

To prove the last part of the claim we show that any submatrix made from ∇G by removing some rows and the corresponding columns has the same properties (1) to (3), including the strict inequalities. The proof then proceeds as before. QED

Proof of corollary 1: (i) Follows directly from propositions 2 and 3. (ii) When only stand-alone bids are permitted, bidders solve the following constrained maximization problem:

$$\max_{b^i} \sum_{s \subseteq S} (b_s^i - c_s) G_s(b) \qquad \text{s.t. } b_s^i = b_{s \setminus t}^i + b_t^i \text{ for all } s \subseteq S \text{ and } t \subset s$$

Substituting for $b_s = b_{s \setminus t} + b_t$ into the objective function reduces the problem to a m dimensional optimization problem in bids for the individual contracts b_s^i for $s \in S$. Assuming differentiability, the first order conditions are:

$$\sum_{t \text{ s.t. } s \subseteq t} G_t(b^i) + \sum_{w \subseteq S} \{(b_w^i - c_w) \sum_{t \text{ s.t. } s \subseteq t} G_w^t(b^i)\} = 0 \qquad \text{for all } s \in S$$

This is a system of m linear equations in $2^m - 1$ variables (the unobserved c_w). This system is under-identified. (iii) Follows from standard results for the single unit auction environments (see, e.g. Guerre et al., 2000). QED

Proof of proposition 4: The claim that any binding constraint introduces a single dimension of underidentification follows directly from (10) since the solution is unique up to the

value of the Lagrangian multiplier involved in that constraint. The rest of the proof uses the following properties of determinants: (1) Determinants are invariant to linear transformations of rows or columns, (2) determinants are invariant to the permutation of rows and the corresponding column, (3)
$\det \begin{bmatrix} a_{11}+b_{11} & .. & a_{1N} \\ .. & & .. \\ a_{N1}+b_{N1} & .. & a_{NN} \end{bmatrix} = \det \begin{bmatrix} a_{11} & .. & a_{1N} \\ .. & & .. \\ a_{N1} & .. & a_{NN} \end{bmatrix} + \det \begin{bmatrix} b_{11} & .. & a_{1N} \\ .. & & .. \\ b_{N1} & .. & a_{NN} \end{bmatrix}$, and (3) the multiplication of any row or column by a constant, multiplies the value of the determinant by that constant.

Let G denote the $2^m - 1$ x 1 vector of G_s, define e_t as the $2^m - 1$ x 1 vector with entry 1 at the row corresponding to contract t and zero elsewhere and $I_{r=t\cup w}$ as the $2^m - 1$ x 1 vector with entry 1 in the row corresponding to contract r and -1 in the rows corresponding to contracts t and w. With these notations,

$$D(b^i;\delta,\mu,\lambda) = -G + \sum_{t\subseteq S}\mu_t e_t + \sum_{t\subseteq S}\delta_t e_t + \sum_{r}\sum_{\substack{t,w\subset r,\ t\cap w=\emptyset, \\ |t|\leq|w|}} \lambda_{r=t\cup w} I_{r=t\cup w}$$

Let $A_s B$ denote matrix A from which the column corresponding to contract s has been replaced by vector B. Cramer's rule together with properties (2) and (3) of determinants imply that

$$\begin{aligned} b_s^{eff} - c_s &= \frac{1}{\det \nabla G} \det \nabla G_s D(b^i;\mu,\lambda,\delta) \\ &= \frac{1}{\det \nabla G}[-\det \nabla G_s G + \sum_{t\subseteq S}\mu_t \det \nabla G_s e_t + \sum_{t\subseteq S}\delta_t \det \nabla G_s e_t \qquad (15) \\ &\quad + \sum_{r\subseteq S}\sum_{\substack{t,w\subset r,\ t\cap w=\emptyset, \\ |t|\leq|w|}} \lambda_{r=t\cup w} \det \nabla G_s I_{r=t\cup w}] \end{aligned}$$

In words, the underlying cost parameter c_s depends linearly on the values of the Lagrangian multipliers.

Claim 1: $\det \nabla G_s e_t > 0$ if $t = s$, and zero otherwise.

