A Markov-Switching Multi-Fractal Inter-Trade Duration Model, with Application to U.S. Equities

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Are big data informative for trends? No

Are big data informative for volatilities? Yes: realized volatility.

For durations? Yes: both trivially and subtly.

– Trivially: trade-by-trade data needed for inter-trade durations
  – Subtly: time deformation links volatilities to durations.
So big data inform us about vols which inform us about durations.
Why More Focus on Durations Now?

- We need better understanding of information arrival, trade arrival, liquidity, volume, market participant interactions, links to volatility, etc.
  - Financial market roots of the Great Recession
  - Purely financial-market events like the Flash Crash

- The duration point process is the ultimate process of interest. It determines everything else, yet it remains incompletely understood.

- Long memory is clearly present in calendar-time volatility and is presumably inherited from conditional intensity of arrivals in transactions time, yet there is little long-memory duration literature.
Stochastic Volatility Model

\[ r_t = \sigma \sqrt{e^{h_t}} \cdot \varepsilon_t \]

\[ h_t = \rho h_{t-1} + \eta_t \]

\[ \varepsilon_t \sim iidN(0, 1) \]

\[ \eta_t \sim iidN(0, \sigma^2) \]

\[ \varepsilon_t \perp \eta_t \]

Equivalently,

\[ r_t|\Omega_{t-1} \sim N(0, \sigma^2 e^{h_t}) \]
Time-deformation model of calendar time (e.g., “daily”) returns:

\[ r_t = \sum_{i=1}^{e^{ht}} r_i \]

\[ h_t = \rho h_{t-1} + \eta_t \]

(Trade-by-trade returns \( r_i \sim iidN(0, \sigma^2) \), daily volume \( e^{ht} \))

\[ \implies r_t | \Omega_{t-1} \sim N(0, \sigma^2 e^{ht}) \]

- Volume/duration dynamics produce volatility dynamics
- Volatility properties *inherited* from duration properties
What Are the Key Properties of Volatility?

In general:

- Volatility dynamics fatten unconditional distributional tails
  e.g., \( r_t | \Omega_{t-1} \sim N(0, \sigma^2 e^{ht}) \implies r_t \sim "fat-tailed" \)
- Volatility dynamics are persistent
- Volatility dynamics are long memory

Elegant modeling framework that captures all properties:

Calvet and Fischer (2008),
*Multifractal Volatility: Theory, Forecasting, and Pricing*, Elsevier
Roadmap

- Empirical regularities in durations
- The MSMD model
- Preliminary empirics
Twenty-Five U.S. Firms Selected Randomly from S&P 100

- Consolidated trade data extracted from the TAQ database
- 20 days, 2/1/1993 - 2/26/1993, 10:00 - 16:00
- 09:30-10:00 excluded to eliminate opening effects

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Table: Stock ticker symbols and company names.
Figure: **Citigroup Duration Distribution.** We show an exponential QQ plot for Citigroup inter-trade durations between 10:00am and 4:00pm during February 1993, adjusted for calendar effects.
Persistent Dynamics

Figure: **Citigroup Duration Time Series.** We show a time-series plot of inter-trade durations between 10:00am and 4:00pm during February 1993, measured in minutes and adjusted for calendar effects.
Figure: **Citigroup Duration Autocorrelations.** We show the sample autocorrelation function of Citigroup inter-trade durations between 10:00am and 4:00pm during February 1993, adjusted for calendar effects.
Roadmap

- Empirical regularities in inter-trade durations ✓
- The MSMD model
- Empirics
A Dynamic Duration Model

\[ d_i \sim \frac{\epsilon_i}{\lambda_i}, \quad \epsilon_i \sim iidExp(1) \]

“Mixture of Exponentials” Representation
Point Process Foundations

\{ T_i(\omega) > 0 \}_{i \in 1,2,...} \text{ on } (\Omega, F, P) \text{ s.t. } 0 < T_1(\omega) < T_2(\omega) < \ldots

\[ N(t, \omega) = \sum_{i \geq 1} 1(T_i(\omega) \leq t) \]

\[ \lambda(t) = \lim_{\Delta t \downarrow 0} \left( \frac{1}{\Delta t} P[N(t + \Delta t) - N(t) = 1 \mid F_{t-}] \right) \]

\[ P(t_1, \ldots, t_n | \lambda(\cdot)) = \prod_{i=1}^{n} \left( \lambda(t_i) \exp \left[ - \int_{t_{i-1}}^{t_i} \lambda(t) dt \right] \right) \]

If \( \lambda(t) = \lambda_i \) on \( [t_{i-1}, t_i) \), then:

\[ P(t_1, \ldots, t_n | \lambda(\cdot)) = \prod_{i=1}^{n} (\lambda_i \exp [- \lambda_i d_i]) \]

\[ d_i \sim \frac{\epsilon_i}{\lambda_i}, \quad \epsilon_i \sim iid\text{Exp}(1) \]

How to parameterize the conditional intensity \( \lambda_i \)?
Markov Switching Multifractal Durations (MSMD)

\[ \lambda_i = \lambda \prod_{k=1}^{\bar{k}} M_{k,i} \]

\( \lambda > 0, \ M_{k,i} > 0, \ \forall k \)

Independent intensity components \( M_{k,i} \)

are Markov renewal processes:

\[ M_{k,i} = \begin{cases} \text{draw from } f(M) & \text{w.p. } \gamma_k \\ M_{k,i-1} & \text{w.p. } 1 - \gamma_k \end{cases} \]

\( f(M) \) is identical \( \forall k \), with \( M > 0 \) and \( E(M) = 1 \)
Modeling Choices

Simple binomial renewal distribution $f(M)$:

$$M = \begin{cases} 
    m_0 & \text{w.p. } 1/2 \\
    2 - m_0 & \text{w.p. } 1/2,
\end{cases}$$

where $m_0 \in (0, 2]$ 

Simple renewal probability $\gamma_k$:

$$\gamma_k = 1 - (1 - \gamma_{\bar{k}})^{b^{k-\bar{k}}}$$

$\gamma_{\bar{k}} \in (0, 1)$ and $b \in (1, \infty)$
Figure: **MSMD Intensity Component Renewal Probabilities.** We show the renewal probabilities \( \gamma_k = 1 - (1 - \gamma k) b^{k-k} \) associated with the latent intensity components \( M_k \), \( k = 3, \ldots, 7 \). We calibrate the MSMD model with \( \bar{k} = 7 \), and with remaining parameters that match our subsequently-reported estimates for Citigroup.
All Together Now

\[ d_i = \frac{\epsilon_i}{\lambda_i} \]

\[ \lambda_i = \lambda \prod_{k=1}^{\bar{k}} M_{k,i} \]

\[ M_{k,i} = \begin{cases} M & \text{w.p. } 1 - (1 - \gamma_{\bar{k}}) b^{k-\bar{k}} \\ M_{k,i-1} & \text{w.p. } (1 - \gamma_{\bar{k}}) b^{k-\bar{k}} \end{cases} \]

\[ M = \begin{cases} m_0 & \text{w.p. } 1/2 \\ 2 - m_0 & \text{w.p. } 1/2 \end{cases} \]

\[ \epsilon_i \sim iid \ exp(1), \ \bar{k} \in \mathbb{N}, \ \lambda > 0, \ \gamma_{\bar{k}} \in (0, 1), \ b \in (1, \infty), \ m_0 \in (0, 2] \]

parameters \( \theta_{\bar{k}} = (\lambda, \gamma_{\bar{k}}, b, m_0)' \)

\( \bar{k} \)-dimensional state vector \( M_i = (M_{1,i}, M_{2,i}, \ldots, M_{\bar{k},i}) \)

2\( \bar{k} \) states
Figure: QQ Plot, Simulated Durations. $N = 22,988$; parameters calibrated to match Citigroup estimates.
Figure: Time-series plots of simulated $M_{1,i}$, ..., $M_{7,i}$, $\lambda_i$, and $d_i$. $N = 22,988$; parameters calibrated to match Citigroup estimates.
Figure: Sample Autocorrelation Function, Simulated Durations. $N = 22,988$; parameters calibrated to match Citigroup estimates.
The MSMD autocorrelation function satisfies

\[
\sup_{\tau \in I_{\bar{k}}} \left| \frac{\ln \rho(\tau)}{\ln \tau^{-\delta}} - 1 \right| \to 0 \quad \text{as} \quad \bar{k} \to \infty
\]

\[
\delta = \log_b E(\tilde{M}^2) - \log_b \{E(\tilde{M})^2\}
\]

\[
\tilde{M} = \begin{cases} 
\frac{2m_0^{-1}}{m_0^{-1} + (2-m_0)^{-1}} & \text{w.p. } \frac{1}{2} \\
\frac{2(2-m_0)^{-1}}{m_0^{-1} + (2-m_0)^{-1}} & \text{w.p. } \frac{1}{2}
\end{cases}
\]
Literature I:  
Mean Duration vs. Mean Intensity

Mean Duration:

\[ d_i = \varphi_i \epsilon_i, \quad \epsilon_i \sim iid(1, \sigma^2) \]

- ACD: Engle and Russell (1998), ...
  - MEM: Engle (2002), ...