Proof: If the reserve price on contract t is binding or if b_t^i is a bid that never wins (which is the only time we need to worry about the sign of $\det \nabla G_s e_t$), then $G_t^s(b_s^{eff}, b_{-s}^i) = 0$ for all $s \neq t$, and $G_t^t(b_t^{eff}, b_{-t}^i) < 0$, that is, the column in matrix ∇G that corresponds to <u>contract t</u> is all zeros but for the (row) entry corresponding to the t^{th} contract. Now, by construction, the column in matrix $\nabla G_s e_t$ that corresponds to <u>contract s</u> is all zeros but for the (row) entry corresponding to the t^{th} contract. If $s \neq t$, matrix $\nabla G_s e_t$ has

two linearly dependent columns (corresponding to contracts s and t) so $\det \nabla G_s e_t = 0$. If $s = t$, $\det \nabla G_t e_t$ is equal to the determinant of matrix ∇G from which the row and column corresponding to contract t have been removed. By lemma 1, $\det \nabla G_t e_t > 0$ (since 2^m rows remain).

Claim 2: $\det \nabla G_s I_{r=t\cup w} > 0$ if $s = r$, it is < 0 if $s = t$ or w.

Proof: Suppose first that $r = s$. Define L_s as the operator that adds the values associated to row s to rows t and w so that $L_s \nabla G_s I_{s=t\cup w}$ becomes a matrix with a zero column at position s except for the "1" entry at row s. Define M as the 2^m x 2^m matrix made of matrix $L_s \nabla G_s I_{s=t\cup w}$ from which the row and the column corresponding to contract s have been removed, and M' as the 2^m x 2^m matrix made of matrix $L_s \nabla G$ from which the row and column corresponding to contract s have been removed. Then $M = M'$ and $\det M' = \det M > 0$ by lemma 1 and property (1). The claim follows from the fact that $\det \nabla G_s I_{s=t\cup w} = \det L_s \nabla G_s I_{s=t\cup w} = \det M$.
Now suppose that $s = t$ or w, and define L_s as the operator that adds row s to row r and substracts row s from row w so that $L_s(\nabla G_s I_{r=s\cup w})$ is now a matrix with a zero column except for an entry -1 at row s. Define M and M' as above. The claim follows from the fact that $\det \nabla G_s I_{r=s\cup w} = -\det M < 0$.

Proposition 4 then follows from (15), claims 1 and 2, and the fact that $\det \nabla G < 0$. QED

Proof of corollary 2: Suppose that $b^i_s = b^i_t + b^i_w$ for some t, w and s with $s = t \cup w$ and $t \cap w = \emptyset$. Define $\alpha_{t\cup w} = c_t + c_w - c_{t\cup w}$, the cost synergy between contracts t and w. With this notation, $b^i_s - c_s = b^i_t - c_t + b^i_w - c_w + \alpha_{t\cup w}$. Let C_s the operator on the columns of ∇G that add column s to columns t and w. We have:

$$\nabla G(b^i)[b^{eff} - c] = C_s \nabla G(b^i) \begin{bmatrix} \ldots \\ b_t - c_t \\ b_w - c_w \\ \alpha_{t\cup w} \\ \ldots \end{bmatrix} = D(b^i; \delta, \mu, \lambda)$$

Given that such transformation on the column of ∇G do not change the value of the determinant, we can use the same type of arguments as in the proof of proposition 4 (in particular, note that $\alpha_{t\cup w}$ is at the position corresponding to contract s) to prove that $\alpha_{t\cup w}$ depends negatively on the value of $\lambda_{\{t\cup w\}=t\cup w}$. Therefore, setting $\lambda_{\{t\cup w\}=t\cup w} = 0$ provides an upper bound to the synergy $\alpha_{t\cup w}$. QED

References

[1] Armstrong, M. and J.C. Rochet (1999), Multi-dimensional Screening: A User's Guide, European Economic Review, 43, 959-979.

[2] Armstrong, M. (2000), Optimal Multi-Object Auctions, Review of Economic Studies, 67, 455-481.

[3] Athey, Susan and Phil Haile (2001), Identification of Standard Auction Models, forth. Econometrica

[4] Ausubel, L., P. Cramton, R.P. McAfee and John McMillan (1997), Synergies in Wireless Telephony: Evidence from the Broadband PCS Auctions, Journal of Economics and Management Strategy, 6(3), 497-527.

[5] Avery, C. and T. Hendershott (2000), Bundling and Optimal Auctions of Multiple Products, Review of Economic Studies, 67, 483-497.

[6] Branco, F. (1997), Multi-object Auctions with Synergies, mimeo.

[7] Cohen A. C. and B. Jones Whitten (1980), Estimation in the Three-Parameter Lognormal Distribution, Journal of the American Statistical Association, 75, 399-404.

[8] Donald, S. and H. Paarsch (1993), Piecewise Pseudo-Maximum Likelihood Estimation in Empirical Models of Auctions, International Economic Review, 34, 121-148.

[9] Donald, S., H. Paarsch and J. Robert (2001), Identification and Estimation in Empirical Models of Sequential, Oral, Ascending Price Auctions, mimeo.

[10] Elyakime, B., J.J. Laffont, P. Loisel and Q. Vuong (1994), First-Price Sealed-Bid Auctions with Secret ReservationPrices, Annales d'Economie et de Statistique, 34, 115-141.

[11] Glaister, S. and M. Beesley (1991), Bidding for Tendered Bus Routes in London, Transportation Planning and Technology, 15, 349-366.

[12] Griffiths D. (1980), Interval Estimation for the Three-Parameter Lognormal Distribution via the Likelihood Function, Applied Statistics, 29, 58-69.

[13] Guerre, I, I. Perrigne and Q. Vuong (2000), Optimal Nonparametric Estimation of First-Price Auctions, Econometrica, 68(3), 525-574.

[14] Hortacsu, Ali (2002), Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market, University of Chicago, mimeo.

[15] Kennedy, D., S. Glaister and T. Travers (1995), London Bus Tendering, The Greater London Group at the London School of Economics and Political Science.

[16] Jofre-Bonet M. and M. Pesendorfer (2001), Estimation of a Dynamic Auction Game, mimeo, Yale University.

[17] London Transport Buses (1999), The Bus Tendering Process, available at http://www.londontransport.co.uk/buses/images/tend_rpt.pdf

[18] Levin, J. (1997), An Optimal Auction for Complements, Games and Economic Behavior, 18, 176-192.

[19] Laffont, J.J., H. Ossard and Q. Vuong (1995), Econometrics of First-Price Auctions, Econometrica, 63(4), 953-980.

[20] Li, Tong, Isabelle Perrigne and Quang Vuong (2000), Structural Estimation of the Affiliated Private Value Auction Model, University of Southern California, mimeo.

[21] McAfee, R.P., J. McMillan and M.D. Whinston (1989), Multiproduct Monopoly, Commodity Bundling and Correlation of Values, Quaterly Journal of Economics, 102(2), 371-383.

[22] Milgrom, Paul (2000), Putting Auction Theory to Work: The Simultaneous Ascending Auction, Journal of Political Economy, 108(2), 245-272.

[23] Moreton, P. and P. Spiller (1998), What's in the Air: Interlicence Synergies and their Impact on FCC Broadband PCS License Auctions, Journal of Law and Economics.

[24] Myerson, R. (1981), Optimal Auction Design, Mathematics of Operation Research, 6, 58-73.

[25] Palfrey, T. (1983), Bundling Decisions by a Multi-product Monopolist with Incomplete Information, Econometrica, 51(2), 463-483.

[26] Whinston, M.D. (1989), Tying, Foreclosure, and Exclusion, American Economic Review, 80(4), 837-859.

figure3 (800x600x2 tiff)