Mean Intensity:

\[ d_i \sim \frac{\epsilon_i}{\lambda_i}, \quad \epsilon_i \sim iidExp(1) \]

- MSMD
- Bauwens and Hautsch (2006)
  - Bowsher (2006)
Literature II: Observation- vs. Parameter-Driven Models

Observation-Driven:

\[ \Omega_{t-1} \text{ observed (like GARCH)} \]

- ACD
- MEM as typically implemented
  - GAS

Parameter-Driven:

\[ \Omega_{t-1} \text{ latent (like SV)} \]

- MSMD
Short Memory: Quick (exponential) duration autocorrelation decay
- ACD as typically implemented
- MEM as typically implemented

Long-Memory: Slow (hyperbolic) duration autocorrelation decay
- MSMD
- FI-ACD: Jasiak (1999)
- FI-SCD: Deo, Hsieh and Hurvich (2010)
Literature IV: Reduced-Form vs. Structural Long Memory

Reduced Form:

\[(1 - L)^d y_t = v_t, \quad v_t \sim \text{short memory}\]

- FI-ACD
- FI-SCD

Structural:

\[y_t = v_{1t} + \ldots + v_{Nt}, \quad v_{it} \sim \text{short memory}\]

- MSMD
Roadmap

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- The MSMD model ✓
- Empirics
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Figure: Distribution of Duration Coefficients of Variation Across Firms. We show a histogram of coefficients of variation (the standard deviation relative to the mean), as a measure of overdispersion relative to the exponential. For reference we indicate Citigroup.
**Figure:** Duration Autocorrelation Function Profile Bundle. For each firm, we show the sample autocorrelation function of inter-trade durations between 10:00am and 4:00pm during February 1993, adjusted for calendar effects. For reference we show Citigroup in bold.
MSMD Likelihood Evaluation (Using $\bar{k} = 2$ for Illustration)

Each $M_k$, $k = 1, 2$ is a two-state Markov switching process:

$$\mathcal{P}(\gamma_k) = \begin{bmatrix} 1 - \gamma_k/2 & \gamma_k/2 \\ \gamma_k/2 & 1 - \gamma_k/2 \end{bmatrix}$$

Hence $\lambda_i$ is a four-state Markov-switching process:

$$\lambda_i \in \{\lambda s_1 s_1, \lambda s_1 s_2, \lambda s_2 s_1, \lambda s_2 s_2\}$$

$$\mathcal{P}_\lambda = \mathcal{P}(\gamma_1) \otimes \mathcal{P}(\gamma_2) \text{ (by independence of the } M_{k,i})$$

Likelihood function:

$$p(d_{1:n}|\theta_{\bar{k}}) = p(d_1|\theta_{\bar{k}}) \prod_{i=2}^{n} p(d_i|d_{1:i-1}, \theta_{\bar{k}})$$

Conditional on $\lambda_i$, the duration $d_i$ is $\text{Exp}(\lambda_i)$:

$$p(d_i|\lambda_i) = \lambda_i e^{-\lambda_i d_i}$$

Weight by state probabilities obtained by the Hamilton filter.
**Figure: Maximized Log Likelihood Profile Bundle.** We show likelihood profiles for all firms as a function of $\bar{k}$, in deviations from the value for $\bar{k} = 7$, which is therefore identically equal to 0. For reference we show Citigroup in bold.
Figure: Distributions of MSMD Parameter Estimates Across Firms, \( \bar{k} = 7 \). We show histograms of maximum likelihood parameter estimates across firms, obtained using \( \bar{k} = 7 \). For reference we indicate Citigroup.
Figure: Estimated Intensity Component Renewal Probability Profile Bundle, $\bar{k} = 7$. For reference we show Citigroup in bold.
Figure: **Empirical CDF of White Statistic $p$-Value, $\bar{k} = 7$.** For reference we indicate Citigroup.
Figure: Distribution of Differences in BIC Values Across Firms. We use \(-BIC/2 = \ln L - k \ln(n)/2\), and we compute differences as MSMD(7) - ACD(1,1). We show a histogram. For reference we indicate Citigroup.
**Figure:** Distribution of Differences in Forecast RMSE Across Firms. We compute differences MSMD(7) - ACD(1,1). For reference we indicate Citigroup.
Roadmap

- Empirical regularities in inter-trade durations ✓
- The MSMD model ✓
- Empirics ✓
Future Directions

- Additional model assessment
- Current data (using ultra-accurate time stamps)
- Panel of trading months.
  Structural change?