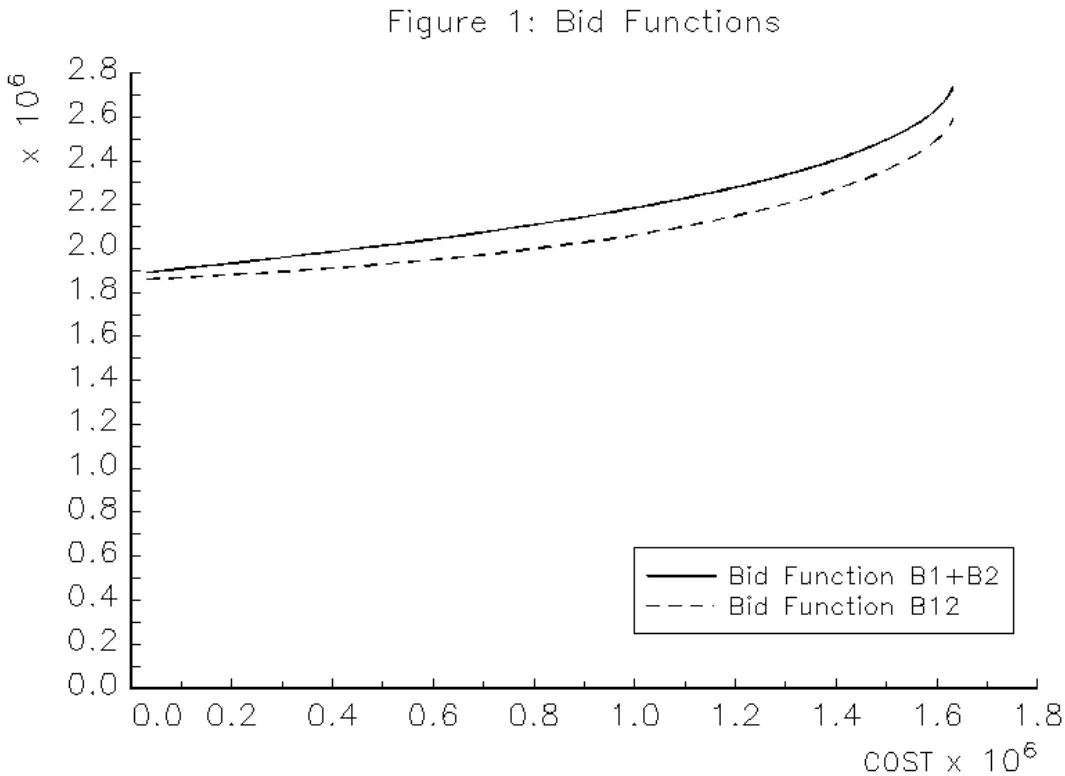

TABLE 3: PARAMETER ESTIMATES OF BID DISTRIBUTION FUNCTION

Data:	Tranches with 1-3 Routes			
Number of Observations:	348			
Log Likelihood:	437,22			
Variables	μ	α_{ii}	α_{ij}	α_{ij}
Constant	-0,8179	-1,7119	-0,0986	-0,0986
	(0,129)	(0,005)	(0,004)	(0,004)
2-Routes	-0,2759			
	(0,025)			
3-Routes	-0,3589			
	(0,023)			
Route-Distance-Non-Overlapping-Routes				0,0215
				(0,001)
Route-Distance-Overlapping-Routes			0,0121	
			(0,000)	
Route-Distance-Standalone-bid	-0,0004			
	(0,001)			
Route-Distance-Combination-bid	-0,0041			
	(0,002)			
No-Bidder-Large	0,0711			
	(0,083)			
No-Bidder-Small	0,1839			
	(0,046)			
ICE	0,0169	0,1037		
	(0,004)	(0,000)		
No-Garages	0,0466			
	(0,009)			

All variables are in logarithm. Standard errors are displayed in parenthesis.

TABLE 4: A SAMPLE OF COST ESTIMATES FROM BIDS ON 2 ROUTE TRANCHES

	BIDS				COSTS			
SAMPLE NUMBER	BID1 =B1 /ICE1	BID2 =B2 /ICE2	BID12 =B12 /ICE12	BID-MARK-DOWN = (B12-B1-B2) /B12	COST1 =C1 /B1	COST2 =C2 /B2	COST12 =C12 /B12	COST-MARK DOWN =(C12-C1-C2) /C12
1	0,66	0,27	0,59	-0,010	0,81	0,86	0,81	-0,012
2	0,83	0,94	0,85	-0,033	0,65	0,73	0,60	-0,144
3	0,78	0,93	0,82	-0,035	0,83	0,86	0,81	-0,039
4	0,85	0,76	0,81	-0,007	0,71	0,81	0,63	-0,190
5	0,92	1,01	0,95	-0,015	0,99	0,99	1,00	0,004
6	1,06	1,09	1,06	-0,017	0,66	-0,77	-0,25	-0,249
7	0,96	1,00	0,98	-0,013	0,85	0,31	0,39	-0,220
8	1,11	1,07	1,05	-0,028	0,81	-0,24	0,12	0,243
9	0,50	0,52	0,46	-0,089	0,93	0,90	0,71	-0,319
10	0,86	1,28	0,84	-0,128	1,00	1,00	1,00	0,002
11	0,79	0,25	0,71	-0,006	1,00	1,00	1,00	0,000
12	0,71	0,52	0,67	-0,029	1,00	1,00	1,00	0,000
Average	0,84	0,80	0,82	-0,03	0,85	0,62	0,65	-0,08

The data are the sample of 2 Route bids in which two stand-alone bids and one combination bid is observed.

Bids are measured as a fraction of the internal cost estimate. Costs are measured as a fraction of the corresponding bid.

The bid mark-down equals (B12-B1-B2)/B12 and the cost mark-down equals (C12-C1-C2)/C12.